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THE SPECTRAL TYPE OF POLYNOMIALS OF THE GAUSSIAN PROCESS

Zoran A. Ivković

Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića br.4, Jugoslavija

ABSTRACT

Let $\xi(t) = \int_0^t g(t,u)d\eta(u)$ be the proper canonical representation of the Gaussian process $\{\xi(t),\ t>0\}$ and let H_n be the linear closure of polynomials $P_n(\xi(t_1),\dots,\xi(t_n))$. The conditional expectation $E_t(\cdot) = E(\cdot|\xi(u),\ u\le t),\ t\ge 0$, is a resolution of the identity in the separable Hilbert space H_n.

It is proved that the measure $||d\eta(u)||^2$ is the uniform maximal spectral type of the infinite multiplicity in H_n.

Let $\{\xi(t),\ t\ge 0\}$ be a Gaussian process with the proper canonical representation ([2]).

$$\xi(t) = \int_0^t g(t,u)d\eta(u),\quad t\ge 0 \tag{1}$$

The process $\{\eta(u),\ u\ge 0\}$ is a martingal with

$$||\eta(t)||^2 = E\eta^2(t) = F(t) = \int_0^t f(u)du,\quad f(u)\ge 0 \quad \text{a.e.}$$

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Let $H^{(1)}(\xi)$ $(H_t^{(1)}(\xi))$ be the linear closure of $\{\xi(u),\ u \geq 0\}$ $(\{\xi(u),\ 0 \leq u \leq t\})$. It follows from (1) that $H_t^{(1)}(\xi) = H_t^{(1)}(\eta)$, $t \geq 0$. The linear time-domain analysis consists in the determination of the measure $dF(t)$. More precisely, the determination of the class of all measures equivalent (by absolute continuity) to $dF(t)$, ([1]).

The conditional expectation $E_t(\cdot) = E(\cdot \mid \xi(u),\ u \leq t)$ as the operator in $H^{(1)}(\xi)$ is the projection onto $H_t^{(1)}(\xi)$. So $\{E_t,\ t \geq 0\}$ is a resolution of the identity in the separable Hilbert space $H^{(1)}(\xi)$. The proper canonical representation (1) means that the space $H^{(1)}(\xi)$ is cyclic with the spectral type $dF(t)$, ([5]).

Now we shall consider the Hilbert space H_n, $(H_n(t))$ as the linear closure of all the polynomials (the degree not exceeding n) of the random variables $\{\xi(u),\ u \geq 0\}$, $(\{\xi(u),\ u \leq t\})$. The space H_n reduces $\{E(t),\ t \geq 0\}$, [6]. So $\{E_t,\ t \geq 0\}$ is a resolution of the identity in H_n. The non-linear time-domain analysis consists in the determination of the spectral type of $\{E_t,\ t \geq 0\}$ in H_n.

THEOREM. *The spectral type of* $\{E_t,\ t \geq 0\}$ *in* H_n *is*

$$dF \geq dF \geq \dots,$$

In the terms of the spectral analysis of selfadjoint operators in the separable Hilbert space, ([5]), the theorem states that the spectral type of the cyclic subspace $H^{(1)}$ is the uniform maximal spectral type in H_n. The multiplicity of dF in H_n is infinite. It is proved in [3] that the spectral type of $\{E_t,\ t \geq 0\}$ in space H_n of the polynomials of the Wiener process $\{W(t),\ t \geq 0\}$ is $dt \geq dt \geq \dots$. The present theorem is a generalization of this result. The idea of the proof is the same as in [3], but the technique is more complicated.

P r o o f. The first step in the proof is the decomposition of H_n in the orthogonal sum of the subspaces $H^{(p)}$, p=1,...,n. $H^{(p)}$ $(H_t^{(p)})$ is the linear closure of the Hermite polynomials $H_p(t_1,\dots,t_p) = H_p(\xi(t_1),\dots,\xi(t_p))$, $0 \le t_p \le \dots \le t_1$ of the degree p. We conclude by the relation ([4])

$$EH_p(\xi(t_1),\dots,\xi(t_p)) = H_p(E_t\xi(t_1),\dots,E_t\xi(t_p)),$$

that $H^{(p)}$ reduces $\{E_t,\ t \ge 0\}$. So it will be sufficient to prove that the spectral type of $\{E_t,\ t \ge 0\}$ in $H^{(p)}$, $p \ge 2$, is $dF(t) \ge dF(t) \ge \dots$.

