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CONNECTEDNESS OF THE NON-COMPLETE EXTENDED p-SUM OF GRAPHS

Dragoš M. Cvetković and Milenko V. Petrić***

**University of Belgrade, Faculty of Electrical Engineering, POB 816, 11001 Beograd, Yugoslavia*

***University of Novi Sad, Advanced Technical School, Školska 1, 21000 Novi Sad, Yugoslavia*

ABSTRACT

We extend the definition of the non-complete extended p-sum (NEPS) of graphs to digraphs (digraphs can have multiple arcs and/ or loops). Using the spectral method we prove a theorem giving the necessary and sufficient condition for a NEPS of strongly connected digraphs to be strongly connected. Some related results are obtained.

Let B be a set of n-tuples $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, of symbols 0 and 1, which does not contain an n-tuple $(0,0,\ldots,0)$.

DEFINITION. *The non-complete extended p-sum (NEPS) with a basis* B *of digraphs* $G_1, G_2, \ldots, G_n$ *is the digraph* G *whose set of vertices is the Cartesian product of the sets*

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of vertices of digraphs $G_1, G_2, \ldots, G_n$. *For two vertices* $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *construct all the possible arc selections of the following type. For each* $\beta \in B$ *and for any* i $(i = 1, 2, \ldots, n)$ *select an arc from* x_i *to* y_i *in* G_i *if* $\beta_i = 1$ *and suppose* $x_i = y_i$ *if* $\beta_i = 0$. *The number of arcs going from* $(x_1, x_2, \ldots, x_n)$ *to* $(y_1, y_2, \ldots, y_n)$ *is equal to the number of such selections.*

If B consists of all the possible n-tuples (of course, without the n-tuple $(0,0,\ldots,0)$) the operation is called a strong product. The incomplete p-sum (complete p-sum, or briefly, p-sum) is obtained if B consists of (all the possible) n-tuples with exactly p 1's. If $p = n$, the p-sum is called a product.

Some special cases of this definition have already appeared in literature (see, for example [1,p.303], [6], [7]).

Let $A \otimes B$ denote the Kronecker product of matrices A and B. Let $(A)_{xy}$ be the element of the matrix A from the row corresponding to vertex x and the column cerresponding to vertex y of a graph which corresponds to A.

THEOREM 1. *The NEPS* G *with the basis* B *of digraphs* $G_1, G_2, \ldots, G_n$, *whose adjacency matrices are* $A_1, A_2, \ldots, A_n$, *has the following adjacency matrix*

$$A = \sum_{\beta \in B} A_1^{\beta_1} \otimes A_2^{\beta_2} \otimes \ldots \otimes A_n^{\beta_n} .$$

P r o o f. In each of the digraphs $G_1, G_2, \ldots, G_n$ let the vertices be ordered (labelled). We shall order, lexicographically, the vertices of G (which represent the ordered n-tuples of the vertices of digraphs $G_1, G_2, \ldots, G_n$) and form the adjacency matrix A according to this ordering.

By virtue of the properties of the Kronecker product of matrices, the entries of A are

$$(1) \quad (A)_{(x_1,\ldots,x_n),(y_1,\ldots,y_n)} = \sum_{\beta \in B} (A_1^{\beta_1})_{x_1 y_1} \ldots (A_n^{\beta_n})_{x_n y_n} .$$

By virtue of the lexicographic ordering, (1) holds if and only if for each $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in B$ there exist $(A_i)_{x_i y_i}$ arcs leading from x_i to y_i in G_i if $\beta_i = 1$, and $x_i = y_i$ if $\beta_i = 0$.

This completes the proof of the Theorem.

The results in [6] are a special case of this theorem.

THEOREM 2. *For* $i = 1, 2, \ldots, n$ *let* G_i *be a digraph with* n_i *vertices, and let* $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}$ *be the spectrum of* G_i. *Then the spectrum of NEPS with the basis* B *of digraphs* $G_1, G_2, \ldots, G_n$ *consists of all the possible values of* $\Lambda_{i_1, \ldots, i_n}$, *where*

$$\Lambda_{i_1, \ldots, i_n} = \sum_{\beta \in B} \lambda_{1i_1}^{\beta_1} \cdots \lambda_{ni_n}^{\beta_n}, \quad (i_k = 1, \ldots, n_k;\ k = 1, \ldots, n) .$$

The proof coincides with the proof in the case of undirect graphs (c.f.Theorem 2.23 in [3]).

It is obvious that NEPS G is not strongly connected if any one of digraphs $G_1, G_2, \ldots, G_n$ is not strongly connected or if B has not the property (D) that for every $j \in \{1, 2, \ldots, n\}$ there exists in B at least one n-tuple $(\beta_1, \beta_2, \ldots, \beta_n)$ with $\beta_j = 1$. (This condition implies that the NEPS, effectively, depends on each G_i.)

