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THEOREMS OF THE GENERALIZED TAUBERIAN TYPE FOR MEASURES

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ABSTRACT

A simple proof of Tauberian type theorems for measures is given. The used limit is general enough to allows the approach not only to the vertex of the cone, but also to any point of the boundary of the conjugate cone.

1. INTRODUCTION

V.S. Vladimorov [7] proved theorems of the Abelian and Tauberian type for positive measures starting from their applications especially in the quantum field theory and also in order to solve some convolution equations. Vladimirov's paper opened up much research in this direction. We shall mention only the results of Yu.N.Drožinov and B. Zavjalov [2], [3]. They proved theorems of the Abelian and Tauberian type for tempered distributions and then, as a special case, they applied these results to measures improving those of Vladimirov.

Our paper [6] relates also to Vladimirov's results. We shall choose another way: we shall prove first some theo-

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rems for measures and then we shall enlarge them to some classes of generalized functions. In such a way we can start with minimum suppositions and with a simpler proof adding new suppositions in relation with the larger class of distributions. In this paper we shall enlarge the limit process in such a way that we can use it to analyse what happens when we approach not only the vertex of cone Γ^* but any point of its boundary.

2. NOTATIONS

Let Γ be a closed and acute cone in R^n with vertex in zero. $\Gamma^* = \{y \in R^n,\ (y,x) \geq 0,\ x \in \Gamma\}$ be the conjugate cone of the cone Γ. We know that for an acute cone Γ $\operatorname{int}\Gamma^* \neq \emptyset$ and we denote it by C; Γ^* is closed and convex; let $\operatorname{pr} C = \{e \in C, |e| = 1\}$.

H_e^+ be the half space $\{t \in R^n,\ (e,t) \geq 0\}$; if $e \in \Gamma^*$, then $\Gamma \subset H_e^+$.

$$J_k = \{1,2,\ldots,k\};\ I^n = \{t \in R^n,\ 0 \leq t_i \leq 1,\ i \in J_n\}\ ;$$

$$I^n(u,v) = \{t \in R^n,\ 0 \leq u_i < t_i < v_i \leq 1,\ i \in J_n\}\ ;$$

$$D_{m,k} = \{x \in I^m,\ x_{k+1} = \ldots = x_m = 1\},$$

$$D_{m,k}(u,v) = \{x \in I^m(u,v), x_{k+1} = \ldots = x_m = 1\}\ .$$

ρ_γ be a regular varying function of the power γ:

$$\lim_{t \to 0^+(\infty)} \rho(ut)/\rho(t) = u^\gamma\ ,\ u > 0.$$

3. THEOREM ON ASYMPTOTIC BEHAVIOUR OF THE LAPLACE TRANSFORM OF A MEASURE

THEOREM 1. *Let us suppose:*

– $\{\sigma_i\}_{i=1}^n$ *be linear independent elements from the convex closed cone* Γ^* ;

– $\rho(r) = \rho_1(r_1)\ldots\rho_m(r_m)$; $\rho_i(r_i)$ *be regular varying functions of powers* $\alpha_1,\ldots,\alpha_k > 0$; $\alpha_{k+1} = \ldots = \alpha_m = 0$, $m \leq n$;

– $g(x) \geq 0$ *be a bounded semicontinuous function over* I^n, *continuous for almost all* $x \in I^n$ *and that the point* $(1,\dots,1)$ *is not an accumulation point of discontinuities;*

– μ *be a nonnegative measure with support in* Γ, $\mu \not\equiv 0$;

– $\tilde{\mu}(iy) = \int_\Gamma e^{-(y,t)}d\mu(t)$ *exists for all* $y \in C = \operatorname{int}\Gamma^*$.