In this way the theorem will be proved when we find the mutually orthogonal martingals $\{\eta_n(t),\ t \ge 0\}$, n=1,2,... in $H^{(p)}$ such that

$$(2) \qquad \sum_{n=1}^{\infty} \oplus H_t^{(1)}(\eta_n) = H_t^{(p)}, \quad t \ge 0,$$

and

$$(3) \qquad \|\eta_n(t)\|^2 = \int_0^t f_n(u)\,dF(u), \quad f_n(u) > 0 \ \text{ a.e. } dF .$$

We recall the fact, [6], that $H^{(p)}$ concides with the set $\{I_p\}$ of Ito-Rozanov integrals

$$I_p = \int_0^\infty \int_0^{t_1} \dots \int_0^{t_{p-1}} \phi(t_1,\dots,t_p)\,d\eta(t_1)\dots d\eta(t_p),$$

$$\|I_p\|^2 = \int_0^\infty \int_0^{t_1} \dots \int_0^{t_{p-1}} \phi^2(t_1,\dots,t_p)\,dF(t_1)\dots dF(t_p).$$

Denote by $S_1(t)$ the section of $\Delta_1 = \{(u_1,\dots,u_p): 0 \le u_p \le \dots \le u_1\} \subset R_p$ at $u_1 = t$ i.e. $S_1(t) = \{(t,u_2,\dots,u_p) \in \Delta_1\}$. The measure of $S_1(t)$ in R_{p-1} is $m(S_1(t)) = \int_{S_1(t)} dF(u_2)\dots dF(u_p)$.

We partition Δ_1 into two subset Δ_2 and Δ_3 such that the measures of the corresponding section $S_2(t)$ and $S_3(t)$ are equal: $m(S_2(t)) = m(S_3(t)) = \frac{1}{2}m(S_1(t))$ for each t. Then partition Δ_2 into Δ_4 and Δ_5 such that $m(S_4(t)) = m(S_5(t)) = \frac{1}{2}m(S_2(t))$,

$t \geq 0$, partition Δ_3 into Δ_6 and Δ_7 such that $m(S_6(t))=m(S_7(t))=$ $=\frac{1}{2}m(S_3(t))$ and so on. Let $I(t)$ be the support of the measure $dF(u_2)\ldots dF(u_p)$ on $S_1(t)$. We suppose that the diameter of $S_n(t) \cap I(t)$ tends to zero as $n \to \infty$, uniformly in t in each finite interval. One construction of Δ_n, n=1,2,... is done after the proof.

Let the partition of Δ_n be Δ_{n1} and Δ_{n2}. We define the processes $\{\eta_n(t),\ t \geq 0\}$, n=1,2,... by

$$\eta_1(t) = \int_0^t \{\int_{S_1(u_1)} d\eta(u_2)\ldots d\eta(u_p)\} d\eta(u_1) ,$$

$$\eta_n(t) = \int_0^t \{\int_{S_{n1}(u_1)} d\eta(u_2)\ldots d\eta(u_p)\} d\eta(u_1) - $$
$$-\int_0^t \{\int_{S_{n2}(u_1)} d\eta(u_2)\ldots d\eta(u_p)\} d\eta(u_1), \quad n=2,3,\ldots .$$

It is easy verify that $\{\eta_n(t)\}$, n=1,2,... are mutually orthogonal martingals. Also

$$\|\eta_n(t)\|^2 = \int_0^t \{\int_{S_n(u_1)} dF(u_2)\ldots dF(u_p)\} dF(u_1) =$$
$$= \int_0^t m(S_n(u_1))dF(u_1) \quad \text{where } m(S_n(u)) > 0 \quad \text{a.s.}$$

with respect to $dF(u)$. Condition (3) is satisfied.

Consider the martingals $\{\zeta_{S_n}(t),\ t \geq 0\}$, n=1,2,...

$$\zeta_{S_n}(t) = \int_0^t \{\int_{S_n(u_1)} d\eta(u_2)\ldots d\eta(u_p)\} d\eta(u_1).$$ It is easy to see that, for each t, $t \geq 0$, and n, n=1,2,..., $\zeta_{S_n}(t)$ is the finite linear combination of $\eta_k(t)$, k=1,2,...:

$$\zeta_{S_1}(t) = \eta_1(t),\quad \zeta_{S_2}(t) = \frac{1}{2}(\eta_1(t)+\eta_2(t)),$$
$$\zeta_{S_3}(t) = \frac{1}{2}(\eta_1(t)-\eta_2(t)),\ \ldots .$$

If $S^*(t) = \sum_{k=1}^{\ell} S_{j_k}(t)$, $t \geq 0$, where $S_{j_k}(t)$, $k=1,\dots,\ell$ are disjoint, then $\zeta_{S^*}(t) = \sum_{k=1}^{\ell} \zeta_{S_{j_k}}(t)$ or $\zeta_{S^*}(t) = \int_0^t \{\int_{S^*(u_1)} d\eta(u_2)\dots d\eta(u_p)\} d\eta(u_1)$.

