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ON n-FINITE FORCING

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ABSTRACT

The main result of this paper is that n-finite forcing companion (of a given theory) can be obtained by an application of (Robinson's) finite forcing (Corollary 3.3). Hence, in particular, it follows: for any theory T defined in a language L there exists its extension defined in a suitable expanded language L', the finite and n-finite forcing companions of which coincide (Theorem 3.8).

INTRODUCTION

In [2] we concluded that the main properties of Robinson's finite forcing are naturally transmitted (in the sense of their "translation for n") to n-finite forcing. (We did not, completely justified make an efort to give the complete proofs for all of them, down to the last one, were inspired by the corresponding ones for finite forcing). Since, however, on that occasion we used, without additional explanation, also the result from [4] (1,5 in [2]) (of a more general character), which is not directly applicable with regard to the fact that for condition we took subsets of the set (C_n) of all senten-

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ces which are equivalent to sentences in prenex normal form with at most n blocks of quantifiers (it is not fulfilled: $\phi \in C_n \to \to \mathrm{sub}(\phi) \subset C_n$) this time we shall deal with this more thoroughly. Simultaneously we shall point out the part of the "assortment", elements of which can be taken equally for a set of the condition, as well as the possible simplifications and improvements of some proofs and propositions. We shall also correct two results from [2].

n-finite forcing itself, for $n > 0$, being considered from a purely "technical" aspect does not offer much new, namely, n-finite forcing companion can be obtained by an application of Robinson's finite forcing as well. We are of the opinion howeover, that doing research on its properties and particularly on properties of the corresponding forcing companion is useful because of many reasons (and from almost all other points of view).

§ 0. Throughout this paper T is a fixed (but otherwise arbitrary) theory defined in a finite language of first order L. $L(A)$ is a normal extension of the language L (i.e. $L(A) = L \cup A$ where A is an infinite set (according to the need of sufficiently large cardinality) of new constants and $L \cap A = \emptyset$). Of course as long as we are in the sphere of syntax we can assume that A is countable infinite.

For the basic logical symbols we take $\neg$ (negation), $\wedge$ (conjuction) and $\exists$ (existential quantifier) (the others are defined by the basic ones in the standard way). FORM (L), SENT(L), AT(L) are, successively, the sets of all formulas, sentences and atomic sentences of the language L. Naturally if F is a set of formulas, SENT(F) will be the set $\{\phi \in F \mid \phi$ is a sentence$\}$. Φ_n ($\Phi_n(A)$), $n \geq 0$ is the set $\{\phi \in \mathrm{FORM}(L)$ $(\mathrm{FORM}(L(A))) \mid \phi$ is logically equivalent to a formula of the language $L(L(A))$ in prenex normal form with at most n blocks of quantifiers$\}$, and $C_n = \{p \mid p$ is a finite subset of the set $\mathrm{SENT}(\Phi_n(A))$ which is consistent with $T\}$. $\mu(T)$ is the class of all models of the theory T and $T \cap \Pi_{n+1}$ (Π_{n+1} segment of T) is $\{\phi \in \mathrm{SENT}(\Phi_n) \mid T \vdash \phi\}$.

We shall assume a knowledge of the definitions of "forcing notions" (forcing system, forcing relation and so on) as well as of their basic general properties (see [1], [5]). Yet, for the coherence of the text we shall quote (for us this time, most relevant) results (more about them as well as about their more complete version can be found in [3], [5]) remarking that in both cases we assume the standard set of logical axioms (for instance the one given in [5]),

THEOREM 0.1. *Let* $\langle C, \Vdash, L\rangle$ *be a forcing system where* L *is a finite language of arbitrary cardinality (i.e. the set of nonlogical symbols is of arbitrary cardinality). Then for each* $p \in C$ *it holds that:*

(1) $T^C[p] = \{\phi \in SENT(L) \mid p \Vdash \neg\neg\phi\}$ *is a consistent and deductively closed set* $(T^C[p] \vdash \phi$ *implies* $\phi \in T^C[p])$;

(2) *If* $\phi(v_0,\ldots,v_m)$ *is a logically valid formula (that is,* $\vdash_L \phi(v_0,\ldots,v_m)$) *then for any closed terms* $t_0,\ldots,t_m$ $\phi(t_0,\ldots,t_m) \in T^C[p]$.

THEOREM 0.2. *Let* T *be a theory of a finite language* L, A *an infinite set of new constants and let* $F \subseteq FORM(L(A))$ *satisfy the following conditions:*

(1) $\phi \in F$ *implies* $sub(\phi) \subseteq F$;

(2) $\phi \in F$ *implies* $\neg\phi \in F$

and (3) *If* $\phi(v) \in F$ *and* t *is a closed term of the language* L(A) *then, also,* $\phi(t) \in F$.

If C is the set $\{p \subset SENT(F) \mid |p| < \aleph_0$ and $T \cup p$ is consistent$\}$ partially ordered by inclusion and the relation $\Vdash$ $(\subseteq C \times SENT(L(A)))$ is determined for $\phi \in AT(L(A))$ by

$$p \Vdash \phi \text{ if and only if } \phi \in p$$

and otherwise is defined with regard to the basic logic symbols as a forcing relation, then for each $p \in C$ and each $\phi \in SENT(F)$ it holds that:

I (a) From $\phi \in p$ follows $p \Vdash \neg\neg\phi$; (b) $p \Vdash \neg\neg\phi$ implies $T \cup p \nvdash \neg\phi$; (c) if $T \cup p \nvdash \neg\phi$ then there exists an element $q \in C$ such that $p \cup \{\phi\} \subseteq q$.