If for $y_r = \sum_{i=1}^{m} r_i\mu_i\sigma_i + \sum_{i=m+1}^{n} \mu_i\sigma_i \equiv y_r^m + \xi$, $\mu_i > 0$, $i \in J_m$; $\mu_i \geq 0$, $i \in J_n$, there exists

$$(1) \qquad \lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r,t)} d\mu(t) = h(y),$$

then

$$\lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r,t)} g(Z_1,\dots,Z_m)d\mu(t) =$$

$$(2) \qquad = \begin{cases} \dfrac{1}{\prod^k \Gamma(\alpha_i)} h(y) \displaystyle\int_{(R^+)^k} \tau_1\cdots\tau_k\, g(\tau_1,\dots,\tau_m) \prod^k t_i^{\alpha_i-1} dt, & k \geq 1, \\ g(1,\dots,1)h(y), & k=0; \end{cases}$$

where $\tau_i = e^{-t_i}$, $i \in J_k$; $\tau_i = 1$, $i \in J_m \setminus J_k$ and $Z_i = e^{-r_i\mu_i(\sigma_i,t)}$.

We will give the proof of this theorem by using three lemmas as follows. The first one is a generalization of a lemma proved by J.Karamata [5].

LEMMA 1. *If* $g(x)$ *is defined over* I^m, *continuous for almost all* $x \in D_{m,k}$ *and bounded over* $D_{m,k}$, *then for* $\alpha_i > 0$, $i \in J_k$, $1 \leq k \leq m$, *and* $\varepsilon > 0$ *there exist polynomials* $p(x_1,\dots,x_m)$ *and* $P(x_1,\dots,x_m)$ *such that*

$$(3) \qquad p(x_1,\dots,x_m) \leq g(x) \leq P(x_1,\dots,x_m)$$

and

$$(4)\qquad \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1}\,[P(\tau_1,\dots,\tau_m) - p(\tau_1,\dots,\tau_m)]\,dt < \varepsilon$$

where

$$\tau_i = e^{-t_i},\ i\in J_k \quad\text{and}\quad \tau_i = 1,\quad i\in J_m\setminus J_k .$$

P r o o f. We shall divide our proof into three parts like J.Karamata. First we suppose that $g(x)$ is of the form:

$$(5)\qquad g(x) = \begin{cases} 1, & x\in D_{m,k}(u,v)\\ 0, & x\in I^m\setminus D_{m,k}(u,v)\ . \end{cases}$$

For every $\omega > 0$ there exist nonnegative numbers ε', ε'' and a continuous function $h(x)$, $0 \le h(x) \le 1$, $x \in I^m$; $g(x) = h(x)$, $x \in D_{m,k}(u,v)$ and $h(x) = 0$, $x \in I^m\setminus D_{m,k}(u-\varepsilon', v+\varepsilon'')$ and such that

$$(6)\qquad \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1}\,[h(\tau_1,\dots,\tau_m)-g(\tau_1,\dots,\tau_m)]\,dt < \omega\ .$$

The Stone Weierstrass theorem says that there exists a polynomial $Q_\varepsilon(x_1,\dots,x_m)$ such that

$$|Q_\varepsilon(x_1,\dots,x_m) - h(x)| < \varepsilon\ ,\quad x\in I^m .$$

We can take now $P(x_1,\dots,x_m) = Q_\varepsilon(x_1,\dots,x_m) + \varepsilon$ and in this case $g(x) \le h(x) \le P(x_1,\dots,x_m)$, $x \in I^m$, and

$$\int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1}\,[P(\tau_1,\dots,\tau_m)-g(\tau)]\,dt \le$$

$$\le \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1}\,|Q_\varepsilon(\tau_1,\dots,\tau_m)-h(\tau)|\,dt +$$

$$(7)\qquad + \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1}\,[h(\tau)-g(\tau)]\,dt +$$

$$+\ \varepsilon\int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1}\,dt \le \omega + 2\varepsilon\int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1}\,dt,$$

$$\tau = (e^{-t_1},\dots, e^{-t_k}, 1,\dots,1)\ .$$

In the same way we can find a polinomial $p(x_1,\ldots,x_m)$ such that $p(x_1,\ldots,x_m) \le g(x)$, $x \in I^n$ and

$$(8)\qquad \int_{(R^+)^k} \tau_1\ldots\tau_k \prod t_i^{\alpha_i-1}[g(\tau)-p(\tau_1,\ldots,\tau_m)]dt \le \omega+2\varepsilon \prod^k \Gamma(\alpha_i).$$

The same can be proved for a step function with a finite number of jumps in every coordinate.

