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THE INVERSE OF A TR-LATTICE NEED NOT BE A TR-LATTICE

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ABSTRACT.

A Tr-lattice is a finite lattice L with the following property: if L is isomorphic to the lattice of cyclic flats of a matroid M, then M is transversal. In this paper we give an example of a Tr-lattice L, such that the inverse lattice L^{-1} is not a Tr-lattice.

PRELIMINARIES

An <u>n-set</u> is a set of cardinality n.

The cardinality of a set X is denoted by "$|X|$".

The denotation "k·K" means "k copies of the set K".

We assume familiarity with the notions "lattice", "inverse lattice", "lattice isomorphism" and "chain" (type of lattice).

A <u>matroid</u> M on a finite set (the ground-set of M) S is an ordered pair (S,B), where B is a family of subsets

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of S, which satisfies the following "exchange" property:

$$(B_1, B_2 \in \mathcal{B} \quad \wedge \quad x \in B_1 \backslash B_2) \Rightarrow$$

$$\Rightarrow (\exists y)(y \in B_2 \backslash B_1 \quad \wedge \quad (B_1 \backslash x) \cup y \in \mathcal{B})$$

The sets of $\mathcal{B}$ are the bases of M.

Two matroids are isomorphic if there is a bijection between their ground-sets, which preserves the bases.

All (including trivial) subsets of bases are the independent sets of M, while the other subsets of S are dependent.

A circuit of M is a minimal dependent set.

A loop of M is a circuit of cardinality 1.

If X is a subset of S, then

$$\text{rank}_M(X) = \max_{B \in \mathcal{B}} |X \cap B|$$

We simply write "rank(X)" when the matroid M is known. It is obvious that a subset X of S is independent if and only if rank(X) = |X|. We define

$$\text{rank}(M) \overset{\text{def}}{=} \text{rank}(S)$$

and this number is known to be the common cardinality of all bases of M.

REMARK: The proofs of this and further unproved statements can be found in [6], unless another reference is cited.

The rank-function of the matroid M satisfies the following ("submodular") inequality for all subsets X and Y of S:

$$\text{rank}(X \cup Y) + \text{rank}(X \cap Y) \leq \text{rank}(X) + \text{rank}(Y)$$

A subset X of S is a flat of M if it satisfies

$$\text{rank}(X \cup y) = \text{rank}(X) + 1, \qquad \text{for all } y \in S \backslash X.$$

The closure of a subset X of S is the minimal flat of M, which contains X.

A flat X of M is cyclic if additionally it satisfies

$$\operatorname{rank}(X\setminus y) = \operatorname{rank}(X), \quad \text{for all } y \in X.$$

It is well-known ([4]) that all cyclic flats of a matroid M constitute a lattice (ordered by set-inclusion), which we call the CF-lattice of M. A matroid is uniquely (up to an isomorphism) determined by the family of its cyclic flats, accompanied by their ranks.

It is also known ([5]) that each finite lattice is the CF-lattice of a matroid.

The family $\{X \subseteq S \mid S\setminus X \in \mathcal{B}\}$ is known to be the family of bases of another, so-called, dual matroid M^* on S. It also holds that a set X is a cyclic flat of M if and only if the set $S\setminus X$ is a cyclic flat of M^* ([1]). A consequence of this fact is that the lattice, which is isomorphic to the CF-lattice of M^*, can be obtained by inversion of the lattice, which is isomorphic to the CF--lattice of M — and conversely.

A coloop of M is a loop of M^*.

Let $\tau = \{T_1,\dots,T_r\}$ be a finite family of finite sets. A set $X = \{x_1,\dots,x_j\}$ is a partial transversal (= a system of distinct representatives of the sets) of τ if there is an injection π of the set $\{1,\dots,j\}$ into the set $\{1,\dots,r\}$, such that

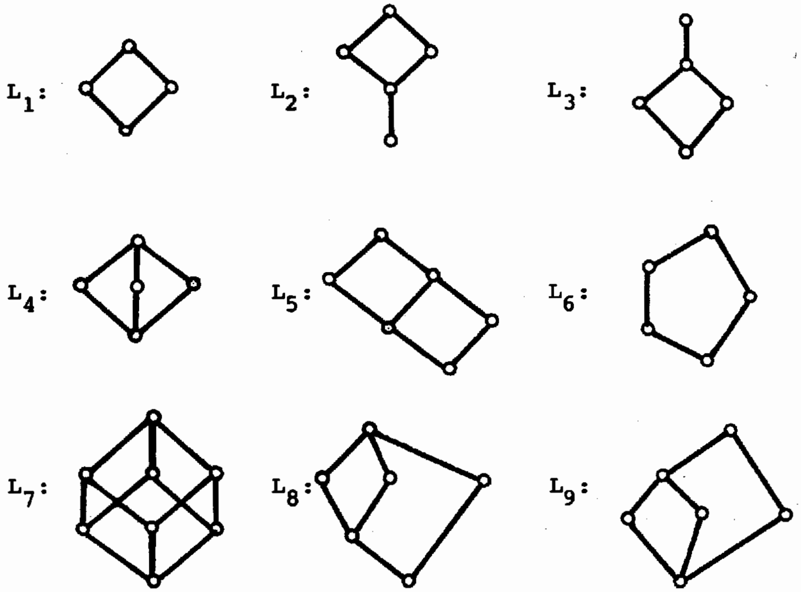
$$x_i \in T_{\pi(i)}, \qquad 1 \le i \le j$$

