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ON FUZZY QUOTIENT ALGEBRAS

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ABSTRACT

For an algebra A and a complete lattice L, one can consider a fuzzy congruence relation $\bar{\rho}$ (defined in [1]). Here we define a quotient algebra $A/\bar{\rho}$. Since every fuzzy congruence relation is a special union of a family of ordinary congruences on the same algebra, it is interesting to consider the relationship between $A/\bar{\rho}$ and the quotient algebra A/ρ by any of the congruences of the family. We prove that there is always a homomorphism from $A/\bar{\rho}$ to A/ρ, and we give the necessary and sufficent conditions for it to be an isomorphism. We also consider the fuzzy subalgebras (defined as in [2]) of A, and A/ρ, and assuming that these mappings preserve the homomorphism, we prove that a fuzzy subalgebra $\bar{A}$ of A induces $\bar{A}/\rho$ (or A/ρ) and vice versa. Using the homomorphism from $A/\bar{\rho}$ onto A/ρ, we finally determine the connection between the corresponding fuzzy subalgebras.

1. Let S be an unempty set, and $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ a complete lattice. A <u>fuzzy set</u> $\bar{S}$ on S (or, a fuzzy subset $\bar{S}$ of S) is any mapping $\bar{S}: S \to L$ ([3]).

If $\bar{S}_1$ and $\bar{S}_2$ are two fuzzy sets on S, then the relation $\subseteq$ and the operations $\cap$ and $\cup$ are defined as follows ([3]):

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$$\bar{S}_1 \subseteq \bar{S}_2 \quad \text{iff for all } x \in S \quad \bar{S}_1(x) \le \bar{S}_2(x) \;;$$

$$\bar{S}_1 \cap \bar{S}_2 : S \to L, \quad \text{and} \quad (\bar{S}_1 \cap \bar{S}_2)(x) = \bar{S}_1(x) \wedge \bar{S}_2(x) \;;$$

$$\bar{S}_1 \cup \bar{S}_2 : S \to L, \quad \text{and} \quad (\bar{S}_1 \cup \bar{S}_2)(x) = \bar{S}_1(x) \vee \bar{S}_2(x) \;;$$

(The operations on the right are those from L).

It is well known ([3]) that for any fuzzy set $\bar{S}$ on S the following equality holds:

1.1. $\bar{S} = \bigcup_{p \in L} p \cdot (S_p)$, where $S_p \subseteq S$, $x \in S_p$ iff $\bar{S}(x) \ge p$,

and $\quad (S_p) : S \to L, \; (S_p)(x) = \begin{cases} 1, & \text{if } x \in S_p \\ 0' & \text{if } x \notin S_p \end{cases}$

(the characteristic function of S_p). Here also

$$(p \cdot (S_p))(x) = p \wedge (S_p)(x) \;.$$

(From now on, we shall identify S_p with its characteristic function (S_p)). Clearly, if $x \in S$,

$$\bar{S}(x) = \bigvee_{p \in L} p \wedge S_p(x).$$

It is also known that from 1.1. it follows that for $p, q \in L$, $p \le q$ implies $S_q \subseteq S_p$.

Let $A = (S, F)$ be an algebra. A <u>fuzzy congruence relation</u> $\bar{\rho}$ on A is a fuzzy relation on S, i.e. ([1])

$$\bar{\rho} : S^2 \to L, \quad \text{such that}$$

(1) $\quad (\forall x \in S)(\bar{\rho}(x,x) = 1) \;;$

(2) $\quad (\forall x, y \in S)(\bar{\rho}(x,y) = \bar{\rho}(y,x)) \;;$

(3) $\quad (\forall x, y, z \in S)(\bar{\rho}(x,y) \ge \bigvee_{z \in S} (\bar{\rho}(x,z) \wedge \bar{\rho}(z,y))) \;;$

(4) If $\bar{\rho}(x_1, y_1) = p_1, \ldots, \bar{\rho}(x_n, y_n) = p_n$, $f \in F(n) \subseteq F$ [1], then $\bar{\rho}(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \ge \bigwedge_{i=1}^{n} p_i$.

