Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 13 (1983)
Review of Research Faculty of Science-University of Novi Sad, Volume 13(1983)

NOTE ON THE STANDARD MATRIX REPRESENTATION OF A MATROID

Dănuţ Marcu
Faculty of Mathematics, University of Bucharest
Academiei, 14, 70109 Bucharest, Romania

ABSTRACT

The main result of this paper is: If $[I_k, A]$ is the SMR of a matroid with respect to a base B, then rank $(A) \geq C(B)$, where $C(B)$ denotes the maximal number of pairwise disjoint fundamental circuits with respect to B.

INTRODUCTION.

The matroid theory terminology and results used in this paper conform to standard literature (see for example [3,4]).

Let E be a finite set, and $M := M(E,r)$ a matroid on E with r as the rank function ($r : 2^E \to \mathbb{N}$, where 2^E is the power set of E, and $\mathbb{N}$ the set of non-negative integers). A subset $S \subseteq E$ is called independent if $r(S) = |S|$, where $|S|$ denotes the cardinality of S. Let $F(M)$ be the family of independent sets of M. A basis of M is a maximal independent subset of E. A subset of E which is not independent is called dependent, and a minimal dependent subset of E is a

---

AMS Mathematics subject classification (1980): 05B35
Key words and phrases: Matroid, standard matrix representation of a matroid (with respect to a base), fundamental circuit.

circuit. We shall denote by $C(M)$ the family of circuits of M. A circuit of cardinality 1 is a loop of M. For every $S \subseteq E$, $\bar{S}$ denotes the span of S, i.e. $\bar{S} = \{e \in E : r(S \cup \{e\}) = r(S)\}$.

If M is a matroid with $k = r(E)$, let $B = \{e_1, e_2, \ldots, e_k\}$ be a basis of M and $E - B = \{f_1, f_2, \ldots, f_m\}$. If M is representable over a field $\mathbb{F}$ (see [1]), it will have a standard matrix representation (see also [2]), with respect to the basis B, of the of the form $R = [I_k, A]$, where I_k is the identity matrix of order k and A is a $k \times m$ matrix with entries belonging to $\mathbb{F}$. If $E_1, E_2, \ldots, E_k$, $F_1, F_2, \ldots, F_m$ denote the column vectors of R, and the map σ, defined by $\sigma(e_i) = E_i$, $i=1, 2, \ldots, k$ and $\sigma(f_i) = F_i$, $i=1,2,\ldots,m$, is a representation of M over $\mathbb{F}$, in the sense that a subset S of E belongs to $F(M)$ if and only if the corresponding vectors of $\sigma(S)$ are linearly independent over $\mathbb{F}$ (see [3]).

If B is any basis of M and $\{f_1, f_2, \ldots, f_m\} = E - B$, then there exists a unique $C_i \in C(M)$, containing f_i and, otherwise, elements of B only. This circuit C_i is called fundamental with respect to f_i and B and will be denoted by $C(f_i, B)$. For a fixed basis B of a representable matroid M, we denote by $c(B)$ the maximal number of pairwise disjoint fundamental circuits with respect to $E - B$ and B. The main result of this paper is the following:

THEOREM. *If* $R = [I_k, A]$ *is the standard matrix representation with respect to* B, *then rank* $\operatorname{rank}(A) \geq c(B)$.

The key lemma. Let B be a fixed basis of M and $e \in E$ arbitrarily chosen. Obviously, $e \in \bar{B}$. Then we can consider the family of sets $D(e,B) = \{S \subseteq B : e \in \bar{S}\}$.

We shall denote by $B(e)$ a minimal (with respect to the inclusion of sets) element of $D(e,B)$. It can be checked that

$$\text{if} \quad r(\{e\}) = 0, \quad \text{then } B(e) = \emptyset,$$
$$\text{if} \quad e \in B, \quad \text{then } B(e) = \{e\}.$$

Suppose that $e \notin B$, and let $C(e,B)$ be the fundamental circuit with respect to e and B. Obviously, by definition, $B(e) \cup \{e\}$ is a circuit of M contained in $B \cup \{e\}$, and therefore $B(e) \cup \{e\} = C(e,B)$. Thus, $B(e)$ is uniquely defined. Throughout, we shall make use of the following fundamental properties of the matroids:

(P1) If $C_1, C_2 \in C(M)$ such that $C_1 \subseteq C_2$, then $C_1 = C_2$.

(P2) If C_1, C_2 are two distinct circuits of M such that $e \in C_1 \cap C_2$, then there exists $C_{12} \in C(M)$ with $C_{12} \subseteq (C_1 \cup C_2) - \{e\}$.

(P3) A subset of E is independent if and only if it does not contain a circuit.

(P4) A subset of E is independent if and only if it is contained in a basis.

LEMMA. *Let B be a fixed basis of M and $F \subseteq E$. If F does not contain any loop and $B(e) \cap B(f) = \emptyset$ for every distinct $e, f \in F$, then $F \in F(M)$.*

P r o o f. Let $F = \{e_1, e_2, \ldots, e_t\}$. If $F \subseteq B$, then the theorem is trivial, by (P4). Thus, two cases must be considered:

(a) $F \cap B = \emptyset$,

(b) $F = H \cup G$, $H \cap B = \emptyset$, $G \subseteq B$ and $G \neq \emptyset$.