(REMARK: $\Vdash$ can, but does not have to be a forcing relation; of course, a sufficient condition for $\Vdash$ to be a forcing relation is that F contains all atomic formulas);

II $p \Vdash \neg\neg\phi$ if and only if $T \cup p \vdash \phi$.

n-finite forcing is a triple $\langle C_n, \Vdash_n, L(A)\rangle$ where the forcing relation is determined (with regard to the basic logical symbols) by:

$$p \Vdash_n \phi \text{ if and only if } \phi \in p \text{ for } \phi \in AT(L(A))$$

Let M be a model of the language L and f a mapping of A into M. We shall say that $\langle A,f\rangle$ is an assignment of constants to M if $f(A)$ is a generative set for M (i.e. no proper substructure of M contains $f(A)$) ([1]). Therefore if $\langle A,f\rangle$ is an assignment of constants to M and each $a \in A$ is interpreted in M by $f(a)$ then each element of M is, let us say, denoted by (at least) one closed term (whose interpretation it is) of the language $L(A)$. Hence $M \in \mu(T \cap \Pi_{n+1})$ if and only if each finite subset of n-elementary diagram of the model M $D_n(M) = \{\phi \in SENT(\Phi_n(A)) \mid M \models \phi\}$ is a condition. By $F_{\langle A,f\rangle}(M)$ we denote the set $\{\phi(v_{i_0},\dots,v_{i_k}) \in FORM(L) \mid$ for any closed terms $t_0,\dots,t_k$ of the language $L(A)$ $M \models \phi(t_0,\dots,t_k)$ iff for some $p \subset D_n(M)$ $p \Vdash \phi(t_0,\dots,t_k)\}$. The well-known theorem says that for any two assignments of constants to M $\langle A,f\rangle, \langle A_1,f_1\rangle$ $F_{\langle A,f\rangle}(M) = F_{\langle A_1,f_1\rangle}(M)$, Thus we can write only $F(M)$. M is, let us recall that too, (T-) n finitely generic model if it is satisfied:

$$(1) \qquad M \in (T \cap \Pi_{n+1})$$

and

$$(2) \qquad F(M) = FORM(L)$$

§ 1. For n-finite forcing, let us say this immediately (the proof will be given in the next paragraph), assertions analogous to I and II from 0.2 hold:

THEOREM 1.1. *Let* $p \in C_n$ *and* $\phi \in SENT(\Phi_n(A))$. *Then* (1) $\phi \in p$; (2) $p \Vdash_n \neg\neg\phi$; (3) $T \cup p \nvdash \neg\phi$ *and* (4) *there exists a condition* q *such that* $p \cup \{\phi\} \subseteq q$ *satisfy:*

(a) (1) → (2); (b) (2) → (3) *and* (c) (3) → (4)

COROLLARY 1.2. *For* $p \in C_n$ *and* $\phi \in SENT(\Phi_n(A))$ *holds:* $p \Vdash_n \neg\neg\phi$ *if and only if* $T \cup p \vdash \phi$.

In [2] we have used more, in fact, (in truth not quite complete) Theorem 1.1 than result 1.5 there mentioned (taken over from [4]), which represents its weakened or strengthened - depending on how one looks at it-version.

The rest of this paragraph is devoted mainly to the correction and improvement of two results from [2].

THEOREM 1.3. *Let* M *be a model of the theory* $T \cap \Pi_{n+1}$ *and* $\langle A,f\rangle$ *an assignment of constants to* M. *Then the set* $F_{\langle A,f\rangle}(M)$ *contains all basic formulas and is closed under conjuction and existential quantifier.*

In particular for any Σ_n, Π_n *formula* $\phi(v_o,\dots,v_m)$ *and any closed terms* $t_o,\dots,t_m$ *of the language* L(A)

$$M \models \phi(t_o,\dots,t_m) \text{ if and only if } p \Vdash_n \neg\neg\phi(t_o,\dots,t_m)$$

for some $p \subset D_n(M)$.

P r o o f. The first part of the theorem obviously holds. In the proof of the second part we apply 1.1 and 1.2.

If $M \models \phi(t_o,\dots,t_m)$ then by 1.1 $\{\phi(t_o,\dots,t_m)\} \Vdash_n \neg\neg\phi(t_o,\dots,t_m)$.

If for some $p \subset D_n(M)$ $p \Vdash_n \neg\neg\phi$ we have with respect to 1.2 $T \cup p \vdash \phi$. Let N be a model of T such that $M <_n N$ Then $N \models \phi$ and hence also $M \models \phi$.

In general $F_{<A,f>}(\mathcal{M})$ is not closed under negation and disjunction. The difference between the definition of a forcing relation given in [1], [6] and ours (which does not take disjunction as the basic logic symbol), which is otherwise eliminated by weak forcing, makes (in [1]) $F_{<A,f>}(\mathcal{M})$ closed under disjunction as well.

Keeping in mind the already mentioned theorem that $F_{<A,f>}(\mathcal{M})$ does not depend on $<A,f>$ (the proof of which is based on 1.3) we shall assume in all the following examples that f is a one-to-one and onto mapping, and use the same notation for element of M and the constant from A corresponding to it. Also we shall write $F(\mathcal{M})$ more simply.