Let h be the greatest common divisor of the lengths of all the cycles in a digraph G. The digraph is called primitive if it is strongly connected and $h = 1$ [5,p.210], and inprimitive if it is strongly connected and $h > 1$. In the second case h is called the index of imprimitivity (h is the index of imprimitivity of the adjacency matrix of the digraph G as well [2,p.183]).

THEOREM 3. *Let* $G_1, G_2, \ldots, G_n$ *be strongly connected digraphs each containing at least two vertices. Suppose also that* $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$ $(\{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, n\})$ *are* *imprimitive with the imprimitivity indices* $h_{i_1}, h_{i_2}, \ldots, h_{i_s}$, *respectively. The NEPS with the basis* B *satisfying condition* (D), *of digraphs* $G_1, G_2, \ldots, G_n$ *is a strongly connected digraph if and only if for every non-empty subset* $\{j_1, j_2, \ldots, j_k\}$ *of* $\{i_1, i_2, \ldots, i_s\}$ *and for every choice of integers* $\ell_{j_1}, \ell_{j_2}, \ldots, \ell_{j_k}$ $(1 \leq \ell_{j_t} \leq h_{j_t}-1;\ t = 1,2,\ldots,k)$, *there exists* $\beta \in B$ *such that* $\{j_1, j_2, \ldots, j_k\} \cap \{r \mid \beta_r = 1\} = \{\nu_1, \nu_2, \ldots, \nu_m\} \neq \emptyset$ *and*

$$\frac{\ell_{\nu_1}}{h_{\nu_1}} + \frac{\ell_{\nu_2}}{h_{\nu_2}} + \ldots + \frac{\ell_{\nu_m}}{h_{\nu_m}}$$

is not an integer.

Moreover, the number of strong components of NEPS is equal to the number of solutions in integers x_i $(0 \leq x_i \leq h_i-1)$, x_β *of the following system of equations*

$$\frac{x_{i_1}}{h_{i_1}}\beta_{i_1} + \frac{x_{i_2}}{h_{i_2}}\beta_{i_2} + \ldots + \frac{x_{i_s}}{h_{i_s}}\beta_{i_s} = x_\beta \quad (\beta \in B) .$$

P r o o f. According to Theorem 0.4 and 0.5 from [3] a digraph G, with an adjacency matrix A, is strongly connected if and only if its index r is a simple eigenvalue and if the positive eigenvectors belong to r both in A and A^T. However, if the index r has a multiplicity p, the other conditions being the same, then G has exactly p strong components.

Let $r_1, r_2, \ldots, r_n$ be indices of digraphs $G_1, G_2, \ldots, G_n$, respectively, and let $u_1, u_2, \ldots, u_n$ $(v_1, v_2, \ldots, v_n)$ be positive eigenvectors [3,p.18] (Theorem of Frobenius) belonging to $r_1, r_2, \ldots, r_n$ in $A_1, A_2, \ldots, A_n$ $(A_1^T, A_2^T, \ldots, A_n^T)$, respectively. Then, from Theorem 1, it immediately follows that $u = u_1 \otimes u_2 \otimes \ldots \otimes u_n$ $(v = v_1 \otimes v_2 \otimes \ldots \otimes v_n)$ is the positive eigenvector belonging to the index $\Lambda = \sum_{\beta \in B} r_1^{\beta_1} r_2^{\beta_2} \cdots r_n^{\beta_n}$ in $A(A^T)$.

By Theorem 2 the index Λ of NEPS can be obtained only from those eigenvalues of the digraphs G_i $(i = 1,2,\dots,n)$ which have a modulus equals to r_i. All these eigenvalues of G_j can be written in the form $r_j \exp(\ell_j \frac{2\pi}{h_j})$, $0 \le \ell_j \le h_j-1$, $(\exp(t) = e^{ti},\ i^2 = -1)$ (Theorem of Frobenius).

By Theorem 2 we have

$$(2)\quad \Lambda = \sum_{\beta\in B} r_1^{\beta_1} r_2^{\beta_2} \cdots r_n^{\beta_n} \exp\Big(\Big(\frac{\ell_{i_1}}{h_{i_1}}\beta_{i_1} + \frac{\ell_{i_2}}{h_{i_2}}\beta_{i_2} + \dots + \frac{\ell_{i_s}}{h_{i_s}}\beta_{i_s}\Big)2\pi\Big) .$$

From (2), it follows that the multiplicity of the index Λ is equal to the number of solutions in integers x_i, $0 \le x_i \le h_i-1$ of the system of equations given above. Furthermore, Λ is a simple eigenvalue if for each choise $\ell_{i_1},\ell_{i_2},\dots,\ell_{i_s}$, $0 \le \ell_{i_t} \le h_{i_t}-1$ $(t = 1,2,\dots,s)$ with at least one $\ell_{i_t} > 0$, at least one sumand in Λ is different from $r_1^{\beta_1} r_2^{\beta_2} \cdots r_n^{\beta_n}$ (i.e. the argument of the operator exp is different from $2k\pi$, $k \in Z$).

From these facts, the statement of the theorem follows.

The strong components of NEPS in this theorem are its components also, i.e. there are no arcs between different strong components (Theorem 7′ from [4,p.376]).