There only remains to suppose that our function $g(x)$ has the properties fixed in the lemma.

We can define δ and ρ in such a way that

$$(9)\qquad 2M\Big[\int_{\delta I^k} + \int_{(R^+)^k\setminus\rho I^k} \tau_1\ldots\tau_k \prod^k t_i^{\alpha_i-1} dt\Big] < \varepsilon/6$$

where $M = \sup|g(x)|$, $x \in I^m$.

Over the bounded set $I^m(\delta,\rho)$ the function

$$(10)\qquad \tau_1\ldots\tau_k g(\tau_1,\ldots,\tau_m) \prod^k t_i^{\alpha_i-1}$$

has the Riemann integral; so we have ([4] p.69) two step functions with a finite number of jumps, $g_1(x)$ and $g_2(x)$, such that $g_1(x) \le g(x) \le g_2(x)$, $x \in D_{m,k}(\delta,\rho)$ and

$$(11)\qquad \int_{I^k(\delta,\rho)} \tau_1,\ldots,\tau_k \prod^k t_i^{\alpha_i-1} [g_2(\tau_1,\ldots,\tau_m) - $$
$$- g_1(\tau_1,\ldots,\tau_k)]dt < \varepsilon/6 .$$

The functions g_1 and g_2 can be extended over $(R^+)^k \setminus I^k(\delta,\rho)$ by the constant M. The first part of this proof says that there exist polynomials $P(x_1,\ldots,x_m)$ and $p(x_1,\ldots,x_m)$ such that

$$p(x_1,\ldots,x_m) \le g_1(x) \le g(x) \le g_2(x) \le P(x_1,\ldots,x_m)$$

and

$$(12)\qquad \int_{(R^+)^k} \tau_1\ldots\tau_k \prod^k t_i^{\alpha_i-1} [P(\tau_1,\ldots,\tau_m)-g_2(\tau)]dt < \varepsilon/3$$

$$(13)\qquad \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^{k} t_i^{\alpha_i-1}\,[g_1(\tau)-p(\tau_1,\dots,\tau_m)]\,dt < \varepsilon/3 .$$

From relations (9),(10),(11),(12) and (13) there follows relation (4) which had to be proved.

LEMMA 2. *Let* $g(x)$ *be a bounded, semicontinuous function over* I^m, *continuous for almost all* $x\in I^m$ *and the point* $(1,\dots,1)$ *not an accumulation point of discontinuities. Then for* $\varepsilon>0$ *there exist polynomials* $p(x_1,\dots,x_m)$ *and* $P(x_1,\dots,x_m)$ *such that* $p(x_1,\dots,x_m)\le g(x)\le P(x_1,\dots,x_m)$ *for* $x\in I^m$ *and* $P(1,\dots,1)-p(1,\dots,1)<\varepsilon$.

P r o o f. From our suppositions it follows that there exist an interval $I^n(1-\omega,1)$ and a continuous function $h(x)$, $x\in I^m$ such that $g(x)\le h(x)$, $x\in I^m$, $g(x)=h(x)$, $x\in I^m(1-\omega,1)$. By the Stone-Weierstrass theorem there exist two polynomials $p(x_1,\dots x_m)$ and $P(x_1,\dots,x_m)$ such that

$$0\le p(x_1,\dots,x_m)-h(x)\le\varepsilon/2,\qquad x\in I^m,$$
$$0\le h(x)-p(x_1,\dots,x_m)<\varepsilon/2,\qquad x\in I^m .$$

These two polynomials satisfy the conditions of our lemma.