EXAMPLE 1.4. Let T be a theory which "says" that R is a relation of the (strict) order (i.e. irreflexive, antisymmetric and transitive relation), $<M,R^{\mathcal{M}}>$ its model where $M = \{a_i \mid i \in w\}$ and $R^{\mathcal{M}} = \{<a_0,a>\}$ and let $\Vdash_0$ be (0-) finite forcing relation. Then $\exists vR(v,a_2), \exists vR(a_2,v) \in$ $\in F(\mathcal{M})$ but $\mathcal{M} \nvDash \exists vR(v,a_2) \vee \exists vR(v,a_2)$ while $p \Vdash \exists vR(v,a_2) \vee \exists vR(a_2,v)$ (that is $p \Vdash \neg(\neg\exists vR(v,a_2) \wedge \neg\exists vR(a_2,v))$ for any $p \subset D_0(\mathcal{M})$. By the way, let us note that $\neg\exists vR(v,a_2) \notin F(\mathcal{M})$ for no condition $p \subset D_0(\mathcal{M})$ forces $\neg\exists vR(v,a_2)$ (as no condition $p \subset D_0(\mathcal{M})$ forces $\exists vR(v,a_2)$).

In case that $\vee$ is taken for basic logic symbol (and $\wedge$ is defined by $\vee$ and $\neg$), $F(\mathcal{M})$ is not necessarily closed under conjunction. For a demonstration of that, one can use the above example. By this we want, at the same time, to point out the error made in [2], where in paragraph 1 we accepted the definition of a forcing relation given in [4], but in the next paragraphs we used the one from [1], [6]. This demands that from 3.3 in [2] we delete "finite conjunction" along with, otherwise, wrong "countable" which should be replaced by "finite" (disjunction).

EXAMPLE 1.5(a). Let T be, as in the previous example, the theory of (strict) order and $<M,R^{\mathcal{M}}>$ its model, where

$M = \{a_i \mid i \in Z\} \cup \{b_j \mid j \in Z\}$ (Z-the set of integers) and $R^M = \{(a_i, a_j) \mid i < j\} \cup \{(b_k, b_\ell) \mid k < \ell\} \cup \{(a_r, b_s) \mid s \geq r+1\}$ and let $\Vdash_1$ be 1-finite forcing relation. Then, obviously, $\exists v \forall u\ (v = u \vee R(v,u) \vee (R(u,v)) \in F(M)$ but, what is less obvious, $\neg\exists v \forall u\ (v = u \vee R(v,u) \vee R(u,v)) \notin F(M)$. In fact there is no condition $p \subset D_1(M)$ which would force that sentence because $T \cup D_1(M) \cup \{\exists v \forall u\ (v = u \vee R(v,u) \vee R(u,v)\}$ is consistent.

(b) Let all conditions be as in (a) with the only exception that now $R^M = \{(a_i, a_j) \mid i < j\} \cup \{(b_i, b_j) \mid i < j\}$. Then, again, $\exists v \forall u\ (v = u \vee R(v,u) \vee R(u,v)) \in F(M)$ but this time also $\neg\exists v \forall u\ (v = u \vee R(v,u) \vee R(u,v)) \in F(M)$. For $\{\phi\} \Vdash_1 \neg\exists v \forall u\ (v = u \vee R(v,u) \vee R(u,v))$ where $\phi \in D_1(M)$ is a (Π_1-) sentence which "claims" that there is no element which is simultaneously in one of the relations $=$, R, R^{-1} (not necessarily the same) with both a_o and b_o.

COROLLARY 1.6. *Let* M *be a model of* $T \cap \Pi_{n+1}$, $\langle A, f \rangle$ *an assignment of constants to* M *and* $\phi(v_o, \ldots, v_m)$ *a* Σ_{n+1} *formula. Then for all closed terms* $t_o, \ldots, t_m$ *(of the language* $L(A)$ *it holds that:*

(a) $M \models \phi(t_o, \ldots, t_m)$ *implies: for some condition* $p \subset D_n(M)$ $p \Vdash_n \neg\neg\phi(t_o, \ldots, t_m)$;

(b) *If for some condition* $p \subset D_n(M)$ $p \Vdash_n \neg\neg\phi(t_o, \ldots, t_m)$ *then there exists an* n*-elementary extension* N *of* M *such that* $N \models T \cup \{\phi(t_o, \ldots, t_m)\}$

P r o o f: (a) is a consequence of theorems 0.1 and 1.3 and the obvious fact: if $p \Vdash_n \neg\neg\psi(t)$ for some closed term t ($\psi(v)$ is an arbitrary formula) then also $p \Vdash_n \neg\neg\exists v\psi(v)$ and (b) is a consequence of Corollary 1.2.

The next example partially corrects result 3.4 from [2] (we have just given the right version). Namely, from $p \Vdash_n \neg\neg\phi$, where $p \subset D_n(M)$ and ϕ is a Σ_{n+1} sentence, $M \models \phi$ does not necessarily follow. At the same time this correction leaves in effect the part of the proof of 3.24 (iii) (from [2] in which 3.4 is used (clearly we also have at our disposal other possibilities to prove that assertion).