1.2. If $\bar{\rho}$ is a fuzzy congruence relation on A, then

$$\bar{\rho} = \bigcup_{p \in L} p \cdot \bar{\rho}_p \;,$$

1) $F(n)$ is a set of n-ary operations from F.

where ρ_p are congruences on A, and $p \leq q$ implies $\rho_q \subseteq \rho_p$. Here, as in 1.1.,

$$\bar{\rho}(x,y) = \bigvee_{p \in L} p \wedge \rho_p(x,y), \rho_p(x,y) = \begin{cases} 1, & \text{if } \bar{\rho}(x,y) \geq p \\ 0, & \text{otherwise. ([1])} \end{cases}$$

The following definitions are similar to those in [2]:

A mapping $\bar{A}:S \to L$ is a <u>fuzzy subalgebra</u> of $A = (S,F)$, iff for every $f \in F(n)$, and for $x_1,\dots,x_n \in S$

$$\bar{A}(f(x_1,\dots,x_n)) \geq \bar{A}(x_1) \wedge \dots \wedge \bar{A}(x_n) .$$

Let $A = (S,F)$ and $B = (T,F)$ be two algebras from the same similarity class. If f is a homomorphism from A to B, then f is said to be a <u>fuzzy homomorphism</u> from a fuzzy subalgebra $\bar{A}$ of A to the fuzzy subalgebra $\bar{B}$ of B iff

$$\bar{A} \subseteq \bar{B} \circ f, \quad \text{i.e. iff for every } x \in S,\ \bar{A}(x) \leq \bar{B}(f(x)) .$$

2. Let $A = (S,F)$ be an algebra, $L = (L,\wedge,\vee,0,1)$ a complete lattice, and $\bar{\rho}$ a fuzzy congruence relation on A. For an $x \in S$, let

$$[x]_{\bar{\rho}}:S \to L, \quad [x]_{\bar{\rho}}(a) = \bar{\rho}(x,a), \text{ for all } a \in S .$$

Let us make the following definition:

$$S/\bar{\rho} = \{[x]_{\bar{\rho}} \ ; \ x \in S\} .$$

If ρ_p, $p \in L$, is one of the congruences from the family determined by $\bar{\rho}$ in 1.2., then let $[x]_{\rho_p}$ be a characteristic function for $|x|_{\rho_p} \in S/\rho_p$, i.e.

$$[x]_{\rho_p}(a) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } a \in |x|_{\rho_p} \\ 0, & \text{if } a \notin |x|_{\rho_p} \end{cases} .$$

Then, $[x]_{\rho_p}(a) = \rho_p(x,a)$. Now we have:

2.1. $$[x]_{\bar{\rho}} = \bigcup_{p \in L} (p \cdot [x]_{\rho_p}), \ x \in S .$$

P r o o f. $[x]_{\bar\rho}(a) = \bar\rho(x,y) = (\bigcup_{p\in L} p\cdot\rho_p)(x,a) =$

$$= \bigvee_{p\in L}(p\cdot\rho_p)(x,a) = \bigvee_{p\in L}(p\cdot\rho_p(x,a)) =$$

$$= \bigvee_{p\in L}(p\cdot[x]\rho_p(a)) = \bigvee_{p\in L}(p\cdot[x]\rho_p)(a) =$$

$$= (\bigcup_{p\in L}(p\cdot[x]\rho_p))(a) .$$

For every $f\in F(n)$, define an operation $\bar f$ on $S/\bar\rho$: If $x_1,\dots,x_n\in S$

$$\bar f([x_1]_{\bar\rho},\dots,[x_n]_{\bar\rho}) \stackrel{\text{def}}{=} \bigcup_{p\in L}(p\cdot f([x_1]\rho_p,\dots,[x_n]_{\rho_p})) ,$$

where, as we noted, we use the characteristic function $[x]\rho_p$ instead of a class $|x|\rho_p$, and thus

$$f([x_1]\rho_p,\dots,[x_n]\rho_p) = [f(x_1,\dots,x_n)]\rho_p .$$

Now we can prove the following statement:

2.2. For all $x_1,\dots,x_n\in S$.

$$\bar f([x_1]_{\bar\rho},\dots,[x_n]_{\bar\rho}) = [f(x_1,\dots,x_n)]_{\bar\rho} .$$

Proof.

$$\bar f([x_1]_{\bar\rho},\dots,[x_n]_{\bar\rho}) = \bigcup_{p\in L}(p\cdot f([x_1]\rho_p,\dots,[x_n]\rho_p)) =$$

$$= \bigcup_{p\in L}(p\cdot[f(x_1,\dots,x_n)]\rho_p) = [f(x_1,\dots,x_n)]_{\bar\rho} .$$

We can thus define a new algebra :

$$A/\bar\rho = (S/\bar\rho,\bar F), \text{ where } F = \{\bar f;\ f\in F\} .$$

The following two propositions deal with some properties of fuzzy equivalence relations, and they will be used in considering the connection between $A/\bar\rho$ and the usual factor algebra A/ρ_p .

2.3. Let $\bar{\rho}$ be a fuzzy equivalence relation on S. Let $a,b \in S$, $a \neq b$, Then for every $x \in S$,

$$\bar{\rho}(a,x) = \bar{\rho}(b,x) \quad \text{iff} \quad \bar{\rho}(a,b) = 1.$$

P r o o f. Let $\bar{\rho}(a,b) = 1$, $a \neq b$. Then, because of (3),

$\bar{\rho}(a,x) \geq \bar{\rho}(a,b) \wedge \bar{\rho}(b,x)$, and thus

(5) $\bar{\rho}(a,x) \geq \bar{\rho}(b,x)$.

Exactly in the same way, using (2) and (3), we get

(6) $\bar{\rho}(b,x) \geq \bar{\rho}(a,x)$.

From (5) and (6), it follows that $\bar{\rho}(a,x) = \bar{\rho}(b,x)$.

Let now $\bar{\rho}(a,x) = \bar{\rho}(b,x)$, $a \neq b$. Then for $x = a$

$1 = \bar{\rho}(a,a) = \bar{\rho}(b,a)$, i.e. $\bar{\rho}(a,b) = 1$.

2.4. Let $\bar{\rho}$ be a fuzzy equivalence relation on S, and $a,b \in S$. Then

$$[a]_{\bar{\rho}} = [b]_{\bar{\rho}} \quad \text{iff} \quad [a]_{\rho_1} = [b]_{\rho_1} \; .$$

P r o o f. Let $[a]_{\bar{\rho}} \neq [b]_{\bar{\rho}}$. Suppose that there is an $x \in S$, such that $\bar{\rho}(a,x) = \bar{\rho}(b,x) = 1$. Using (3), we get that $\bar{\rho}(a,b) = 1$. But then for every $x \in S$, by 2.3., $\bar{\rho}(a,x) = \bar{\rho}(b,x)$, i.e. $[a]_{\bar{\rho}} = [b]_{\bar{\rho}}$, which is a contradiction.

If $[a]_{\bar{\rho}} = [b]_{\bar{\rho}}$, then for every $p \in L$, $[a]_{\rho_p} = [b]_{\rho_p}$, and hence $[a]_{\rho_1} = [b]_{\rho_1}$.

Consider now the above defined factor algebra $A/\bar{\rho} = (S/\bar{\rho}, F)$, for a given algebra A, by a fuzzy congruence relation $\bar{\rho}$, and a factor algebra $A/\rho_p = (S/\rho_p, F)$, where ρ_p, $p \in L$, is any of the congruences from the collection defined in 1.2.

