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# ASYMPTOTIC ANALYSIS OF A NONLINEAR SECOND ORDER DIFFERENCE EQUATION II

*Mirko Budinčević* *)
*Prirodno-matematički fakultet. Institut za matematiku*
*21000 Novi Sad, ul. dr Ilije Djuričića br. 4, Jugoslavija*
*M.R.S. Kulenović* **)
*Univerzitet u Sarajevu, Prirodno-matematički fakultet*
*Institut za matematiku, Sarajevo, Jugoslavija*

## ABSTRACT

The paper deals with the oscillatory and asymptotic behaviour of solutions of the nonlinear second order difference equation and gives the characterization of oscillation and its extreme solutions. Also, some specific results for the Emden-Fowler equation and some stability results for the linear equation are given.

## 1. INTRODUCTION

This paper deals with the oscillatory and asymptotic behaviour of solutions of the second order nonlinear difference equation of the form

(E) $\quad \Delta(r(n)\Delta y(n)) + p(n+1)f(y(n+1)) = 0, \quad n=0,1,\dots,$

where $\{r(n)\}_0^\infty$ and $\{p(n)\}_0^\infty$ are the given real sequences and $r(n) > 0$ for $n=0,1,\dots$, and $\Delta$ is the forward difference operator defined by $\Delta y(n) = y(n+1)-y(n)$.

Moreover, the function f is considered subject to condition

(O) $\quad f$ is nondecreasing and $uf(u) > 0$ for $u \neq 0$.

It is known (see [2]) that the general linear difference operator of the second order can be presented as the first term of (E).

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By a solution of (E) we mean a real sequence $\{y(n)\}_0^\infty$ satisfying (E). Obviously, the solution $\{y(n)\}_0^\infty$ of (E) is uniquely determined by the initial values $y(0)$ and $y(1)$ or equivalently by any two successive values $y(k)$ and $y(k+1)$ and can be defined for all $n=0,1,\ldots$ .

The following notions will be used in the sequel.

A real sequence $\{g(n)\}_0^\infty$ eventually has some property if there exists $N \geq 0$ such that $g(n)$ has this property for $n = N, N+1, \ldots$ . Throughout this paper, we shall usually refer to a solution $\{y(n)\}_0^\infty$ of (E) simply as a solution $y$ and considered only the nontrivial solutions $y$. A nontrivial solution $y$ of (E) is said to be oscillatory if $y(n)$ changes the sign infinitely many times. Otherwise, $y$ is said to be nonoscillatory.

Equation (E) is called oscillatory if each of its solution is oscillatory. Otherwise, it is called nonoscillatory.

The detailed discussion on various problems of the qualitative theory of the difference equation, including the oscillation and stability problems, can be found in [1] and [8]. We also cite papers [2],[3] and [4] for some further study on the problems of oscillation and asymptotic behaviour of solutions of (E).

Section 2. contains a fixed point theorem according to Knaster and a useful estimate for nonoscillatory solutions of (E). In Section 3., assuming that $p(n) \geq 0$ for $n \geq 0$, we obtain the characterization of some extreme types of nonoscillatory solutions. In Section 4. we give some oscillation results for (E), under the condition $\sum_0^\infty (r(n))^{-1} < \infty$ and we obtain the characterization of oscillation of (E).

Section 5. contains some specific results for the Emden-Fowler difference equation concerning the asymptotic behaviour of nonoscillatory solutions as well as some stability results for the linear equation (E) i.e. for the case where $f(u) = u$. The last section contains some extensions of the presented results.

The results of this paper are a continuation of our previous results obtained in |7| which hold in the case $\sum_{0}^{\infty} (r(n))^{-1} = \infty$.

Throughout this paper, the phrase without loss of generality is abbreviated as WLOG.

## 2. PRELIMINARIES

In what follows, we shall use the following simple fixed point theorem according to Knaster.