If $j = r$, then X is a transversal of τ. It is well-known that the maximal partial transversals of τ are the bases of a matroid M_τ. The family τ is a transversal representation of the matroid M_τ. A matroid is transversal if it has a transversal representation. Each transversal matroid has a transversal representation in which the bases are transversals.

A <u>Tr-lattice</u> is a finite lattice L, which satisfies:

If L is isomorphic to the CF-lattice of a matroid M, then M is transversal.

We shall introduce nine special finite lattices $L_1, L_2, \ldots, L_9$, which are given in the following table:



INTRODUCTION

It is proved in paper [2] that all chains (of arbitrary length) and the lattice L_1 are Tr-lattices. We also know that the lattices L_2 and L_3 are Tr-lattices. However, the very simple lattice L_4 is not a Tr-lattice.

The matroids with the CF-lattices isomorphic to L_5, L_6 and L_7 are rather numerous in the catalogue [3] and all such examples are transversal matroids. We conjecture for that reason that the lattices L_5, L_6 and L_7 are also Tr-lattices.

The following example from the same catalogue attracted our particular attention:

All 37 non-isomorphic matroids on at most 8 elements, the CF-lattices of which are isomorphic to L_8, ——— are transversal. However, among the 37 dual matroids, the CF-lattices of which are isomorphic to $L_9 = L_8^{-1}$ (the CF-lattices of mutually dual matroids, when considered in a pure lattice - theoretical sense, are mutually inverted), ——— there are only 14 transversal matroids. This is the "smallest" example of this kind that we know and it motivated the assertion, which coincides with the title of this paper. Such an assertion is closely connected with (but it by no means directly follows from) the well-known fact that the dual of a transversal matroid need not be a transversal matroid.

In this paper we shall prove that the lattice L_8 is a Tr-lattice (this is the main difficulty) and we shall give an example of a non-transversal matroid with the CF-lattice isomorphic to $L_9 = L_8^{-1}$.

RESULTS

We prove two lemmas primarily:

LEMMA 1. *The closure of a circuit of a matroid is a cyclic flat.*

P r o o f. If we suppose that the closure of a circuit C is a non-cyclic flat X, then there exists $e \in X$ such that $\operatorname{rank}(X\setminus e) = \operatorname{rank}(X) - 1$.

If $e \in C$, then

$$\operatorname{rank}(X) = \operatorname{rank}(C) = \operatorname{rank}(C\setminus e) \leq \operatorname{rank}(X\setminus e) = \operatorname{rank}(X) - 1$$

a contradiction.

If $e \in X\setminus C$, then we have the same contradiction; the only difference is in that the inequality $\operatorname{rank}(C) \leq \operatorname{rank}(X\setminus e)$ follows directly. □

LEMMA 2. *Let* M *be a matroid on* S. *A subset* X *of* S *is independent in* M *if and only if* $|X \cap F| \leq$ $\leq \mathrm{rank}(F)$ *for each cyclic flat* F *of* M.

P r o o f. Let F be a cyclic flat of M such that $|X \cap F| > \mathrm{rank}(F)$. Then we also have $|X \cap F| >$ $> \mathrm{rank}(X \cap F)$, that is, the set $X \cap F$ is dependent in M. It follows that the superset X is also dependent in M.

Conversely, let X be a subset of S, which is dependent in M. Then X contains a circuit C. The closure of the circuit C is by Lemma 1 a cyclic flat F. Then by

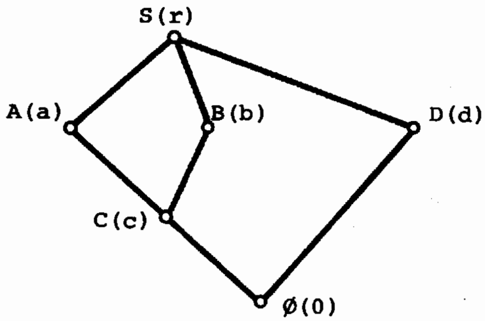
$$|X \cap F| \geq |C| > |C| - 1 = \mathrm{rank}(C) = \mathrm{rank}(F)$$

we have that the cyclic flat F of M satisfies $|X \cap F| > \mathrm{rank}(F)$. □

We shall proceed with proving the main theorem of this paper:

THEOREM 1. L_8 *is a Tr-lattice.*

P r o o f. Let M (on S) be an arbitrary matroid having the CF-lattice isomorphic to L_8. We denote the cyclic flats of M and their ranks (in brackets) as follows:



REMARK: We may assume that M has no loops or coloops, since they do not influence transversality (the coloops appear in each base of a matroid and the loops in none). It is for this reason that we may assume that the minimal element in L_8 is the empty set (necessarily of rank 0), while the maximal element coincides with the ground-set S of M.