EXAMPLE 1.7. Once again T is the theory of (strict) order. For its model we take $\langle Z,<\rangle$ while $\Vdash_1$ is 1-finite forcing relation. Now $Z \nVdash \exists v \forall u\,(u = v \vee R(u,v))$ though $p \equiv \{\forall u \forall v\,(u = v \vee R(u,v) \vee R(v,u))\} \Vdash_1 \neg\neg \exists v \forall u\,(u = v \vee R(u,v))$. The latter claim holds because if $q \supseteq p$ and $M \models T \cup q$ then M is linearly ordered whence either it itself has the last element or it is existentially complete in some model with the last element.

§ 2. Let, for a given $n > 0$, $\Phi'_n(A)$, $\Phi''_n(A)$ and C'_n, C''_n be, respectively, sets $\{\phi \in \Phi_n(A) \mid sub(\phi) \subset \Phi_n(A)\}$, $\{\phi \in \Phi_n(A) \mid \phi$ is in prenex normal form$\}$, that is $\{p' \subset SENT(\Phi'_n(A)) \mid p'$ is a finite subset consistent with $T\}$, $\{p'' \subset SENT(\Phi''_n(A)) \mid p''$ is a finite set consistent with $T\}$. Further, let $\Vdash'_n \subseteq C'_n \times SENT(L(A))$, $\Vdash''_n \subseteq C''_n \times SENT(L(A))$ be forcing relations defined in the same way as n-finite forcing relation (consequently, the sets C'_n, C''_n are ordered by inclusion and for $\phi \in AT(L(A))$ and $p \in C'_n(C''_n)$ $p \Vdash'_n \phi$ $(p \Vdash''_n \phi)$ iff $\phi \in p$).

LEMMA 2.1. *For all conditions* $p = \{\phi_1,\ldots,\phi_k\} \in C_n$, $p' = \{\phi'_1,\ldots,\phi'_k\} \in C'_n$ *and* $p'' = \{\phi''_1,\ldots,\phi''_k\} \in C''_n$ *such that* ϕ_i, ϕ'_i *and* ϕ''_i, $i = 1,\ldots,k$ *are equivalent with regard to* T, *and for each sentence* $\phi \in SENT(L(A))$ *it holds that:*

$$p \Vdash_n \neg\neg\phi \quad iff \quad p' \Vdash'_n \neg\neg\phi \quad iff \quad p'' \Vdash''_n \neg\neg\phi.$$

P r o o f: By induction on the complexity of ϕ. We shall only prove, just for an illustration

$$p \Vdash_n \neg\neg\phi \quad \text{iff} \quad p' \Vdash'_n \neg\neg\phi.$$

If ϕ is an atomic formula, we can check directly

$$p \Vdash_n \neg\neg\phi \;\text{ iff }\; T \cup p \vdash \phi \;\text{ iff }\; T \cup p' \vdash \phi \;\text{ iff }\; p' \Vdash'_n \neg\neg\phi.$$

The case ϕ is $\phi_1 \wedge \phi_2$ is trivial.

If ϕ is $\neg\phi_1$, $p \Vdash_n \neg\phi_1$ and $q' \supseteq p'$, $q' \in C_n'$ then q' cannot force $\phi_1 (q' \nVdash_n' \phi_1)$ (for on the contrary, by inductive assumption, $q = p \cup (q'-p') \Vdash_n \neg\neg\phi_1$) whence $p' \Vdash_n' \neg\phi_1$. Also, from $p' \Vdash_n' \neg\phi_1$ follows $p \Vdash_n \neg\phi_1$.

Let there now be $\phi \equiv \exists v\phi_1(v)$, $p \Vdash_n \neg\neg\exists v\phi_1(v)$ and $q' \supseteq p'$. For $q = p \cup (q'-p')$ there exists a condition $r \in C_n$, $r \supseteq q$ and a closed term t such that $r \Vdash_n \phi_1(t)$. By the inductive hypothesis $r' = q' \cup (r-q)' \Vdash_n' \neg\neg\phi_1(t)$. Thus $p' \Vdash_n' \neg\neg\exists v\phi_1(v)$. Analogously one would prove the opposite.

COROLLARY 2.2. $T^{f_n} = T^{f_n'} = T^{f_n''}$ *where* $T^{f_n}, T^{f_n'}$ *and* $T^{f_n''}$ *are forcing companions (of* T*) corresponding to, respectively, the forcing relations* $\Vdash_n$, $\Vdash_n'$ *and* $\Vdash_n''$.

Let us remark that with 2.1 we have proved also the following result: if $p = \{\phi_1,\dots,\phi_k\}$ and $q = \{\theta_1,\dots,\theta_k\}$ are elements of the set $C_n(C_n', C_n'')$ and if $T \vdash \phi_i \leftrightarrow \theta_i$, $i = 1,\dots,k$ then for any sentence ψ of the language $L(A)$ $p \Vdash_n \neg\neg\psi$ iff $q \Vdash_n \neg\neg\psi$ ($p \Vdash_n' \neg\neg\psi$ iff $q \Vdash_n' \neg\neg\psi_1$, $p \Vdash_n'' \neg\neg\psi$ iff $q \Vdash_n'' \neg\neg\psi$).