LEMMA 3. *Let us suppose that* $\{\sigma_i\}_{i=1}^n$ *are linear independent elements from* Γ^*; $g(x)\ge 0$ *is a bounded and upper (lower) semicontinuous function for* $x\in I^m$.

If for all $y\in C$, $y=\sum_{i=1}^n \mu_i\sigma_i$, $\mu_i\ge 0$ *there exists the integral*

$$(14)\qquad \int_\Gamma e^{-(y,t)}\,d\mu(t)$$

then the integral

$$(15)\qquad \int_\Gamma e^{-(y,t)}g(z_1,\dots,z_m)\,d\mu(t)$$

exists too, where $z_i=e^{-\mu_i(\sigma_i,t)}$, $i=1,\dots,m$.

P r o o f. By supposition on $g(x)$, $g(e^{-\mu_1(\sigma_1,t)},\ldots$ $\ldots,e^{-\mu_m(\sigma_m,t)})$, for a fixed $y\in C$, is upper (lower) semicontinuous in $t\in\Gamma$ and therefore ([4] p. 96) μ-measurable on every closed and bounded subset E of Γ and the integral

$$\int_E e^{-(y,t)} g(z_1,\ldots,z_m)\,d\mu(t)$$

exists for every such E ([4], p. 112). Let us denote by $y_p^m = \sum_{i=1}^{m} p_i\mu_i\sigma_i$. We know that $y_p^m\in C$ for all $p\geq 0$; for $\xi\in C$, $y_p^m+\xi$ belongs to C too, because Γ^* is convex. The integral (14) exists for all $y\in C$ and

$$\int_\Gamma e^{-(y_{p+1}^m+\xi,t)}\,d\mu(t) = \int_\Gamma e^{-(y,t)}\,e^{-(y_p^m,t)}\,d\mu(t)\ .$$

From this relation it follows that for every polynomial $P(x_1,\ldots,x_m)$, $m\leq n$ there exists the following integral too:

$$\int_\Gamma e^{-(y,t)} p(z_1,\ldots,z_m)\,d\mu(t)$$

where $z_i = e^{-\mu_i(\sigma_i,t)}$.

Let $P(x_1,\ldots,x_m)$ be the polynomial from Lemma 1. then

$$\int_E e^{-(y,t)} g(z_1,\ldots,z_m)\,d\mu(t) \leq \int_\Gamma e^{-(y,t)} P(z_1,\ldots,z_m)\,d\mu$$

for every $E\subseteq\Gamma$ which shows that the integral

$$\int_\Gamma e^{-(y,t)} g(z_1,\ldots,z_m)\,d\mu(t)$$

exists.

P r o o f of Theorem 1. Case $k\geq 1$. Let us suppose that $P(x_1,\ldots,x_m)$ and $p(x_1,\ldots,x_m)$ are polynomials from Lemma 1, then by our suppositions and Lemma 3 we have:

$$\begin{aligned}(16)\qquad &\int_\Gamma e^{-(y_p^m+\xi,t)} p(z_1,\ldots,z_m)\,d\mu(t) \leq \int_\Gamma e^{-(y_p^m+\xi,t)} g(z_1,\ldots,z_m)\,d\mu(t)\\ &\leq \int_\Gamma e^{-(y_p^m+\xi,t)} P(z_1,\ldots,z_m)\,d\mu(t)\ ,\end{aligned}$$

where $z_i = e^{-p_i\mu_i(\sigma_i,t)}$; similarly:

$$\int_{(R^+)^k} \tau_1\cdots\tau_k \prod^k t_i^{\alpha_i-1} p(\tau_1,\dots,\tau_m)dt \le$$

$$(17) \qquad \le \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^k t_i^{\alpha_i-1} g(\tau)dt \le$$

$$\le \int_{(R^+)^k} \tau_1\cdots\tau_k \prod^k t_i^{\alpha_i-1} P(\tau_1,\dots,\tau_m)dt$$

where $\tau = (e^{-t_1},\dots,e^{-t_k},1,\dots,1)$.