We shall prove that the matroid M is transversal, by exhibiting its explicit transversal representation. Namely, we claim that the family

$$\Phi = \{(r - a)\cdot(S\setminus A),\ (r - b)\cdot(S\setminus B),$$
$$(a + b - r - c)\cdot(S\setminus C),\ (r - d)\cdot(S\setminus D),\ (c + d - r)\cdot S\}$$

is a transversal representation of M.(*)

To prove this, it suffices to prove the following two statements:

(i) each dependent r-subset X of S is not a transversal of Φ

(ii) each base of M is a transversal of Φ.

P r o o f of (i): Following Lemma 2, we conclude that $|X \cap F| > \text{rank}(F)$ for some $F \in \{A,B,C,D\}$. If $F = A$, then X contains less than $r-a$ elements in the set $S\setminus A$. Thus X does not contain a transversal of the subfamily $\{(r-a)\cdot(S\setminus A)\}$ of Φ, which implies that X is not a transversal of Φ. We apply a very similar reasoning if $F = B$ or $F = D$. If $F = C$, then we observe that

$$(r - a) + (r - b) + (a + b - r - c) = r - c$$

and that $S\setminus A \subset S\setminus C$ and $S\setminus B \subset S\setminus C$.
Since X contains less than $r-c$ elements in $S\setminus C$, it does

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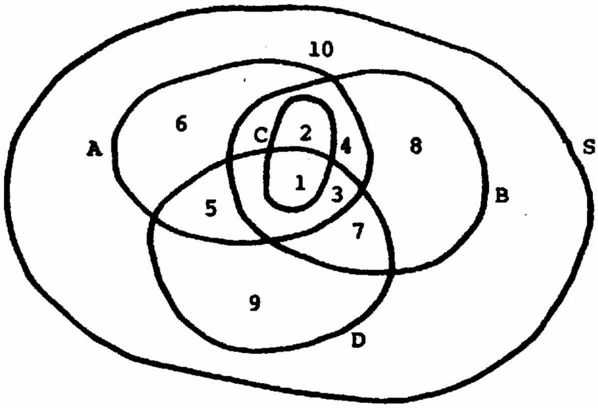
(*) *this general transversal representation was "experimentally" derived by using [3].*

not contain a transversal of the subfamily
$\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (a+b-r-c)\cdot(S\setminus C)\}$ of Φ.

P r o o f of (ii): The set S is partitioned by its subsets A,B,C,D into ten pairwise disjoint subsets, which is shown by the following diagram:



The number i in the diagram corresponds to the subset denoted by O_i, $1 \le i \le 10$.

Let X be a base of M and let $x_i = |X \cap O_i|$, $1 \le i \le 10$. We derive and numerate nine inequalities and one equality, which are satisfied by the numbers $x_1,\dots,x_{10}$:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le a \qquad (1)$$
$$x_1 + x_2 + x_3 + x_4 + x_7 + x_8 \le b \qquad (2)$$
$$x_1 + x_2 \le c \qquad (3)$$
$$x_1 + x_3 + x_5 + x_7 + x_9 \le d \qquad (4)$$
$$x_1 + x_2 + x_3 + x_4 \le a + b - r \qquad (5)$$
$$x_1 + x_3 + x_5 \le a + d - r \qquad (6)$$
$$x_1 + x_3 + x_7 \le b + d - r \qquad (7)$$
$$x_1 \le c + d - r \qquad (8)$$
$$x_1 + x_3 \le a + b + d - 2r \qquad (9)$$
$$x_1 + x_2 + \dots + x_{10} = r \qquad (10)$$

The first four inequalities, follow from the fact that the set X is independent, by the use of Lemma 2 applied to the cyclic flats A,B,C,D respectively. The next four inequalities are derived by means of

$$|X \cap (F \cap G)| \leq \text{rank}(F \cap G) \leq \text{rank}(F) + \text{rank}(G) - \text{rank}(F \cup G)$$

where $\{F,G\}$ is equal to $\{A,B\}$, $\{A,D\}$, $\{B,D\}$, $\{C,D\}$ respectively. In a similar way (9) is obtained, but the last inequality (the submodular law) should be applied twice, e.g. separately to the pairs $\{A,B\}$ and $\{A \cap B,D\}$. Finally, (10) is equivalent to $|X| = r$.

REMARK: The right-hand sides of the inequalities (5) - (9) are non-negative, otherwise the submodular law for the rank-function of the matroid M would be violated. We point out that this nonnegativity of different coefficients, which we adjoin to the sets of Φ in the course of proving, follows either from the relations (1) - (10) or from some additional assumptions.