For given forcing relations there also holds (apparently stronger) the proposition (because of 2.1 it is sufficient to give the proof for forcing relation $\Vdash_n$):

LEMMA 2.3. *Let* p *and* q *be conditions (from* C_n*) and* $T \vdash \bigwedge p \leftrightarrow \bigwedge q$. *Then for each* $\phi \in SENT(L(A))$ $p \Vdash_n \neg\neg\phi$ *iff* $q \Vdash_n \neg\neg\phi$. *In particular* $T^{f_n}[p] = T^{f_n}[q]$.

P r o o f. Let us suppose $p(c_o,\dots,c_m) \Vdash_n \neg\neg\phi(c_o,\dots,c_m)$ where $c_o,\dots,c_m$ are all constants from A which occur in either the sentences of p or in ϕ. By the known theorem (the proof of which, among other thing, uses corollary 1.2) $\forall v_o \dots \forall v_m (\bigwedge p(v_o,\dots,v_m) \rightarrow \phi(v_o,\dots,v_m)) \in T^{f_n} \subseteq T^{f_n}[q]$ and so $q \Vdash_n \neg\neg(\bigwedge p(c_o,\dots,c_m) \rightarrow \phi(c_o,\dots,c_m)$. Since for each $\psi \in p$ $T \cup q \vdash \psi$ according to 2.5 $q \Vdash_n \neg\neg\psi$, whence $q \Vdash_n \wedge\neg\neg p$ and consequently $q \Vdash_n \neg\neg\wedge p$. It follows that $q \Vdash_n \neg\neg\phi$.

Still we are obliged to prove theorem 1.1 and corolary 1.2. But the analogies of these propositions hold for the forcing relations $\Vdash'_n$ (and set $\Phi'_n(A)$) (Theorem 0.2), hence the proofs of 1.1 and 1.2 follow from 0.1 and 2.1. Let us demostrate it for the case: (for $p \in C_n$ and $\phi \in \Phi_n(A)$) $\phi \in p$ implies $p \Vdash_n \neg\neg\phi$.

Let $p = \{\phi_1,\ldots,\phi_k\}$ and $\phi \equiv \phi_i$ for some i, $1 \leq i \leq k$ and let $p' = \{\phi'_1,\ldots,\phi'_k\} \in C'_n$ where $\vdash \phi_j \leftrightarrow \phi'_j$ $j = 1,\ldots,k$. By 0.2 $p' \Vdash'_n \neg\neg\phi'_i$ and by 0.1 $p' \Vdash'_n \neg\neg(\phi'_i \rightarrow \phi_i)$. Thus $p' \Vdash'_n \neg\neg\phi_i$ and then also (2.1) $p \Vdash \neg\neg\phi_i$.

Clearly the assertions corresponding to 1.1 and 1.2 hold for forcing relation $\Vdash''_n$ (and set $\Phi''_n(A)$) as well.

Let now C, C_o, C' and C'_o be, in order, the sets (of conditions)

$\{p|p$ is a finite set of the basic sentences of the language $L(A)$, consistent with $T\}$

$\{p|p$ is a finite subset of $SENT(\Phi_o(A))$, consistent with $T\}$,

$\{p|p$ is a finite set consistent with T, elements of which are conjunction of basic sentences of the language $L(A)\}$

and $\{p|p$ is a finite set of quantifier free sentences of the language $L(A)$ in disjunctive normal form, consistent with $T\}$

and let $\Vdash$, $\Vdash_o$, $\Vdash'$ and $\Vdash'_o$ and T^f, T^{f_o}, $T^{f'}$ and $T^{f'_o}$ be, respectively, corresponding to these sets, forcing relations i.e. forcing companions (in any case we recall: the sets are ordered by inclusion and in all cases p forces an atomic sentence ϕ if (and only if) ϕ belongs to p); $\Vdash$ is Robinson's finite forcing relation and $\Vdash_o$ is O-finite forcing relation.

By Lemma 2.1 it follows immediately that:

LEMMA 2.4. $T^{f_o} = T^{f'_o}$.

LEMMA 2.5. *For* $p' \in C'$ *let* $p(\in C)$ *be the set of all basic sentences which occur as subformulas of sentences of* p'. *Then it holds that for each* $p' \in C'$ *and each* $\phi \in SENT(L(A))$

$$p' \Vdash' \neg\neg\phi \text{ if and only if } p \Vdash \neg\neg\phi.$$

P r o o f. By induction on the complexity of formula ϕ.

COROLLARY 2.6. $T^f = T^{f'}$.

LEMMA 2.7. *If* $p' = \{\bigvee \Phi_{1_i}, \ldots, \bigvee \Phi_{k_i}\} \in C'_o$ (Φ_{m_i} *is the conjunction of basic sentences) and* $p \in C' \subseteq C'_o$ *is a condition, elements of which are arbitrarily chosen "representative" disjuncts from each sentence of* p' *(it can happen that some "representatives" coincide) then for any sentence* $\phi \in SENT(L(A))$

$$p' \Vdash'_o \neg\neg\phi \text{ implies } p \Vdash'_o \neg\neg\phi.$$

P r o o f. Obviously, for if $p \subseteq q' \in C'_o$ then $q' \cup p'$ is a condition, too.

COROLLARY 2.8. *If* $p \in C' \subseteq C'_o$ *and* $\phi \in SENT(L(A))$ *then* $p \Vdash' \neg\neg\phi$ *if and only if* $p \Vdash'_o \neg\neg\phi$.

