Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 13 (1983)
Review of Research Faculty of Science-University of Novi Sad, Volume 13 (1983)

GRAPHS WHOSE ENERGY DOES NOT EXCEED 3

Aleksandar Torgašev
Institute of Mathematics, Faculty of Science
Studentski trg 16a, 11000 Beograd, Jugoslavija

ABSTRACT

In this paper, we determine all the finite connected graphs, whose energy (i.e. the sum of all positive eigenvalues) does not exceed 3. To do this, we consistently apply the method of forbidden subgraphs.

INTRODUCTION

Throughout the paper, we shall consider only finite connected graphs, having no loop or multiple edges. The spectrum of such a graph G is the set of eigenvalues of its 0-1 adjacency matrix $A(G)$. The sum of all its positive eigenvalues is denoted by $S(G)$, and called the energy of G.

For any real $a \geq 1$, we consider the class of graphs

$$P(a) = \{G | S(G) \leq a\} ,$$

and, in this paper, we shall completely describe the class $P(3)$.

Briefly, any graph $G \in P(3)$ is here called - admissible, and any other graph - impossible.

AMS Mathematics subject classification (1980): 05C40
Key words and phrases: Finite connected graphs, the energy.

Let next G' be any connected (induced) subgraph of a graph G, which is denoted by $G' \subseteq G$. Since by the known interlacing theorem [1, p.19] $S(G') \leq S(G)$, we have that any connected subgraph of an admissible graph is admissible, too. This implies that the method of forbidden subgraphs can be consistently applied.

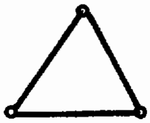
Throughout the paper, K_n, P_n, C_n will be the complete graph, the path and the cycle with $\underline{n}$ vertices, respectively, while $K_{n,m}$ is the complete bipartite graph with $n+m$ vertices.

In this paper, without special mention, we shall often use the lists of spectra of all connected graphs with 2, 3, 4 or 5 vertices (see [1]), or connected graphs with 6 vertices (112 graphs; an internal publication). So, using these lists, for each particular graph with this number of vertices, we shall determine whether it is admissible or not.

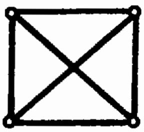
RESULTS

Denote by a circle any set of isolated vertices, and by the line between two circles the fact that there are all edges between these circles.

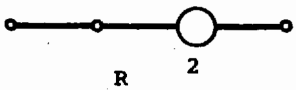
Then, by the direct inspection of the spectra of all the connected graphs with 2, 3, 4, 5 or 6 vertices, we have that all the admissible graphs with at most 6 vertices, belong to one of the following classes of graphs:



K_3



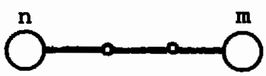
K_4



R

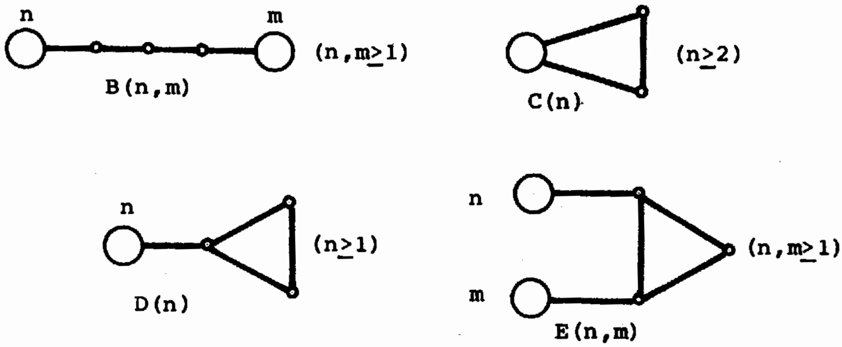
$(n,m \geq 1)$

K(n,m)



$(n,m \geq 1)$

A(n,m)



Now, we shall determine the exact values of parameters for which the above graphs are admissible.

LEMMA 1. *The graph* K(n,m) $(1 \leq n \leq m)$ *is admissible exactly in the following cases:*

1) $n = 1, \quad m \leq 9$;

2) $n = 2, \quad m = 2,3,4$;

3) $n = m = 3$.

P r o o f. As is easily seen, this graph is admissible if and only if $nm \leq 9$ holds, whence the statement is immediate. □

LEMMA 2. *The graph* A(n,m) $(1 \leq n \leq m)$ *is admissible exactly in the following cases:*

1) $n = 1, \quad m = 1,2,3$;

2) $n = m = 2$.