We are going to show that X is a transversal of Φ. This will be done step-by-step. We shall gradually make the sets $X \cap O_i$, $1 \leq i \leq 10$, to be some pairwise disjoint partial transversals of Φ. The elements of a set $X \cap O_i$ can represent only those sets in Φ, which are supersets of of O_i. We list such supersets for each O_i, $1 \leq i \leq 10$:

O_1: S ; O_2: $S\setminus D, S$; O_3: $S\setminus C, S$;

O_4: $S\setminus C, S\setminus D, S$; O_5: $S\setminus B, S\setminus C, S$; O_6: $S\setminus B, S\setminus C, S\setminus D, S$;

O_7: $S\setminus A, S\setminus C, S$; O_8: $S\setminus A, S\setminus C, S\setminus D, S$;

O_9: $S\setminus A, S\setminus B, S\setminus C, S$; O_{10}: $S\setminus A, S\setminus B, S\setminus C, S\setminus D, S$

We use the abbreviation "x_i is covered" to denote that the elements of the set $X \cap O_i$ are appropriately represented by some sets of Φ ("appropriately" means: by some sets which have not been already used). Our proof is over at the moment when all x_i - s, $1 \leq i \leq 10$, are covered.

x_1 can be directly covered by (8).

x_2 can be covered by reason of the inequality

$$x_2 \leq (r - d) + (c + d - r - x_1)$$

which is equivalent to (3) (in what follows we just put down the number of the equivalent inequality). Namely, for covering x_2 we may use only some of $r-d$ sets $S\setminus D$ and the remaining $c+d-r-x_1$ sets S in Φ. However, we give an "advantage" to the sets $S\setminus D$. We use some of the sets S only if all the sets $S\setminus D$ are exhausted. The reason is that the sets S are more "universal": they can be used for covering all x_i, $3 \leq i \leq 10$. In what follows, we shall always give an adventage in covering to less universal sets of Φ.

We observe that the sets $S\setminus C$ can also cover all x_i, $3 \leq i \leq 10$. Therefore we need not make any difference between the sets S and $S\setminus C$ in the rest of our proof. Let "W" denote "any of the sets S and $S\setminus C$". Then we replace $(a+b-r-c)\cdot(S\setminus C)$ and $(c+d-r-x_1)\cdot S$ in Φ by $(a+b+d-2r-x_1)\cdot W$, while the new families of supersets for O_i, $3 \leq i \leq 10$, are:

O_3: W; O_4: $S\setminus D, W$; O_5: $S\setminus B, W$; O_6: $S\setminus B, S\setminus D, W$;

O_7: $S\setminus A, W$; O_8: $S\setminus A, S\setminus D, W$; O_9: $S\setminus A, S\setminus B, W$;

O_{10}: $S\setminus A, S\setminus B, S\setminus D, W$

It should be stressed that the rest of our proof will include a kind of branching process. The first branching depends on whether the sets $S\setminus D$ are exhausted while covering x_2 or not. We denote the first possibility by <1> and the second one by <2>. We shall also use the denotations "1" and "2" in further branchings, depending on whether the less universal sets are exhausted or not. Thus each particular branch of the branching will be denoted by a binary vector (with components 1,2) in brackets "< >". This vector will be followed by the condition (an inequality in the brackets "( )"), which corresponds to the last branching. After that the remaining subfamily of Φ will be listed and one or more relations (equivalent to some of the relations (1)-(10)), which correspond to the coverings on that branch.

$$<1> : (x_2 \geq r - d)$$

After covering x_2, the following subfamily remains from Φ for further coverings:

$$\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (a+b-r-x_1-x_2)\cdot W\}$$

We observe that after eliminating the sets $S\setminus D$, some families of supersets coincide as follows:

$O_3 \cup O_4 : W$; $O_5 \cup O_6 : S\setminus B, W$;
$O_7 \cup O_8 : S\setminus A, W$; $O_9 \cup O_{10} : S\setminus A, S\setminus B, W$

We cover together the corresponding x_i - S:

$$x_3+x_4 \leq a+b-r-x_1-x_2 \quad , \quad (5)$$

The next remaining subfamily is:

$$\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (a+b-r-x_1-x_2-x_3-x_4)\cdot W\}$$

$$x_5+x_6 \leq (r-b) + (a+b-r-x_1-x_2-x_3-x_4) \quad , \quad (1)$$

The next branching depends on the way of covering x_5 and x_6:

$$<1,1> : (x_5+x_6 \geq r-b)$$

The remaining subfamily is:

$$\{(r-a)\cdot(S\setminus A),\ (a-x_1-x_2-x_3-x_4-x_5-x_6)\cdot W\}$$

We should further cover x_i, $7 \leq i \leq 10$, by using, solely, the sets $S\setminus A$ and W. This is possible on the basis of

$$x_7+x_8+x_9+x_{10} = (r-a) + (a-x_1-x_2-x_3-x_4-x_5-x_6) \quad , \qquad (10)$$

<1,2> : $(x_5+x_6 < r-b)$

The remaining subfamily is:

$\{(r-a)\cdot(S\setminus A),\ (r-b-x_5-x_6)\cdot(S\setminus B),\ (a+b-r-x_1-x_2-x_3-x_4)\cdot W\}$

We further have

$$x_7+x_8 \leq (r-a) + (a+b-r-x_1-x_2-x_3-x_4) \quad , \qquad (2)$$

and regardless of the way of covering x_7 and x_8:

$$x_9+x_{10} = ((r-a) + (a+b-r-x_1-x_2-x_3-x_4) - (x_7+x_8)) + \\ + (r-b-x_5-x_6) \quad , \qquad (10)$$

covers x_9 and x_{10}.

<2> : $(x_2 < r-d)$

The covering x_2 does not exhaust the sets $S\setminus D$ and after covering x_3 with the sets W (it is possible because of

$$x_3 \leq a+b+d-2r-x_1 \quad , \qquad (9)\ ),$$

the remaining subfamily is:

$\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (r-d-x_2)\cdot(S\setminus D)\ ,$
$(a+b+d-2r-x_1-x_3)\cdot W\}$

We cover primarily x_4:

$$x_4 \leq (r-d-x_2) + (a+b+d-2r-x_1-x_3) \quad , \qquad (5)$$

We have the following branching, depending on the way of covering x_4:

<2,1> : $(x_4 \geq r-d-x_2)$

The remaining subfamily is:

$\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (a+b-r-x_1-x_2-x_3-x_4)\cdot W\}$

This is the same family as in case <1> and we can cover x_i, $5 \le i \le 10$, in the same way as before (the only difference between cases <1> and <2,1> is in the way of covering x_4: in the first case only the sets W are used, while in the second one the sets $S\setminus D$ are primarily exhausted).

<2,2> : $(x_4 < r-d-x_2)$

The remaining subfamily after covering x_i, $1 \le i \le 4$, is

$\{(r-a)\cdot(S\setminus A),\ (r-b)\cdot(S\setminus B),\ (r+d-x_2-x_4)\cdot(S\setminus D),$
$(a+b+d-2r-x_1-x_3)\cdot W\}$

We can cover x_5 on the basis of

$$x_5 \le (r-b) + (a+b+d-2r-x_1-x_3) \quad , \qquad (6)$$

The way of covering this determines the next branching:

<2,2,1> : $(x_5 \ge r-b)$

The remaining subfamily is:

$\{(r-a)\cdot(S\setminus A),\ (r-d-x_2-x_4)\cdot(S\setminus D),\ (a+d-r-x_1-x_3-x_5)\cdot W\}$

Since the sets $S\setminus B$ are exhausted, the scheme of supersets for further coverings is:

O_6 : $S\setminus D, W$; $O_7 \cup O_9$: $S\setminus A, W$
$O_8 \cup O_{10}$: $S\setminus A, S\setminus D, W$

We proceed with covering x_6:

$$x_6 \le (r-d-x_2-x_4) + (a+d-r-x_1-x_3-x_5) \quad , \qquad (1)$$

The next branching depends on whether the sets $S \setminus D$ are exhausted or not:

$\langle 2,2,1,1 \rangle : (x_6 \geq r-d-x_2-x_4)$

The following subfamily remains from Φ for further coverings:

$$\{(r-a)\cdot(S\setminus A),\ (a-x_1-x_2-x_3-x_4-x_5-x_6)\cdot W\}$$

Since the sets $S\setminus A$ and W are the supersets of all sets O_i, $7 \leq i \leq 10$, it follows that the equality:

$$x_7+x_8+x_9+x_{10} = (r-a) + (a-x_1-x_2-x_3-x_4-x_5-x_6) \quad , \qquad (10)$$

completes the necessary coverings in this case.

$\langle 2,2,1,2 \rangle : (x_6 < r-d-x_2-x_4)$

After covering x_i, $1 \leq i \leq 6$, the following subfamily remains from ϕ:

$$\{(r-a)\cdot(S\setminus A),\ (r-d-x_2-x_4-x_6)\cdot(S\setminus D),\ (a+d-r-x_1-x_3-x_5)\cdot W\}$$

Simultaneously we cover x_7 and x_9 with

$$x_7+x_9 \leq (r-a) + (a+d-r-x_1-x_3-x_5) \quad , \qquad (4)$$

Without regard to the way of this covering, the coverings are completed by

$$x_8+x_{10} = ((r-a) + (a+d-r-x_1-x_3-x_5) - (x_7+x_9)) + \\ + (r+d-x_2-x_4-x_6) \quad , \qquad (10)$$

for any of the sets $S\setminus A, S\setminus D$ and W can be used for covering x_8 and x_{10}.