COROLLARY 2.9. $T^{f'} = T^{f'_o}$.

THEOREM 2.10. $T^f = T^{f_o}$.

Therefore if we look at the (finite) forcing relation, before all, as a tool for obtaining the forcing companion and other, for this theory, relevant results, we are free in the choice of any of the four given sets (C, C_o, C', C'_o) for the set of conditions, that is when an n-finite forcing relation is in question, $n > 0$, we can choose between C_n, C'_n and C''_n (of course in both cases we could enlarge the assortment of the sets of conditions).

§ 3. For obtaining n-finite forcing companion $(n > 0)$ we do not need more than a finite forcing relation. Moreover each theory T of a language L is contained in (some) theory defined in (an appropriate) expansion of L for which the finite forcing and n-finite forcing companions coincide. There follow the proofs of these assertions.

Let us join to each formula $\phi(v_{i_1},\dots,v_{i_m})$ from Φ_n, where $fv(\phi) = \{v_{i_1},\dots,v_{i_m}\}$ and $(v_{i_1},\dots,v_{i_m})$ is uniquely determined (for instance, by a sequence of free occurences of variables v_{i_k}, $i \le k \le m$ in ϕ), a new relation symbol $R_{\phi,\tilde{v}}$ of length m ($R_{\phi,\tilde{v}}(t_1,\dots,t_m)$ is then always interpreted as a result of substituting in $R_{\phi,\tilde{v}}(v_{i_1},\dots,v_{i_m})$ the terms $t_1,\dots,t_m$ for occurences of $v_{i_1},\dots,v_{i_m}$, respectively); in case ϕ is a sentence its corresponding relation symbol (now in notation just R_ϕ) is of length one. In the language L' obtained by extension of the language L by the set of these new relation symbols let T' be the following set of sentences:

$$T \cup \{\forall v_{i_1},\dots,\forall v_{i_m}(\phi(v_{i_1},\dots,v_{i_m}) \leftrightarrow R_{\phi,\tilde{v}}(v_{i_1},\dots,v_{i_m}) \mid$$
$$\mid \phi \in \Phi_n - SENT(\Phi_n)\} \cup \{(\phi \leftrightarrow \forall v_o R_\phi(v_o)) \wedge (\forall v_o R_\phi(v_o) \vee \forall v_o \neg R_\phi(v_o)) \mid$$
$$\mid \phi \in SENT(\Phi_n)\}.$$

LEMMA 3.1. T' *is consistent.*

P r o o f. Clearly. Any model M of T can be extended to a model M' (with the same domain) for T' by interpreting the new relation symbols $R_{\phi,\tilde{v}}$ by $R^{M'}_{\phi,\tilde{v}}$ where $(a_1,\dots,a_m) \in R^{M'}_{\phi,\tilde{v}}$ if and only if $M \models \phi[a_1,\dots,a_m]$, while $R^{M'}_\phi = M$ if $M \models \phi$, otherwise $R^{M'}_\phi = \emptyset$.

We immediately notice the following

$$T' \vdash \forall v_o R_\phi(v_o) \leftrightarrow \exists v_o R_\phi(v_o)$$

hence also

$$T' \vdash \forall v_o R_\phi(v_o) \leftrightarrow R_\phi(t) \quad \text{for any closed term } t.$$

Let us note that for any basic sentence of the language $L'(A)$ there exists an atomic sentence of the form $R_{\phi,\tilde{v}}(t_1,\dots,t_m)$ or $R_\phi(t)$ equivalent to it with regard to T' (this is implied by the fact that the set Φ_n is closed under negation).

Let C' be the set of conditions of Robinson's finite forcing relation ($\Vdash'$) for the theory T' and language $L'(A)$ and let $C'' = \{p' \in C' \mid$ the elements of p' are sentences of the form $R_{\phi,\tilde{v}}(t_1,\dots,t_m)$ or $R_\phi(c)$ where $t_1,\dots,t_m$ are (closed) terms of $L'(A)$ and c is a fixed, but otherwise, arbitrarily chosen constant from $L(A)$ $\Vdash_n$ and C_n will remain the signs of n-finite forcing and its set of conditions for the theory T and language $L(A)$. For $p' = \{R_{\phi_1,\tilde{v}^1}(\tilde{t}^1),\dots,$ $R_{\phi_m,\tilde{v}^m}(\tilde{t}^m), R_{\psi_1}(c),\dots,R_{\psi_k}(c)\} \in C''$ $(m \geq 0,\ k \geq 0)$ let $f(p') = \{\phi_1(\tilde{t}^1),\dots,\phi_m(\tilde{t}^m),\psi_1,\dots,\psi_k\} (\in C_n)$. Obviously f is a surjective mapping of C'' onto C_n (in case L is a language without constants f is injective as well). $q' \in f^{-1}(p)$ means that $f(q') = p$. Clearly for $p',q' \in f^{-1}(p)$ $T' \vdash \bigwedge p' \leftrightarrow \bigwedge q'$.

THEOREM 3.2. *For each* $\phi \in \mathrm{SENT}(L(A))$ *and each* $p' \in C''$ *it holds that*

$$f(p') \Vdash_n \neg\neg\phi \quad \textit{if and only if} \quad p' \Vdash' \neg\neg\phi.$$

P r o o f. By induction, on the complexity of ϕ.