P r o o f. Immediately, this graph is admissible if and only if $\sqrt{n} + \sqrt{m} \leq 2\sqrt{2}$, whence the statement is obvious. □

LEMMA 3. *The graph* B(n,m) $(1 \leq n \leq m)$ *is admissible exactly in the following cases:*

1) $n = 1, \quad m = 1,2,3$;

2) $n = m = 2$.

P r o o f. As is easily seen, the graph $B(n,m)$ is an admissible graph if and only if $n + m + 2\sqrt{n+m} \leq 9$, whence the statement. □

LEMMA 4. *The graph* $C(n)$ $(n \geq 2)$ *is admissible iff* $n = 2,3$.

P r o o f. Indeed, since it is a complete 3-partite graph, it has exactly one positive eigenvalue $r_n = r(C(n))$, and $r_n = (1 + \sqrt{1 + 8n})/2 \leq 3$ iff $n = 2,3$. □

LEMMA 5. *The graph* $D(n)$ $(n \geq 1)$ *is admissible iff* $n = 1,2$.

P r o o f. The graphs $D(1)$, $D(2)$ are admissible, and $D(3)$ is not so. Whence, all $D(n)$ $(n \geq 3)$ are non-admissible, also. □

LEMMA 6. *The graph* $E(n,m)$ $(n,m \geq 1)$ *is admissible iff* $n = m = 1$.

P r o o f. Indeed, since $E(1,1)$ is admissible, and $E(1,2)$ is an impossible graph, we have that $E(n,m)$ for $n \geq 2$ or $m \geq 2$, are impossible graphs. □

Now, we shall prove the main result of the paper.

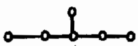
THEOREM 1. *Each admissible graph* G *is one of the graphs displayed in Figure 1.*

P r o o f. We distinguish the next three cases:

I. There is no C_3 or C_4 in G as a subgraph;

II. There is a C_3 in G;

III. There is C_4 but no C_3 as an induced subgraph in G.

CASE I. Since C_n $(n \geq 5)$ cannot be a subgraph of an admissible G, we conclude that, in this case, there is no contour in G; thus G is a tree. Since, next, there is no P_5 or  in G, we have that G must be one of the graphs $K(1,n)$ $(n \geq 1)$, $A(n,m)$ $(n,m \geq 1)$, $B(n,m)$ $(n,m \geq 1)$.

CASE II: Denote the vertices of the $L = C_3$ in G by 1,2,3. Next, denote by T_i $(i = 1,2,3)$ the vertices of G which are (with respect to L) adjacent exactly to the vertex $\underline{i}$; The denotations T_{12}, T_{13}, T_{23} and T_{123} have a similar meaning. Put

$$T = T_1 + T_2 + T_3 + T_{12} + T_{13} + T_{23} + T_{123} .$$

Next, denote by $\tilde{T}_i$ the vertices of G (non-adjacent to L), which are (with respect to T), adjacent exactly to some vertices of T_i; the denotations $\tilde{T}_{ij}$ $(i \neq j)$ and $\tilde{T}_{123}$ have a similar meaning.

Now, we are interested to determine the edge structure of each particular subset between T_i, T_{ij} and T_{123}, as well as the edge structure between these subsets.

For any two subsets A,B, we use the denotation $A/A = 0$ if A consists of the isolated vertices only, $A/A = 1$ if it is complete, $A/B = 0$ or 1 or $\emptyset$ or $*$, if there is no edge between A and B, or there are all such edges, or A and B are not consistent, or we cannot determine this structure, respectively.

All the above information is obtained by choosing two arbitrary vertices $a \in A$, $b \in B$, then testing the subgraph 123ab in the two possible cases - whether a,b are adjacent or not.

So, by the impossible graphs or order 5, we obtain the following relations easily:

$$T_i/T_i = 0 \quad (i = 1,2,3), \qquad T_{ij}/T_{ij} = 0 \quad (i \neq j),$$

$$|T_{123}| \leq 1, \quad T_i/T_j = 0 \quad (i \neq j), \quad T_i/T_{ij} = \emptyset,$$

$$T_i/T_{jk} = \emptyset, \quad T_i/T_{123} = \emptyset, \quad T_{ij}/T_{ik} = \emptyset,$$

$$T_{ij}/T_{123} = \emptyset .$$

Next, testing the graph 123abc ($a \in T_1$, $b \in T_2$, $c \in T_3$), we also obtain that the 3-tuple T_1, T_2, T_3 is not consistent in G.