2.5. The mapping $h: S/\bar{\rho} \to S/\rho_p$, defined with

$$h([a]_{\bar{\rho}}) = [a]_{\rho_p} \; , \quad a \in S \; ,$$

is a homomorphism from $A/\bar{\rho}$ onto A/ρ_p .

P r o o f. Since for every $a \in S$, and $[a]_{\rho_p} \in S/\rho_p$ there is an original in $S/\bar{\rho}$, namely $[a]_{\bar{\rho}}$, h is onto. We also have

$$h(\bar{f}([x_1]_{\bar{\rho}},\dots,[x_n]_{\bar{\rho}})) = h([f(x_1,\dots,x_n)]_{\bar{\rho}}) =$$
$$= f([x_1]_{\rho_p},\dots,[x_n]_{\rho_p}) = f(h([x_1]_{\bar{\rho}}),\dots,h([x_n]_{\bar{\rho}})) ,$$

proving that h is a homomorphism.

2.6. $A/\bar{\rho} \cong A/\rho_1$, $1 \in L$.

P r o o f. We have to prove that h (defined in 2.5.) is 1-1. By 2.4. we have

$$[a]_{\bar{\rho}} \neq [b]_{\bar{\rho}} \quad \text{iff} \quad [a]\rho_1 \neq [b]\rho_1 ,$$

proving that h satisfies this property.

A fuzzy relation $\bar{\rho}:S^2 \to L$ is <u>strongly reflexive</u>, if the following is satisfied:

$$\bar{\rho}(x,y) = 1 \quad \text{iff} \quad x = y.$$

2.7. $A/\bar{\rho} \cong A$ iff $\bar{\rho}$ is a strongly reflexive fuzzy congruence relation on A.

P r o o f. If $A/\bar{\rho} \cong A$, then by 2.6. $A/\rho_1 = A$, i.e. ρ_1 is a diagonal.

On the other hand, if $\bar{\rho}$ is strongly reflexive, then ρ_1 is a daigonal, and thus $|a|_{\rho_1} = a$. Now, by 2.4., $a \neq b$ implies $[a]_{\bar{\rho}} \neq [b]_{\bar{\rho}}$ i.e. the mapping h $(h([a]_{\bar{\rho}}) = a)$ is an isomorphism.

3. Consider now the algebras $A = (S,F)$, $A/\bar{\rho} = (S/\bar{\rho},\bar{F})$, and for every $p \in L$, $A/\rho_p = (S/\rho_p, F)$, where $\bar{\rho} = \bigcup_{p \in L} p \cdot \rho_p$ is a fuzzy congruence relation on A, ρ_p $(p \in L)$ is an ordinary congruence relation (as in 1.2.), and $L = (L,\wedge,\vee,0,1)$ is a complete lattice. For each of these algebras one can define the corresponding fuzzy subalgebras, as in 1.3. Here we shall discuss the relationship between these fuzzy structures.

3.1. Let $\bar{A}:S\to L$ be a fuzzy subalgebra of A. Let also ρ be an ordinary congruence relation on A. Define the mapping $\bar{A}/\rho:S/\rho\to L$, so that $\bar{A}/\rho([x]_\rho) \overset{\text{def}}{=} \bigvee_{y\in[x]_\rho} \bar{A}(y)$. Now, if L is distributive, then

a) $\bar{A}/\rho$ is a fuzzy subalgebra of A/ρ, and

b) $f_\rho = \ker\rho$ is a fuzzy homomorphism from A/ρ onto $\bar{A}/\rho$.