If ϕ is an atomic sentence then $T \cup f(p') \vdash \phi$ iff $T' \cup p' \vdash \phi$, hence also $f(p') \Vdash_n \neg\neg\phi$ iff $p' \Vdash' \neg\neg\phi$.

The case ϕ is $\phi_1 \wedge \phi_2$ is trivial.

Let us suppose now that $\phi \equiv \neg\phi_1$ and $f(p') \Vdash_n \neg\phi_1$ but $p' \nVdash' \neg\phi_1$. According to the already concluded facts (and Lemma 2.3) there exists $q' \in C''$ such that $q' \supseteq p'$ and $q' \Vdash' \neg\neg\phi_1$. It follows by the inductive assumption $f(q') \Vdash' \neg\neg\phi_1$, which is, however, in contradiction with $f(p') \Vdash_n \neg\phi_1$. On the other hand, if $p' \nVdash \neg\phi_1$ but not $f(p') \Vdash_n \neg\phi_1$ then for some $q \in C_n$, $q \supseteq f(p')$ $q \Vdash_n \neg\phi_1$. Let $q' \in C'$ be such a condition that $q' \in f^{-1}(q)$ and $p' \subseteq q'$. Then (again we keep in mind 2.3 and the above remarks) $q' \Vdash' \neg\neg\phi_1$ and there is a contradiction.

The proof for case $\phi \equiv \exists v\phi_1(v)$ is only technically somewhat more complicated. Let us suppose $f(p') \Vdash_n \neg\neg\exists v\phi_1(v)$ and let $p' \subseteq q'' \in C'$. Let us obtain a condition $(p' \subseteq) q' \in$ $\in C''$ by substituting the basic sentences of $L(A)$ and negations of the atomic sentences of $L'(A)$ in q'' by the equivalent to them (with regard to T') atomic sentences of the form $R_{\phi,\tilde{v}}(\tilde{t})$, that is, $R_\psi(c)$ and by substituting c for t in sentences of the form $R_\theta(t)$. If $r \in C_n$ and (a closed term) t are such that $f(q') \subseteq r$ and $r \Vdash_n \phi(r)$ and r' is a condition such that $q' \subseteq r' \in f^{-1}(r)$, then $r' \Vdash' \neg\neg\phi_1(t)$ whence $r'' = q'' \cup (r'-q') \Vdash' \neg\neg\phi_1(t)$ and so $p' \Vdash' \neg\neg\exists v\phi_1(v)$. Finally, let $p' \Vdash' \neg\neg\exists v\phi_1(v)$ and $q \supseteq f(p')$. If $p' \subseteq q' \in f^{-1}(q)$ there exists a condition $r'' \in C'$ and a closed term t sich that $q' \subseteq r''$ and $r'' \Vdash' \phi_1(t)$. Then $(q' \subseteq) r' \Vdash' \neg\neg\phi_1(t)$ where the condition $r' \in C'$ is obtained (from r'') in the same way as q' (from q'') in the previous case and thus by the inductive hypothesis as well $(q \subseteq) f(r') \Vdash_n \neg\neg\phi_1(t)$. Consequently, $f(p') \Vdash_n \neg\neg\exists v\phi_1(v)$.

COROLLARY 3.3. $T^{f_n} = (T')^f \cap SENT(L)$.

Let us define, recursively (and simultaneously), the sequences of languages L^k and theories T^k, $k \in \omega$ in the following way:

$$L^o = L, \quad T^o = T$$

$L^{k+1} = (L^k)'$, $T^{k+1} = (T^k)'$ (where it is assumed that $(L^k)'$ and $(T^k)'$ are being formed by extension of the language L^k, that is, theory T^k, in the way analogous to obtaining L' and T' (in the previous proposition) from L and T).

Let $L^{\omega} = \bigcup_{k \in \omega} L^{k}$ and $T^{\omega} = \bigcup_{k \in \omega} T^{k}$.

LEMMA 3.4. T^{ω} *is consistent.*

LEMMA 3.5. *Let* C^{ω} *and* C^{k}, $k \in \omega$, *be, respectively, the sets of conditions of Robinson's finite forcing relations for theory* T^{ω} *and language* $L^{\omega}(A)$, *that is, for theory* T^{k} *and language* $L^{k}(A)$. *Then* $C^{\omega} = \bigcup_{k \in \omega} C^{k}$

P r o o f. Clearly. Any model of theory T^{k} can be expanded to a model of theory T^{ω}.

Surely the following holds as well

LEMMA 3.6. *Let* C_n^{ω} *and* C_n^{k}, $k \in \omega$ *be, one after another, the sets of conditions of* n*-finite forcing relations for theory* T^{ω} *and language* $L^{\omega}(A)$, *that is, for theory* T^{k} *and language* $L^{k}(A)$. *Then* $C_n^{\omega} = \bigcup_{k \in \omega} C_n^{k}$.

LEMMA 3.7. *For each* $\phi \in \mathrm{SENT}(L^{\omega}(A))$ *and each* $p \in C^{\omega}$

$$p \Vdash \neg\neg\phi \quad \textit{if and only if} \quad p \Vdash_n \neg\neg\phi$$

where $\Vdash$ *and* $\Vdash_n$ *are finite, that is,* n*-finite forcing relation for theory* T^{ω} *and language* $L^{\omega}(A)$.

P r o o f. By induction on the complexity of formula ϕ.