THEOREM 2. 1. *([9]) Let* $\leq$ *be a partial ordering with field* A *, and suppose that every* $B \subseteq A$ *has a least upper bound. Suppose that* F *maps* A *into* A *in such a way that for all* x,y *in* A $x \leq y$ *implies that* $Fx \leq Fy$. *Then* $Fx = x$ *for some* $x \in A$ .

The following result gives useful information about the bounds for nonoscillatory solution of (E).

LEMMA 2. 1. *Consider* (E) *subject to the conditions*

($C_1$) $p(n) \geq 0$ *for* $n=0,1,\ldots$ *and* $p(n)$ *is not eventually zero,*

*and*

($C_2$) $$\sum_{n=o}^{\infty} \frac{1}{r(n)} < \infty$$

*Then, every nonoscillatory solution* y *of* (E) *satisfies eventually the following estimate*

(1) $$a\rho(n) \leq |y(n)| \leq A,$$

*for some positive constants* a *and* A *(depending on* y*) where*

$$\rho(n) = \sum_{i=n}^{\infty} \frac{1}{r(i)} .$$

P r o o f. WLOG we suppose that $y > 0$ eventually, which implies that $\Delta(r(n)\Delta y(n)) \leq 0$ eventually, which gives that $\Delta y(n)$ is eventually of constant sign.

First, we suppose that $\Delta y(n) \geq 0$ eventually, i.e. $\Delta y(n) \geq 0$ for $n \geq N \geq 0$. Then, obviously $y(n) \geq y(N)$ for $n \geq N$ and moreover for $n \geq N$

$$0 \leq r(n)\Delta y(n) \leq C \Rightarrow y(n) \leq y(N) + C \sum_{i=N}^{n-1} \frac{1}{r(i)} \leq$$

$$\leq y(N) + C \sum_{i=o}^{\infty} \frac{1}{r(i)} = C_1 .$$

Secondly, we assume that $\Delta y(n) < 0$ eventually, i.e. $\Delta y(n) \leq 0$ for $n \geq N \geq 0$. Obviously, we may suppose that $y(n) > 0$ for $n \geq N$. Thus, we get $y(n) \leq y(N)$, $n \geq N$ and $r(n)\Delta y(n) \leq -C$ for some constant $C > 0$ and $n \geq N$. Now, we have for $n \geq N$

$$y(k) - y(n) \leq -C \sum_{i=n}^{k-1} \frac{1}{r(i)} \Rightarrow y(n) \geq y(k) + C \sum_{i=n}^{k-1} \frac{1}{r(i)} \geq$$

$$\geq y(\infty) + C \sum_{i=n}^{k-1} \frac{1}{r(i)} ,$$

which by $k \to \infty$, yields $y(n) \geq C_1 > 0$ if $y(\infty) > 0$ and $y(n) \geq a\rho(n)$ if $y(\infty) = 0$.

Taking into account Lemma 2.1., it is natural to introduce the following terminology: a (nonoscillatory) solution asymptotically equivalent to $a\rho(n)$ $(n \to \infty)$ will be called the solution of the <u>minimal type</u>, while the solution which is asymptotic to a nonzero constant will be called the solution of the <u>maximal type</u>.

## 3. CHARACTERIZATION OF THE MINIMAL AND MAXIMAL TYPE SOLUTIONS

In this section we shall give the necessary and sufficent conditions which provide the existence of the minimal and maximal types solutions.

THEOREM 3. 1. *Consider the equation* (E) *subject to conditions* $(C_1)$ *and* $(C_2)$. *Then* (E) *has a nonoscillatory solution of the maximal type iff the following condition holds*

$$(C_3) \qquad \sum_{n=o}^{\infty} \rho(n)p(n) < \infty .$$

P r o o f. First, we suppose that $(C_3)$ holds. Then, there exist $N \geq 0$ such that

$$f(2K) \sum_{n=N}^{\infty} \rho(n) p(n) \leq K ,$$

where $K > 0$ is some arbitrary constant.