Let $S(t)$ be a measurable subset of $S_1(t)$. We apply the standard limit procedure: if $S_m^*(t) \uparrow S(t)$ as $m \to \infty$, then $\zeta_{S_m^*}(t) \to \zeta_S(t)$ or $\zeta_S(t) = \int_0^t \{\int_{S(u_1)} d\eta(u_2)\dots d\eta(u_p)\} d\eta(u_1)$, $t \geq 0$.

We conclude that

$$\zeta_S(t) \in \sum_{n=1}^{\infty} H_t^{(1)}(\eta_n), \quad t \geq 0.$$

Let D be a bounded measurable subset of Δ_1 and let $s = \inf\{u_1 : (u_1,\dots,u_p) \in D\}$, $t = \sup\{u_1 : (u_1,\dots,u_p) \in D\}$. Consider a partition $s = s_0 < s_1 < \dots$ $s_m = t$ of $[s,t]$. Denote by $\{\zeta_j(t), t \geq 0\}$ the martingal $\{\zeta_{S_{(j)}}(t), t \geq 0\}$ where $S_{(j)}(u_1)$ for $u_1 = s_j$ is the section of D at $u_1 = s_j$. Let

$$\xi_m = \sum_{j=0}^{m-1} \int_{s_j}^{s_{j+1}} d\zeta_j(u) = \sum_{j=0}^{m-1} [\zeta_j(s_{j+1}) - \zeta_j(s_j)]$$

It follows that $\xi_m \to \int_D d\eta(u_1)\dots d\eta(u_p)$, when $\max_{0 \leq j \leq m-1} (s_{j+1} - s_j) \to 0$.

So we have proved that

$$\int_D d\eta(u_1)\dots d\eta(u_p) \in \sum_{n=1}^{\infty} \oplus H_t^{(1)}(\eta_n) \text{ or}$$

$$\int_0^t \int_0^{u_1} \dots \int_0^{u_{p-1}} \phi(u_1,\dots,u_p) d\eta(u_1)\dots d\eta(u_p) \in$$

$$\sum_{n=1}^{\infty} \oplus H_t^{(1)}(\eta_n), \quad t \geq 0.$$

This complets the proof.

<u>Construction of the sets</u> Δ_n, $n=2,3\dots$ First we consider the case $p=2$. Let $\phi_0(u) = 0$, $\phi_1(u) = u$, $u \geq 0$ and $\phi_2(u) = F^{-1}(\frac{1}{2}(F(\phi_0(u)) + F(\phi_1(u)))) = F^{-1}(\frac{1}{2}F(u))$, $u \geq 0$. Remark that

$\phi_2(u)$, $u \geq 0$ is continuous and nondecreasing. Also $0 < \phi_2(u) < u$, $u > 0$. (Of course, we suppose that $t=0$ is the increasing point of $F(t)$). We may put $S_1(t) = [0,t]$, $S_2(t) = [0,\phi_2(t)]$, $S_3(t) = [\phi_2(t), 0]$. Indeed, $m(S_1(t)) = F(t)$, $m(S_2(t)) = \int_{S_2(t)} dF(u) = F(\phi_2(t)) = \frac{1}{2} F(t)$. Now let $\phi_3(u) = F^{-1}(\frac{1}{2}(F(\phi_o(u)) + F(\phi_2(u))))$. Partition $S_2(t)$ in $S_4(t) = [0,\phi_3(t)]$ and $S_5(t) = [\phi_3(t),\phi_2(t)]$. Generally, for $S_n(t) = [\phi_{n_1}(t),\phi_{n_2}(t)]$ let

$$\phi_{n*}(u) = F^{-1}\left(\frac{1}{2}(F(\phi_{n_1}(u)) + F(\phi_{n_2}(u)))\right), \quad u \geq 0 .$$