The cases: ϕ is an atomic sentence and $\phi \equiv \phi_1 \wedge \phi_2$ are trivial. The case where ϕ is $\neg\phi_1$ is slightly more difficult, and the case where ϕ is $\exists v\phi_1(v)$ demands (as well as in 3.2) a litle more patience, which we shall show again.

Let $p \Vdash \neg\neg\exists v\phi_1(v)$ and $p \subseteq q \in C_n^{\omega}$, $q = p \cup \{\psi_1(\tilde{t}^1), \ldots, \psi_k(\tilde{t}^k)\}$, $k \geq 0$. Further let $\bar{\psi}_1(\tilde{t}^1), \ldots, \bar{\psi}_k(\tilde{t}^k)$ be atomic sentences such that $T^{\omega} \vdash \psi_i(\tilde{t}^i) \leftrightarrow \bar{\psi}_i(\tilde{t}^i)$, $i = 1, \ldots, k$ (obviously we have at our disposal a wide choice of such sentences, but any choice is equally good for this proof), and let $r \in C^{\omega}$ be a condition and t a closed term such that

$p \cup \{\bar{\Psi}_1(\tilde{t}^1),\ldots,\bar{\Psi}_k(\tilde{t}^k)\} \subset r$ and $r \Vdash \phi_1(t)$. By the inductive hypothesis $r \Vdash_n \neg\neg\phi_1(t)$, whence, according to 2.3, $q \cup r - \{\bar{\Psi}_1(\tilde{t}^1),\ldots,\bar{\Psi}_k(\tilde{t}^k)\} \Vdash_n \neg\neg\phi_1(t)$. Accordingly $p \Vdash_n \neg\neg\exists v\phi_1(v)$. But if $p \Vdash_n \neg\neg\exists v\phi_1(v)$ and $p \subseteq q \in C^{\omega}$ then for some $r \in C_n^{\omega}$ and some closed term t, $q \subseteq r$ and $r \Vdash_n \phi_1(t)$. Let r' be a condition made by substituting sentences from $r-q$ by atomic sentences, which are equivalent to them with regard to T^{ω}. Then $r' \Vdash_n \neg\neg\phi_1(t)$ and also (by the inductive assumption) $r' \Vdash \neg\neg\phi_1(t)$. We conclude $p \Vdash \neg\neg\exists v\phi_1(v)$.

THEOREM 3.8. $(T^{\omega})^f = (T^{\omega})^{f_n}$

P r o o f. An immediate consequence of Lemmas 2.3 and 3.7.

REFERENCES

[1] K.J. Barwise, A. Robinson, *Completing Theories by Forcing, Annals of Mathematical Logic, Vol.2,(1970), 119-142.*

[2] M. Grulović, *Primedba o forcingu, Zbornik radova Prirodno--matematičkog fakulteta u Novom Sadu, br.10 (1980), 161-171.*

[3] M. Grulović, *A Note on Forcing and Weak Interpolation Theorem for Infinitary Logics, Zbornik radova Prirodno-matematičkog fakulteta u Novom Sadu, ser.mat. Vol. 12(1982), 327-348.*

[4] H.J. Keisler, *Forcing and Omitting Types Theorem, Studies in Mathematics (ed. A. Morley), Vol. 8, 96-133.*

[5] J-P. Keller, *Abstract Forcing and Applications, Ph.D. Thesis, New York University 1977.*

[6] A. Robinson, *Forcing in Model Theory, Proceedings of the Colloquium on Model Theory, Rome 1969, 69-82.*

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REZIME

O n-KONAČNOM FORSINGU

U [2] smo manje-više samo konstatovali da se (osnovna) svojstva Robinsonovog forsinga prirodno prenose na n-konačni forsing (u smislu njihovog "transliranja za n") (oko dokaza se sasvim prirodno nismo mnogo trudili jer su do poslednjeg inspirisani odgovarajućim za konačni forsing). Kako smo, medjutim, tom prilikom, bez posebnog objašnjenja koristili i (jedan) rezultat iz [4] (1.5 u [2]) (opšteg karaktera) koji, pak, nije direktno primenljiv, s obzirom da smo za uslove uzimali podskupove skupa svih rečenica koje su ekvivalentne rečenicama u preneks normalnoj formi sa najviše n blokova kvantifikatora (takav skup (C_n) ne ispunjava: $\phi \in C_n \rightarrow \mathrm{sub}(\phi) \subset C_n$) ovde ćemo se time podrobnije pozabaviti. Ukazaćemo ujedno na deo "asortimana" čiji elementi nam se (ravnopravno) nude za skup uslova kao i moguću simplifikaciju i poboljšanje pojedinih dokaza i stavova, ali i delom korigovati dva rezultata iz [2].

Sam n-konačni forsing za $n > 0$, gledano sa čisto "tehničkog" aspekta ne nudi nam mnogo novog, naime n-konačno forsing pridruženje (date teorije) se može dobiti i primenom Robinsonovog konačnog forsinga. Posledica toga je i da za svaku teoriju T definisanu u jeziku L postoji proširenje T' definisano u adekvatnom proširenju jezika $L(L')$ za koju se konačno i n-konačno forsing pridruženje podudaraju. No mišljenja smo, izučavanje njegovih svojstava i posebno svojstava korespodentnog forsing pridruženja korisno je iz više razloga (i iz gotovo svih ostalih aspekata).