The following theorem is a specialization of the preceding one.

THEOREM 4. *Let* $G_1,G_2,\dots,G_n$ *be strongly connected digraphs each containing at least two vertices and let* $G_{i_1},G_{i_2},\dots,G_{i_s}$ $(\{i_1,i_2,\dots,i_s\} \subset \{1,2,\dots,n\})$ *be imprimitive with imprimitivity indices* $h_{i_1},h_{i_2},\dots,h_{i_s}$*, respectively. Then the p-sum of* $G_1,G_2,\dots,G_n$ *is a strongly connected digraph if and only if one of the following condition holds:*

1° $n - s \ge p - 1$;

2° $n - s < p - 1$,

and for every non-empty subset $\{j_1,j_2,\ldots,j_k\}$ *of* $\{i_1,i_2,\ldots,i_s\}$ $(n-p+2 \le k \le s)$ *and for each choise of integers* $\ell_{j_1},\ell_{j_2},\ldots,\ell_{j_k}$; $1 \le \ell_{j_t} \le h_{j_t}-1$ $(t = 1,2,\ldots,k)$ *there exists a non-empty subset* $\{\nu_1,\nu_2,\ldots,\nu_m\}$ *of* $\{j_1,j_2,\ldots,j_k\}$ $(p+k-n \le m \le \min(k,p))$ *such that*

$$\frac{\ell_{\nu_1}}{h_{\nu_1}} + \frac{\ell_{\nu_2}}{h_{\nu_2}} + \ldots + \frac{\ell_{\nu_m}}{h_{\nu_m}}$$

is not an integer.
The number of strong components in the p-sum is equal to the number of solution in integers x_i $(0 \le x_i \le h_i-1)$, $x_{j_1j_2\ldots j_p}$ *of the following system of equations*

$$\frac{x_{j_1}}{h_{j_1}} + \frac{x_{j_2}}{h_{j_2}} + \ldots + \frac{x_{j_p}}{h_{j_p}} = x_{j_1j_2\ldots j_p}$$

where $\{j_1,j_2,\ldots,j_p\}$ *runs over all p-subsets of* $\{1,2,\ldots,n\}$.

For $p = n$, from this theorem, it follows that the product of digraphs $G_1,G_2,\ldots,G_n$ is strongly connected if and only if $h_{i_1},h_{i_2},\ldots,h_{i_s}$ are the relative prime in pairs (which is well known [7]) and have as many strong components as is the number of solutions in integers x_i $(0 \le x_i \le h_i-1)$, x of the equation

$$\frac{x_{i_1}}{h_{i_1}} + \frac{x_{i_2}}{h_{i_2}} + \ldots + \frac{x_{i_s}}{h_{i_s}} = x\,.$$

It can be easily shown that this equation has exactly [1)]

$$\frac{h_{i_1}\cdot h_{i_2}\ldots h_{i_s}}{\text{l.c.m.}(h_{i_1},h_{i_2},\ldots,h_{i_s})}$$

soluitons, which implies the result from [7].

[1)] *l.c.m. denotes the lowest common multiple*

Finally, we shall prove a simple result concerning regularity properties. A digraph is caled a regular of degree r if each indegree and each outdegree equals r. It is easy to see that a digraph is regular if the eigenvector $(1,1,\dots,1)$ belongs to its index both in the adjacency matrix and its transpose.

THEOREM 5. *The NEPS of regular digraphs is a regular digraph.*

P r o o f. The vector $u_1 \otimes u_2 \otimes \dots \otimes u_n$, where $u_1, u_2, \dots, u_n$ are eigenvectors of indices of $G_1, G_2, \dots, G_n$, is an eigenvector belonging to the index of NEPS.

REFERENCES

[1] C. Berge, *Graphes et hypergraphes, DUNOD, Paris 1970.*

[2] D.M. Cvetković, *Kombinatorna teorija matrica, Naučna knjiga, Beograd 1980.*

[3] D.M. Cvetković, M. Doob, H. Sachs, *Spectra of graphs--Theory and application, Deutscher Verlag der Wissenschaften-Academic Press, Berlin-New York 1980; second edition 1982.*

[4] Ф.Р. Гантмахер, Теория матриц, Наука, Москва 1966.

[5] F. Harary, *Graph Theory, Addison-Wesley, Reading 1972.*

|6| G.B. Martini, *Somme cartesiane di n-grafi e loro matrici associata, Riv.Mat.Univ.Parma (u) 1(1975),79-87.*

[7] M.H. McAndrew, *On the product of directed graphs, Proc. Amer.Math.Soc., 14(1963), 600-606.*

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REZIME

POVEZANOST NEPOTPUNE PROŠIRENE
p-SUME GRAFOVA

U radu je proširena definicija nepotpune proširene p-sume (NEPS) grafova na digrafove. Korišćenjem spektralnog metoda dokazana je teorema (Theorem 3) koja daje potrebne i dovoljne uslove da NEPS jako povezanih digrafova bude jako povezan digraf.