P r o o f. a) Let $f\in F(n)$, $x_1,\dots,x_n\in S$. Then

$$\bar{A}/\rho(f([x_1]_\rho,\dots,[x_n]_\rho)) = \bar{A}/\rho([f(x_1,\dots,x_n)]_\rho) =$$
$$\bigvee(\bar{A}(y);\ y\in[f(x_1,\dots,x_n)]_\rho) \geq$$
$$\bigvee(\bar{A}(f(y_1,\dots,y_n);\ y_1\in[x_1]_\rho,\dots,y_n\in[x_n]_\rho) \geq$$
$$\bigvee(\bar{A}(y_1)\wedge\dots\wedge\bar{A}(y_n);\ y_1\in[x_1]_\rho,\dots,y_n\in[x_n]_\rho) =$$
$$\bigvee_{y_1\in[x_1]_\rho}(\bar{A}(y_1)\wedge\dots\wedge\bigvee_{y_n\in[x_n]_\rho}\bar{A}(y_n) \quad ,$$

where we use the definition of a fuzzy subalgebra, and the fact that L is distributive.

b) Let $x\in S$, and $f\ =\ker\rho$. Then

$$\bar{A}/\rho(f_\rho(x)) = \bar{A}/\rho([x]_\rho) = \bigvee_{y\in[x]_\rho}\bar{A}(x) \geq \bar{A}(x) \quad , \quad \text{i.e.}$$
$$\bar{A}\subseteq\bar{A}/\rho\circ f_\rho \quad .$$

The following corollary shows that, in the family of all fuzzy subalgebras of A/ρ, the one defined in 3.1. (and provided that L is distributive) is the smallest one - as a fuzzy set - satisfying 3.1. (b).

3.2. Let $\bar{A}/\rho$ be as in 3.1. If $\bar{A}_1/\rho:S/\rho\to L$ is any fuzzy subalgebra of A/ρ satisfying 3.1. (b), then

$$\bar{A}/\rho \subseteq \bar{A}_1/\rho \ .$$

P r o o f. By the definition of fuzzy homomorphism, if $x\in S$, $\bar{A}_1/\rho([x]_\rho)\geq\bar{A}(x)$. But then for every $y\in[x]_\rho$,

$$\bar{A}_1/\rho([y]_\rho)\geq\bar{A}(y) \quad , \quad \text{and hence}$$
$$\bar{A}_1/\rho([x]_\rho)\geq\bigvee_{y\in[x]_\rho}\bar{A}(y) = \bar{A}/\rho([x]_\rho) \ .$$

If we start with a fuzzy subalgebra of A/ρ, then it induces a fuzzy subalgebra of A in the following way.

3.3. Let ρ be a congruence relation on A, and $\bar{A}/\rho : S/\rho \to L$ an arbitrary fuzzy subalgeba of A/ρ. If $\bar{A}$ is a mapping from S to L, such that for $x \in S$ $\bar{A}(x) \overset{\mathrm{def}}{=} \bar{A}/\rho([x]_\rho)$, then:

a) $\bar{A}$ is a fuzzy subalgebra of A,

and

b) $f_\rho = \ker \rho$ is a fuzzy homomorphism from $\bar{A}$ onto $\bar{A}/\rho$.

Proof. a) For $x_1,\dots,x_n \in S$, $f \in F(n)$,

$$\bar{A}(f(x_1,\dots,x_n)) = \bar{A}/\rho([f(x_1,\dots,x_n)]_\rho) =$$

$$\bar{A}/\rho(f([x_1]_\rho,\dots,[x_n]_\rho)) \geq$$

$$\bar{A}/\rho([x_1]_\rho) \wedge \dots \wedge \bar{A}/\rho([x_n]_\rho) = \bar{A}(x_1) \wedge \dots \wedge \bar{A}(x_n),$$

since $\bar{A}/\rho$ is a fuzzy subalgebra of A/ρ, by assumption.

b) For every $x \in S$,

$\bar{A}(x) \leq \bar{A}/\rho(f_\rho(x))$, since the equality holds by definition.

3.4. Let $\bar{A}$ be as in 3.3. If $\bar{A}_1 : S \to L$ is any fuzzy subalgebra of A satisfying 3.3. (b), then $\bar{A}_1 \subseteq A$.