$\langle 2,2,2 \rangle : (x_5 < r-b)$

The remaining subfamily after covering x_i, $1 \leq i \leq 5$, is

$\{(r-a)\cdot(S\setminus A),\ (r-b-x_5)\cdot(S\setminus B),\ (r-d-x_2-x_4)\cdot(S\setminus D),$
$(a+b+d-2r-x_1-x_3)\cdot W\}$

We cover primarily x_7 (before x_6), for it can be covered only by two kinds of sets, $S\setminus A$ and W:

$$x_7 \leq (r-a) + (a+b+d-2r-x_1-x_3) \quad , \qquad (7)$$
$$\langle 2,2,2,1\rangle : (x_7 \geq r-a)$$

The following subfamily remains:

$\{(r-b-x_5)\cdot(S\setminus B),\ (r-d-x_2-x_4)\cdot(S\setminus D),\ (b+d-r-x_1-x_3-x_7)\cdot W\}$

The corresponding scheme of supersets is:

$O_8 : S\setminus D, W \quad ; \quad O_9 : S\setminus B, W \quad ; \quad O_6 \cup O_{10} : S\setminus B, S\setminus D, W$

The next covering and branching are related to x_8:

$$x_8 \leq (r-d-x_2-x_4) + (b+d-r-x_1-x_3-x_7) \quad , \qquad (2)$$
$$\langle 2,2,2,1,1\rangle : (x_8 \geq r-d-x_2-x_4)$$

It remains

$\{(r-b-x_5)\cdot(S\setminus B),\ (b-x_1-x_2-x_3-x_4-x_7-x_8)\cdot W\}$

Since x_6, x_9 and x_{10} can be covered by any of the sets $S\setminus B$ and W, the covering is completed by

$$x_6+x_9+x_{10} = (r-b-x_5) + (b-x_1-x_2-x_3-x_4-x_7-x_8) \quad , \qquad (10)$$
$$\langle 2,2,2,1,2\rangle : (x_8 < r-d-x_2-x_4)$$

The remaining subfamily is

$\{(r-b-x_5)\cdot(S\setminus B),\ (r-d-x_2-x_4-x_8)\cdot(S\setminus D),\ (b+d-r-x_1-x_3-x_7)\cdot W\}$

We cover x_9 with

$$x_9 \leq (r-b-x_5) + (b+d-r-x_1-x_3-x_7) \quad , \qquad (4)$$

Disregarding the relation between x_9 and $r-b-x_5$, x_6 and x_{10} can be covered on the basis of

$$x_6+x_{10} = ((r-b-x_5) + (b+d-r-x_1-x_3-x_7) - x_9) + + (r-d-x_2-x_4-x_8) \quad , \qquad (10)$$

for all three kinds of sets: $S\setminus B$, $S\setminus D$ and W, can be used for their covering.

$\langle 2,2,2,2\rangle$: $(x_7 < r-a)$

The following subfamily remains from ϕ:

$$\{(r-a-x_7)\cdot(S\setminus A),\ (r-b-x_5)\cdot(S\setminus B),\ (r-d-x_2-x_4)\cdot(S\setminus D),\ (a+b+d-2r-x_1-x_3)\cdot W\}$$

We recall that the corresponding scheme of supersets is:

O_6 : $S\setminus B, S\setminus D, W$; O_8 : $S\setminus A, S\setminus D, W$

O_9 : $S\setminus A, S\setminus B, W$; O_{10} : $S\setminus A, S\setminus B, S\setminus D, W$

that is, each of x_6, x_8, x_9, x_{10} can be covered by using three (respectively four) different kinds of sets. We cover primarily x_6 by use of

$$x_6 \leq (r-b-x_5) + (r-d-x_2-x_4) + (a+b+d-2r-x_1-x_3) \quad , \qquad (1)$$

This time our branching is somewhat different: it depends on whether the BOTH less universal kinds of sets, $S\setminus B$ and $S\setminus D$, are exhausted or not.