Now, in relation (2) instead of r_i, $i \in J_m$, we write $(n_i+1)r_i$, $i \in J_m$, then

$$\lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} e^{-(y_{nr}^m,t)} d\mu(t) =$$

$$(18) \qquad = h(y) = \frac{1}{\prod^k (n_i+1)^{\alpha_i}} .$$

This relation shows that for any polynomial $P(x_1,\dots,x_m)$ we have

$$\lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} P(Z_1,\dots,Z_m)d\mu(t) =$$

$$(19) \qquad = \frac{h(y)}{\prod^k \Gamma(\alpha_i)} \int_{(R^+)^k} \tau_1\cdots\tau_k P(\tau_1,\dots,\tau_m) \prod^k t_i^{\alpha_i-1} dt .$$

From relations 16-19 it follows:

$$\frac{h(y)}{\prod^k \Gamma(\alpha_i)} \int_{(R^+)^k} \tau_1\cdots\tau_k p(\tau_1,\dots,\tau_m) \prod^k t_i^{\alpha_i-1} dt \le$$

$$(20) \quad \le \lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} g(Z_1,\dots,Z_m)d\mu(t) \le$$

$$\frac{h(y)}{\prod^k \Gamma(\alpha_i)} \int_{(R^+)^k} \tau_1\cdots\tau_k P(\tau_1,\dots,\tau_m) \prod^k t_i^{\alpha_i-1} dt .$$

Now, it is enough to use the properties of our polynomials p and P from Lemma 1, and we shall have relation (2) that we had to prove.

The difference of the proof in the case when no $\alpha_i \neq 0$ is only in the fact that we shall use Lemma 2 instead of Lemma 1.

Relation (18) in this case becomes

$$(21)\qquad \lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} e^{-(y_{nr}^m,t)} d\mu(t) = h(y) .$$

For any polynomial $P(x_1,\dots,x_m)$ we have

$$(22)\qquad \lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} P(Z_1,\dots,Z_m) d\mu(t) =$$

$$= h(y)P(1,\dots,1) ,$$

where $Z_i = e^{-r_i\mu_i(\sigma_i,t)}$, $i \in J_m$.

Now the proof follows as in the first case.

4. THEOREMS OF THE TAUBERIAN TYPE FOR THE LAPLACE TRANSFORM OF A MEASURE

THEOREM 2. *Let us suppose:*

- $\{\sigma_i\}_{i=1}^n$ *are linear independent elements from the convex closed cone* Γ^* *;*
- $\rho(r) = \rho_1(r_1)\dots\rho_m(r_m)$; $\rho_i(r_i)$ *are regular varying functions of powers* $\alpha_1,\dots,\alpha_k > 0$; $\alpha_{k+1} = \dots = \alpha_m = 0$, *respectively.*
- μ *is a nonnegative measure with a support in* Γ, $\mu \not\equiv 0$;
- $\tilde{\mu}(iy)$ *exists for all* $y \in C$.

If there exists for a fixed $y \in C$, $y = \sum_{i=1}^n \mu_i\sigma_i$, $\mu_i \geq 0$

$$(23)\qquad \lim_{r\to o^+(\infty)} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} d\mu(t) = h(y) ,$$

where $y_r^m = \sum_{i=1}^n r_i\mu_i\sigma_i$ *and* $\xi = \sum_{i=m+1}^n \mu_i\sigma_i$, *then*

$$(24)\qquad \lim_{r\to\infty(o^+)} \rho(1/r) \int_{\Gamma\cap\left(\bigcap_{i}^{m} r_{i}\frac{\sigma_i}{\mu_i|\sigma_i|^2}-H^+_{\sigma_i}\right)} e^{-(\xi,t)}d\mu(t) =$$

$$= \begin{cases} \dfrac{h(y)}{\prod^{k}\Gamma(\alpha_i+1)}, & k\geq 1 \\ \\ h(y), & k=0\ . \end{cases}$$

To prove this theorem we shall use our Theorem 1 and the lemma as follows.