The function $\phi_{n*}(u)$, $u \geq 0$ is continuous and nondecreasing. Also, $\phi_{n_1}(u) < \phi_{n*}(u) < \phi_{n_2}(u)$, $u > 0$. The partitions of $S_n(t)$ are $S_{n'}(t) = [\phi_{n_1}(t), \phi_{n*}(t)]$ and $S_{n''}(t) = [\phi_{n*}(t),\phi_{n_2}(t)]$, because $m(S_{n'}(t)) = \int_{\phi_{n_1}(t)}^{\phi_{n*}(t)} dF(u) = F(\phi_{n*}(t)) - F(\phi_{n_1}(t)) = \frac{1}{2}(F(\phi_{n_2}(t) - F(\phi_{n_1}(t)) = \frac{1}{2} m(S_n(t))$. Let $p=3$ and $S_1(t) = \{(t_1,t_2,t_3) \in \Delta_1\}$. Consider the subsets $S_1'(t) = \{(t\ ,t_2,t_3) : 0 \leq t_3 \leq t_2 \leq \frac{1}{2} t\}$ and $S_1''(t) = \{(t\ ,t_2,t_3) : 0 \leq t_3 \leq t_2, \frac{1}{2}t \leq t_2 \leq t\}$. Partition $S_1'(t)$ in $S_2'(t) = \{(t\ ,t_2,t_3) : 0 \leq t_3 \leq \phi_2(t_2), 0 \leq t_2 \leq \frac{1}{2}t\}$ and $S_3'(t) = \{(t, t_2,t_3) : \phi_2(t_2) \leq t_3 \leq t_2, 0 \leq t_2 \leq \frac{1}{2} t\}$. We have $m(S_2'(t)) = m(S_3'(t))$ $(= \frac{1}{4} F^2(\frac{1}{2} t))$, $t \geq 0$ because $m(S_2'(t)) = \int_0^{t/2} \int_0^{\phi_2(u_1)} dF(u_1)dF(u_2)$ and $m(S_3'(t)) = \int_0^{t/2} \int_{\phi_2(u_1)}^{u_1} dF(u_1)dF(u_2)$. Similarly, we partition $S_1''(t)$ in $S_4'' = \{(t\ ,t_2,t_3) : 0 \leq t_3 \leq \phi_2(t), \frac{1}{2}t \leq t_2 \leq t\}$ and $S_5''(t) = \{(t\ ,t_2,t_3) : \phi_2(t_2) \leq t_3 \leq t_2, \frac{1}{2}t \leq t_2 \leq t\}$. Define

$$\eta_1 = \int_o^t \int_{S_1}, \quad \eta_2 = \int_o^t \int_{S_2'} - \int_o^t \int_{S_3'}, \quad \eta_3 = \int_o^t \int_{S_4''} - \int_o^t \int_{S_5''} .$$

In the next step we partition $S_2'(t)$ ($(S_3'(t))$) by $\frac{1}{4}t$ and $\phi_3(u)$ ($\phi_4(u)$). We partition the set $S_4''(t)$ ($S_5''(t)$) by $\frac{3}{4}t$ and $\phi_3(u)$ ($\phi_4(u)$) and so on. Define $\eta_4(\eta_5)$, η_6 (η_7), and so on.

Passing to the case p=4 we use the sets $S(t) = \{(t_1, t_2, t_3, t_4) \in S^{(3)}(t_2)\}$ where $S^{(3)}(t_2)$ belongs to the sets involving the partitions in the case p=3. At the same time we partition $S_1(t) = \{(t_1, t_2, t_3, t_4) \in \Delta_1\}$ in $S_1'(t) = \{0 \le t_2 \le \frac{1}{2}t\}$ and $S_1''(t) = \{\frac{1}{2}t \le t_2 \le t\}$ and so on. The procedure for arbitrary p, $p \ge 5$ follows by induction.

A consequence. Consider the Hilbert space H (H_t) of all random variables η, $E\eta = 0$, $E\eta^2 < +\infty$ measurable with respect to σ-field generated by $\{\xi(u),\ u \ge 0\}$ $(\{\xi(u),\ 0 \le u \le t\})$. Noting the relation $H_t = \sum_{p=0}^{\infty} \oplus H_t^{(p)}$, $t \ge 0$, ([6]), we have: the spectral type of $\{E_t,\ t \ge 0\}$ in H is $dF \ge dF \ge \dots$.

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REZIME

SPEKTRALNI TIP POLINOMA GAUSOVOG PROCESA

Neka je $\xi(t)=\int_0^t g(t,u)d\eta(u)$ čisto kanonička reprezentacija Gausovog procesa $\{\xi(t),\ t\geq 0\}$ i neka je H_n linearna zatvorenost polinoma $P_n(\xi(t_1),\ldots,\xi(t_n))$. Uslovno očekivanje $E_t(.)=E_t(.|\xi(u),\ u\leq t)$ je razlaganje jedinice u separabilnom Hilbertovom prostoru H_n.

Dokazuje se da je mera $\|d\eta(u)\|^2$ uniformni maksimalni spektralni tip beskonačnog multipliciteta u H_n.