$\langle 2,2,2,2,1\rangle$: $(x_6 > (r-b-x_5) + (r-d-x_2-x_4))$

The remaining subfamily is

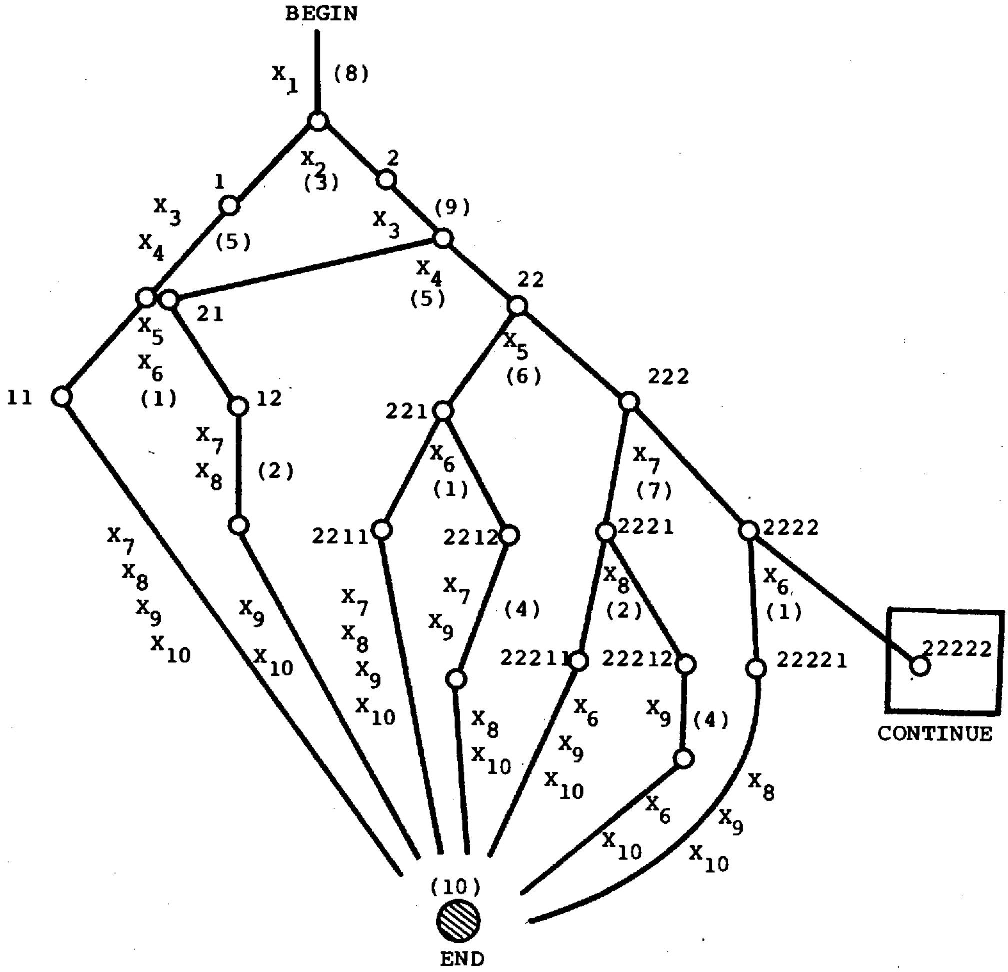
$$\{(r-a-x_7)\cdot(S\setminus A),\ (a-x_1-x_2-x_3-x_4-x_5-x_6)\cdot W\}$$

Since x_8, x_9 and x_{10} can be covered by both $S\setminus A$ and W, it follows that the equality

$$x_8+x_9+x_{10} = (r-a-x_7) + (a-x_1-x_2-x_3-x_4-x_5-x_6), \qquad (10)$$

completes the coverings.

We give a diagram of the "branching part" of our proof:



The coverings are denoted by the coresponding x_i-s. Each non-denoted vertex corresponds to a realized single covering step. The vertices denoted by binary vectors (without brackets and commas) follow the corresponding branchings. In both cases the inequalities used for the coverings are denoted. The "double" vertex 21 can be reached in two ways. The vertex "END" can be reached from seven different vertices, always by an application of (10). It denotes the end of the proof on each of the corresponding seven branches[(*)]. However, such an end is reached from the vertex 22222 in a special and more elastic "parametric" way:

$$\langle 2,2,2,2,2\rangle : (x_6 \leq 2r-b-d-x_2-x_4-x_5)$$

We do not give an adventage, in covering x_6, to any of the "less universal" sets $S\setminus B$ and $S\setminus D$. Let g $(0 \leq g \leq x_6)$ sets $S\setminus B$ and x_6-g sets $S\setminus D$ be used for covering x_6. The value of the parameter g will be determined after taking into account some inequalities satisfied by x_8 and x_9.

We need not make any difference between $S\setminus A$ and W in the course of covering x_8, x_9, x_{10} and so we introduce the denotation "Y" to mean "any of the sets $S\setminus A$ and W".

The further scheme of supersets is:

$$O_8 : S\setminus D, Y \;;\qquad O_9 : S\setminus B, Y \;;\qquad O_{10} : S\setminus D, S\setminus B, Y$$

The remaining subfamily from Φ after covering x_i, $1 \leq i \leq 7$, is:

$$\{(r-b-x_5-g)\cdot(S\setminus B),\ (r-d-x_2-x_4-x_6+g)\cdot(S\setminus D),\ (b+d-r-x_1-x_3-x_7)\cdot Y\}$$

We underline that the number of sets in the remaining

(*) *We observe that (10) completes the proof whenever all the kinds of the remaining sets in* Φ *can be used for the remaining coverings.*

subfamily is equal to the number of still uncovered elements in X, that is, $x_8+x_9+x_{10}$. The sets $S\setminus B$ cannot be used for covering x_8 and the same holds for the sets $S\setminus D$ and x_9. Since the sets $S\setminus D$, respectively the sets $S\setminus B$, have an advantage in covering x_8, respectively x_9, we conclude that there cannot be a "deficiency" of the sets Y. Thus the only further conditions, necessary and sufficient for a full covering, are

$$x_8 \leq (r-d-x_2-x_4-x_6+g) + (b+d-r-x_1-x_3-x_7) \qquad \text{(I)}$$

and

$$x_9 \leq (r-b-x_5-g) + (b+d-r-x_1-x_3-x_7) \qquad \text{(II)}$$

In order to complete the proof of our Theorem, we should establish the existence of an integer g, which fulfils conditions (I), (II) and $0 \leq g \leq x_6$ simultaneously.