Similarly, we obtain that

$$\tilde{T}_i = \emptyset , \quad \tilde{T}_{ij} = \emptyset , \quad \tilde{T}_{123} = \emptyset ,$$

whence follows that each admissible G, in case II, consists of $L = C_3$ and eventually of the classes T_i, T_{ij}, T_{123}.

In view of all the above results, excluding the symmetric cases, we have that G, in case II, consists of one of the following subsets: only $L = C_3$, $L + T_1$, $L + T_{12}$, $L + T_{123}$, $L + T_1 + T_2$.

Consequently, G is one of the following graphs: C_3, K_4, $C(n)$ $(n \geq 2)$, $D(n)$ $(n \geq 1)$, $E(n,m)$ $(n,m \geq 1)$.

CASE III. Denote the vertices of $L = C_4$ in G by 1,2,3,4.

Then, similarly as in the previous case, we have the subsets T_i, T_{ij}, T_{ijk} and T_{1234} in G.

By assumption, or by forbidden subgraphs, we conclude easily that

$$T_{12} = T_{23} = T_{34} = T_{14} = \emptyset, \quad T_{ijk} = \emptyset \text{ and } T_{1234} = \emptyset.$$

so in T there remain only the subsets T_i $(i = 1,2,3,4)$ and T_{13}, T_{24}.

By the impossible graphs of order 6 (and by the assumption), we conclude that $\tilde{T}_i = \emptyset$ and $\tilde{T}_{13} = \tilde{T}_{24} = \emptyset$, thus, in this case, each admissible G consists only of the subsets L, T_1, T_2, T_3, T_4 T_{13} and T_{24}.

As in case II, we conclude that

$$|T_i| \leq 1, \qquad T_{13}/T_{13} = T_{24}/T_{24} = 0 ,$$

$$T_i/T_j = \emptyset, \quad T_i/T_{13} = T_i/T_{24} = \emptyset, \quad T_{13}/T_{24} = 1.$$

Hence, excluding the symmetric cases, we have that G consists of one of the following subsets: $L = C_4$, $L + T_1$, $L + T_{13}$, $L + T_{13} + T_{24}$.

Consequently, in case III, G must be one of the following graphs: $K(2,2)$, R, $K(2,n)$ $(n \geq 3)$, $K(n,m)$ $(n,m \geq 3)$, which completes the proof. □

Note that Theorem 1 and Lemma 1-6 describe the class $P(3)$ completely.

Note, still, that the previous results imply that class $P(a)$ is finite if $a = 3$. In the following theorem, we shall prove this for any $a \geq 1$.

THEOREM 2. *The class* $P(a)$ *is finite, for any* $a \geq 1$.

P r o o f. Choose an arbitrary graph $G \in P(a)$ and any (not necessarily induced) subgraph $K(1,n)$. Then, since $a \geq S(G) \geq r(G) \geq \sqrt{n} = r(K(1,n))$, where $r(G)$ is the spectral radius of G (see Theorem 0.9 [1, p.15] for the last inequality), we conclude that $K(1,n) \in P(a)$, thus that all such n's are uniformly bounded by $b = a^2$. Hence, the degrees of all the vertices in G cannot exceed the constant b.

Next, choose any path P_n. Since, for an arbitrary $q \in N$, its q-th positive eigenvalue tends to 2 as $n \to \infty$, we get that $S(P_n) \to \infty$ as $n \to \infty$. Consequently, for any path $P_n \in P(a)$ we have that all n's are uniformly bounded by a constant $k = f(a)$.

Now, assume, contrarily to the statement, that the set $P(a)$ is finite for an $a \geq 1$. Then, it is seen easily that either there is a sequence of the complete bipartite graphs $K(1,n_i) \in P(a)$ $(n_1 < n_2 < \ldots)$, or there is a sequence of paths $P(n_i)$ $(n_1 < n_2 < \ldots)$, and both these cases give the contradictions.

This proves the theorem. □

REFERENCES

[1] D.M. Cvetković, M. Doob, H. Sachs, *Spectra of graphs – – Theory and Application*, *VEB Deutscher Verlag der Wissen., Berlin, 1980; Academic Press, New York, 1980.*

Received by the editors May 27, 1984.

REZIME

GRAFOVI ČIJA ENERGIJA NE PRELAZI 3

U ovom radu su odredjeni svi konačni povezani grafovi, čija energija (zbir svih pozitivnih svojstvenih vrednosti) ne prelazi 3.