LEMMA 4. *Let* $g(y)$ *be the function defined on* I^m *:*

$$g(y) = \begin{cases} \prod^{m} y_i^{-1}, & e^{-1}\leq y_i\leq 1, \text{ for all } i\in J_m \\ 0, & 0\leq y_i < e^{-1}, \text{ for one } i\in J_m \end{cases}$$

then

$$z_1\ldots z_m\, g(z_1,\ldots,z_m) = \emptyset_{\Gamma\cap\left[\bigcap^{m}\left(\frac{1}{q_i}e_i - H^+_{e_i}\right)\right]}$$

for $e_i\in \mathrm{pr}\,\Gamma^*$, $i\in J_m$ *and* $t\in\Gamma$, *where* $\emptyset_F$ *is the characteristic function of the set* F *and*

$$z_i = e^{-(e_i,q_i t)}\ ,\quad q_i > 0\ .$$

P r o o f. By our supposition on g we have:

$$z_1\ldots z_m g(z_1,\ldots,z_m) = \begin{cases} 1, \text{ if } 0\leq (e_i,tq_i)\leq 1, \text{ for all } i\in J_m \\ 0, \text{ if } 1<(e_i,tq_i), \text{ for one } i\in J_m\ . \end{cases}$$

The inequality $0\leq(e_i,tq_i)\leq 1$ is equivalent with

$$0\leq\left(\frac{1}{q_i}e_i,t\right)\leq\left(\frac{1}{q_i}\right)^2\ .$$

The first part is always satisfied for $t\in\Gamma$. For the second part we have

$$\left(\frac{1}{q_i}e_i,\ \frac{1}{q_i}e_i - t\right)\geq 0\ ,$$

whence

$$\frac{1}{q_i} e_i - t \in H^+_{e_i} \quad \text{or} \quad t \in \left(\frac{1}{q_i} e_i - H^+_{e_i} \right).$$

It follows that t belongs to Γ and to every halfspace $\frac{1}{q_i} e_i - H^+_{e_i}$.

P r o o f of Theorem 2. If we take in Theorem 1 the function g as in our Lemma 4, then from relation (2) with $r_i = 1/r_i$ we have:

$$\lim_{r\to\infty(o^+)} \rho(1/r) \int_\Gamma e^{-(\xi,t)} Z_1 \dots Z_m g(Z_1 \dots Z_m)\, d\mu(t) =$$

$$= \begin{cases} \dfrac{h(y)}{\prod^k \Gamma(\alpha_i)} \displaystyle\int_{(R^+)^k} \tau_1 \dots \tau_k g(\tau_1, \dots, \tau_m) \prod^k t_i^{\alpha_i - 1}\, dt \\ h(y)\, g(1, \dots, 1), \end{cases}$$

where $Z_i = e^{-(\sigma_i, r_i^{-1} \mu_i t)}$; $\tau_i = e^{-t_i}$, $i \in J_k$ and $\tau_i = 1$, $i \in J_m \setminus J_k$.

By Lemma 4 we have:

$$\lim_{r\to\infty(o^+)} \rho(1/r) \int_{\Gamma \cap \left[\bigcap^m \left(r_i \frac{\sigma_i}{\mu_i |\sigma_i|^2} - H^+_{\sigma_i} \right) \right]} e^{-(\xi,t)} d\mu(t) =$$

$$= \begin{cases} \dfrac{h(y)}{\prod^k \Gamma(\alpha_i)} \displaystyle\int_{I^k} \prod^k t_i^{\alpha_i - 1}\, dt, & k \geq 1 \\ h(y), & k = 0. \end{cases}$$

whence follows relation (24) of our theorem.