We prove now the lemma in case (a). Let us denote $B(e_i) = B_i$. We have seen above that $B_i \cup \{e_i\} = C(e_i, B)$.

Suppose that $F \notin F(M)$. By (P3), there exists $C \in C(M)$ such that $C \subseteq F$. Without loss of generality, we can consider C of the form $C = \{e_1, e_2, \ldots, e_s\}$. By (P2), there exists a circuit $C_1 \subseteq (C \cup C(e_1,B)) - \{e_1\}$. If $C_1 \subseteq C(e_1,B)$, then $C_1 = C(e_1,B)$ by (P1), and this is impossible. Thus, $C_1 \subseteq (B_1 \cup C) - \{e_1\}$ and $C_1 \neq C$. Similarly, there exists $C_2 \in C(M)$ with $C_2 \subseteq (B_2 \cup C) - \{e_2\}$ and $C_2 \neq C$. Suppose that $e_1 \in C_2$. By (P2), there exists $\tilde{C} \in C(M)$

such that $\tilde{C} \subseteq (C \cup C_2) - \{e_1\}$, $\tilde{C} \neq C_2$. Since $B_1 \cap B_2 = \emptyset$ (by hypothesis), we then have $\tilde{C} \neq C_1$. If $e_2 \in \tilde{C} \cap C_1$, then, by (P2), there exists $\tilde{\tilde{C}} \in C(M)$ with $\tilde{\tilde{C}} \subseteq (\tilde{C} \cup C_1) - \{e_1, e_2\}$. Therefore, there exists $C_o \in C(M)$ such that $C_o \subseteq (B_1 \cup B_2 \cup C) - \{e_1, e_2\}$.

On the other hand, $C_3 = B_3 \cup \{e_3\}$ is a circuit of M and $C_3 \neq C_o$. If $e_3 \in C_o$, then, by (P2), there exists $C_o' \in C(M)$ with $C_o' \subseteq (C_o \cup C_3) - \{e_1, e_2, e_3\}$. Thus, there exists C_o (if $e_3 \in C_o$, we take $C_o := C_o'$) in $C(M)$ such that $C_o \subseteq (B_1 \cup B_2 \cup B_3 \cup C) - \{e_1, e_2, e_3\}$. Repeating the above a finite number of times (s-3 times), thus, gives rise to a circuit C_o with $C_o \subseteq B_1 \cup B_2 \cup \ldots \cup B_s$, i.e. a subset of B contains a circuit, contradicting (P3) and (P4). Hence, $F \in F(M)$.

Now, we shall prove the lemma in case (b). Suppose that $F \notin F(M)$, i.e. by (P3), there exists $C \in C(M)$ such that $C \subseteq F$. Obviously, C must contain at least an element of H as otherwise $G \notin F(M)$, which is in contradiction with (P4). Considering C of the form $C = \{e_1, e_2, \ldots, e_s\} \cup T$ with $\{e_1, e_2, \ldots, e_s\} \subseteq H$ and $T \subseteq G$, and repeating the above judgement (as in case (a)) for the set $\{e_1, e_2, \ldots, e_s\}$, we obtain a similar contradiction: a subset of B contains a circuit. Therefore, the lemma is entirely proved. (QED).

COROLLARY. *Let* B *be a fixed basis of* M, *and* $\{f_{i_1}, f_{i_2}, \ldots, f_{i_t}\} \subseteq E - B$ *such that* $C(f_{i_j}, B)$ *are pairwise disjoint. Then* $\{f_{i_1}, f_{i_2}, \ldots, f_{i_t}\} \in F(M)$.

P r o o f. It readily follows form tha lemma (QED).

Proof of the theorem. Let M be a representable matroid over the field $\mathbb{F}$, B a fixed basis of M and $R = [I_k, A]$ the standard matrix representation with respect to B. By the above corollary, it follows that the column vectors

$F_{i_1}, F_{i_2}, \ldots, F_{i_t}$, are linearly independent over $\mathbb{F}$, i.e. $\text{rank}(A) \geq t$ (QED).

Acknowledgement. I would like to express my gratitude the referee for suggesting the present form of the stated material.

REFERENCES

[1] A.W. Ingleton, *Representation of matroids, Proc. of the Conference on Combinatorial Mathematics and its Applications, Oxford, 1969.*

[2] W.T., Tutte, *Introduction to the theory of matroids, (Elsevier, New York, 1971).*

[3] D.J.A. Welsh, *Matroid theory, (Academic Press, New York, 1976).*

[4] R.J. Wilson, *An introduction to matroid theory, Amer.Math.Monthly, 80 (1973), 500-525.*

Received by the editors January 30, 1984.

REZIME

NOTA O STANDARDNOJ MATRIČNOJ REPREZENTACIJI MATROIDA

Osnovni rezultat ovog rada je: Ako je $[I_k, A]$ SMR matroida u odnosu na bazu B, tada je $\text{rank}(A) \geq C(B)$, gde je $C(B)$ maksimalan broj.