Proof. By the definition of a fuzzy homomorphism, for any $\bar{A}_1$ and $x \in S$

$$\bar{A}_1(x) \leq \bar{A}/\rho([x]_\rho) = \bar{A}(x), \quad \text{i.e.} \quad \bar{A}_1 \subseteq \bar{A}.$$

As a direct consequence of the definition of fuzzy homomorphism, we have the following two lemmas.

3.5. If f is an isomorphism from algebra A to A_1, and f and f^{-1} are both fuzzy homomorphism relative to the subalgebras $\bar{A}$ and $\bar{A}_1$ of A and A_1 respectively, then for every $x \in S$ $\bar{A}(x) = \bar{A}_1(f(x))$.

3.6. If ρ and σ are two congruences on $A = (S,F)$ and $\rho \subseteq \sigma$, and if the homomorphism $h: S/\rho \to S/\sigma$ is also a fuzzy homomorphism from a fuzzy subalgebra $\bar{A}/\rho$ of A/ρ to the fuzzy subalgebra $\bar{B}/\sigma$ of A/σ, then for every $x \in S, \bar{A}/\rho([x]\rho) \leq \bar{B}/\rho([x]\sigma)$.

Consider now an arbitrary fuzzy congruence relation $\bar{\rho} = \bigcup_{p \in L} p \cdot \rho_p$ on $A = (S,F)$. In part 2 we have proved that there is a homomorphism from $A/\bar{\rho}$ to A/ρ_p, for every $p \in L$, and that $A/\bar{\rho} \cong A/\rho_1$. If we considered the fuzzy subalgebras of these structures, we get the following consequences of the preceding propositions.

3.7. Let $\bar{A}/\bar{\rho}$ be a fuzzy subalgebra of $A/\bar{\rho}$, and for every $p \in L$, take a fuzzy subalgebra $\bar{A}/\rho_p$ of A/ρ_p, such that the homomorphism $h: A/\bar{\rho} \to A/\rho_p$ is preserved under the formation of fuzzy subalgebras (i.e. h is the corresponding fuzzy homomorphism), then for every $x \in S$, $\bar{A}/\bar{\rho}([x]_{\bar{\rho}}) \leq \bar{A}/\rho_p([x]_{\rho_p})$, and $\bar{A}/\bar{\rho}([x]_{\bar{\rho}}) = \bar{A}/\rho_1([x]\rho_1)$.

P r o o f. By 2.5., 2.6., 3.5., and 3.6. .

REFERENCES

[1] Vojvodić,G.,Šešelja,B., *O strukturi slabih relacija ekvivalencije i slabih relacija kongruencije, Matematički Vesnik, 1(14) (29) 1977, pp. 147-152.*

[2] Delorme,M., *Sous-groupes flous, Seminaire:"Mathematique flous", 1978-79.*

[3] Kaufmann,A., *Introduction á la theorie des sous-ensembles flous, Paris, 1973.*

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REZIME

O RASPLINUTIM FAKTOR ALGEBRAMA

U vezi sa algebrom A i kompletnom mrežom L, posmatraju se rasplinute kongruencije (definisane u [1]). Uvodi se pojam faktor algebre po rasplinutoj kongruenciji: $A/\bar{\rho}$. Kako je svaka rasplinuta kongruencija posebna unija familije običnih kongruencija na istoj algebri, od interesa je posmatrati odnos izmedju $A/\bar{\rho}$ i A/ρ, za proizvoljnu kongruenciju ρ familije. Dokazano je da uvek postoji homomorfizam izmedju $A/\bar{\rho}$ i A/ρ i dati su potrebni i dovoljni uslovi za koje je to izomorfizam. Razmatraju se i rasplinute podalgebre (definisane slično kao u [2]) algebri A i A/ρ i pretpostavljajući da ta preslikavanja očuvavaju homomorfizam, dati su uslovi pod kojima jedno od njih indukuje drugo i obrnuto. Takodje je opisana veza izmedju rasplinutih podalgebri od $A/\bar{\rho}$ i A/ρ.