REMARKS. The nonnegativity of the measure μ in Theorem 2 can be replaced by a less restrictive condition as follows:

Let $\rho'(r)$ be the product of regular varying functions $\rho_i'(r_i)$ of powers $\alpha_i' > 0$, $i \in J_k$. We know that every real measure μ is the difference of two nonnegative measures μ^+ and μ^-, $\mu = \mu^+ - \mu^-$.

Theorem 2 remains valid if instead of the nonnegativity of μ we suppose that μ^- is with support in Γ and that there exists ρ' such that

a) $\alpha_i' \le \alpha_i$ and

$$(25) \qquad \lim_{r\to o^+} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} d\mu^-(t) = h^-(y) \neq 0$$

or

b) $\alpha_i \le \alpha_i'$ and

$$(26) \qquad \lim_{r\to\infty} \rho(r) \int_\Gamma e^{-(y_r^m+\xi,t)} d\mu^-(t) = h^-(y) \neq 0 .$$

In case a) in our Theorem 2 we have use the limit only with the first value, and in case b) with the one which is in brackets.

The following function shows the interest of our Theorem 2. The two-dimensional Laplace transform of the function $\frac{1}{\sqrt{x+y}} J_1(2\sqrt{x+y})$ is ([1], p.241): $\frac{e^{-(1/u)}-e^{-(1/v)}}{u-v}$ where J_1 is the Bessel function. There is no $\alpha \ge 0$ such that

$$\lim_{\rho\to o^+} \rho^\alpha \frac{e^{-\frac{1}{\rho u}} - e^{-\frac{1}{\rho v}}}{\rho(u-v)} = h \neq 0$$

$u,v \neq 0$, $u \neq v$.

But we can use our Theorem 2. (See Remarks). We have to take that $\Gamma^* = (R^+)^2, \sigma_1 = (1,0)$, $\sigma_2 = (0,1)$. It is easy to see that

$$\lim_{r\to o^+} \frac{e^{-\frac{1}{ru}} - e^{-\frac{1}{v}}}{ru-v} = \frac{e^{-\frac{1}{v}}}{v}$$

and

$$\lim_{r\to o^+} \frac{e^{\frac{1}{u}} - e^{-\frac{1}{rv}}}{u-rv} = \frac{e^{-\frac{1}{u}}}{u} .$$

There exists one and only one element t_o which belongs to all hyperplanes $q_i\sigma_i - H_{\sigma_i}$, $q_i \neq 0$, $i \in J_n$, because the system

$$(q_i\sigma_i, t) = (q_i)^2, \quad q_i \neq 0, \quad i \in J_n \tag{27}$$

has one and only one solution t_o. The set $\bigcap\limits^{n} (q_i\sigma_i - H^+_{\sigma_i})$ is a cone translated in the point t_o. To show this let us suppose that $t \in \bigcap\limits^{n} (q_i\sigma_i - H^+_{\sigma_i})$, then $t_o - t$ belongs to $\bigcap\limits^{n} H^+_{\sigma_i}$:

$$(q_i\sigma_i, t_o - t) = (q_i\sigma_i, t_o - t) + (q_i\sigma_i, q_i\sigma_i - t_o)$$
$$= (q_i\sigma_i, q_i\sigma_i - t) \geq 0, \quad i \in J_n .$$

In the case $\Gamma = (R^+)^n$, $\sigma_i = e_i = (0,\ldots,1,\ldots,0)$ the set $\bigcap\limits^{n} q_ie_i - H^+_{e_i}$ is the cone $-(R^+)^n$ translated in the point $(q_1, \ldots, q_n)$.

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REZIME

GENERALISANE TAUBEROVE TEOREME ZA MERE

Dat je jednostavan dokaz Tauberove teoreme za meru koja je nenegativna ili zadovoljava dodatni uslov. Granični proces je opštiji i dozvoljava da se ispituje šta se dešava kada se približimo ne samo vrhu konjugovanog konusa, već i bilo kojoj tački njegovog ruba.