We observe that the conjunction of inequalities (I) and (II) is equivalent to

$$x_1+x_2+x_3+x_4+x_6+x_7+x_8-b \leq g \leq d-x_1-x_3-x_5-x_7-x_9$$

Denote the left bound for g, contained in this inequality, by L, and the right one by R. We claim that the interval [L,R] is non-empty, that is, that $L \leq R$. Since by (10)

$$R = d-r+x_2+x_4+x_6+x_8+x_{10}$$

it follows that $L \leq R$ is equivalent to

$$x_1+x_3+x_7 \leq b+d-r+x_{10}$$

This last inequality is necessarily true, for it is (because of $x_{10} \geq 0$) weaker than (7). Hence our claim is proved and all the integers g in the interval [L,R] satisfy the conditions (I) and (II).

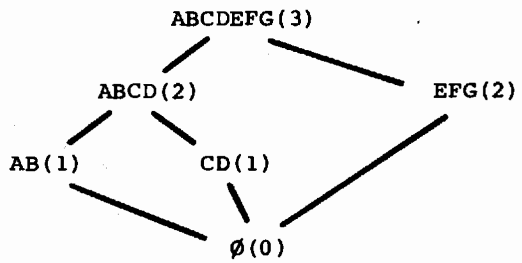
We should only prove that at least one of these g belongs to the interval $[0,x_6]$, that is, that

$$[0,x_6] \cap [L,R] \neq \emptyset$$

To prove this, it suffices to show that $O \leq R$ and $L \leq x_6$. We see immediately that the first of the last two inequalities is equivalent to (4), while the second is equivalent to (2). Thus the proof of Theorem 1 is completed.□

THEOREM 2. *There exists a non-transversal matroid with the CF-lattice isomorphic to* L_9.

P r o o f. Let M be a rank 3 matroid on the set $\{A,B,C,D,E,F,G\}$ with the following CF-lattice:



The cyclic flats of M are denoted without brackets and commas; we shall adopt this convention for all sets in the rest of our proof. The numbers in brackets denote the ranks of the corresponding cyclic flats; the submodular law for the rank-function is not violated.

It is obvious that the CF-lattice of M is isomorphic to L_9. We are going to prove that M is not a transversal matroid.

Suppose, on the contrary, that M has a transversal representation $\tau = \{T_1,T_2,T_3\}$. Since rank(A) = rank(B) = = rank(C) = rank(D) = rank(AB) = rank(CD) = 1, while rank(ABCD) = 2, it follows that the elements A and B appear (together) in only one of the sets T_1,T_2,T_3, and so do the elements C and D, but it is not true that all the four elements A,B,C,D appear in the same set of τ. We may assume, without any loss of generality, that

$A, B \in T_1 \setminus (T_2 \cup T_3)$ and
$C, D \in T_2 \setminus (T_1 \cup T_3)$

Let (W_1, W_2, W_3) be an arbitrary permutation of (T_1, T_2, T_3). Since $\mathrm{rank}(EF) = 2$, it follows that the set EF is a partial transversal of τ. We may assume that $E \in W_1$ and $F \in W_2$. We claim that $EFG \cap W_3 = \emptyset$.

The fact that $\mathrm{rank}(EFG) = 2$ implies that $G \notin W_3$ and this gives $G \in W_1 \cup W_2$ (because of $\mathrm{rank}(G) = 1$). If $G \in W_1$, then $\mathrm{rank}(EFG) = 2$ implies $E \notin W_3$. It follows that the rank 2 set EG is a transversal of $\{W_1, W_2\}$, which implies that $F \notin W_3$. If $G \in W_2$, then the proof of the claim is analogously completed.

If $W_3 \equiv T_3$, then the set T_3 is empty, which is a contradiction with $\mathrm{rank}(M) = 3$. Now suppose that $W_3 \equiv T_1$. Then $T_1 = AB$ and the base CEF is not a transversal of τ, a contradiction. The assumption $W_3 \equiv T_2$ and the base AEF lead to a similar contradiction. We conclude that τ cannot exist.□

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REZIME

INVERZNA MREŽA TR-MREŽE
NE MORA BITI TR-MREŽA

Tr-mreža je konačna mreža L, koja ima sledeće svojstvo: Ako je L izomorfna mreži cikličkih potprostora nekog matroida M, onda je matroid M transverzalan. U ovom radu dajemo primer Tr-mreže L, takve da inverzna mreža L^{-1} nije Tr-mreža.