Let $S$ be the set of all nonincreasing sequences $x$ with the property

$$K \leq x(n) \leq 2N \quad \text{for} \quad n \geq N .$$

The set $S$ is considered endowed with the usual point-wise ordering $\leq$ :

$$x_1 \leq x_2 \iff (\forall n \geq N)\ x_1(n) \leq x_2(n) .$$

Moreover, we consider the mapping $F$ on $S$ defined as follows:

$$(Fx)(n) = K + \rho(n) \sum_{i=N}^{n-1} p(i+1) f(x(i+1)) +$$

$$+ \sum_{i=n}^{\infty} \rho(i+1) p(i+1) f(x(i+1)), \quad n = N, N+1, \ldots .$$

Obviously, all conditions of Theorem 2.1. are satisfied, which implies that there exists $w \in S$ such that $Fw = w$, i.e.

$$w(n) = K + \rho(n) \sum_{i=N}^{n-1} p(i+1) f(x(i+1)) +$$

$$+ \sum_{i=n}^{\infty} \rho(i+1) p(i+1) f(x(i+1)), \quad n \geq N.$$

Now, it is easy to see that $w$ is a required nonoscillatory solution of (E). Next, we suppose that (E) has a nonoscillatory solution $y$ such that $\lim_{n\to\infty} y(n) = A > 0$. Then, by Lemma 2.1., we conclude that $\Delta y(n)$ is eventually of constant sign. Hence, there exists $N \geq 0$ such that

$$n \geq N \Longrightarrow y(n) \geq \frac{A}{2} \quad \text{and} \quad \Delta y(n) \geq 0 \quad \text{or} \quad \Delta y(n) \leq 0 .$$

If $\Delta y(n) \geq 0$, then we introduce the sequence $U(n) = \rho(n) r(n) \Delta y(n)$ and get

$$\Delta U(n) = \rho(n+1)\Delta(r(n)\Delta y(n)) - \Delta y(n) =$$

$$= -\rho(n+1)p(n+1)f(y(n+1)) - \Delta y(n) \ ,$$

which by summation from N to k-1, k > N, yields

$$0 \leq U(k) \leq U(N) + y(N) - y(k) - \sum_{n=N}^{k-1} \rho(n+1)p(n+1)f(y(n+1)) \leq$$

$$\leq C - \sum_{n=N}^{k-1} \rho(n+1)p(n+1)f(y(n+1)) \ ,$$

where $C = U(N) + \frac{A}{2}$. Thus, we obtain

$$f\left(\frac{A}{2}\right) \sum_{n=N}^{\infty} \rho(n+1)p(n+1) \leq \sum_{n=N}^{\infty} \rho(n+1)p(n+1)f(y(n+1)) \leq C \ .$$

In the case where $\Delta y(n) \leq 0$ we again obtain the relation

$$U(k) \leq C - \sum_{n=N}^{k-1} \rho(n+1)p(n+1)f(y(n+1)), \quad k > N,$$

which implies

$$U(k) \leq C - f(A) \sum_{n=N}^{k-1} \rho(n+1)p(n+1), \quad k > N.$$

If $(C_3)$ fails, the last relation gives $U(k) \leq -1$ for $k \geq M \geq N$, which immediately implies

$$\Delta y(k) \leq -\frac{1}{r(k)\rho(k)} \ , \quad k \geq M,$$

and summing from M to n > M we have

$$y(n) \leq y(M) - \sum_{k=M}^{n-1} \frac{1}{r(k)\rho(k)} = y(M) +$$

$$+ \sum_{k=M}^{n-1} \frac{\Delta\rho(k)}{\rho(k)} \sim y(M) + \ln \rho(n-1) \quad \text{as} \quad n \to \infty \ .$$

So, $\lim_{n\to\infty} y(n) = -\infty$, which is an immediate contradiction.

THEOREM 3.2. *Consider the difference equation* (E) *subject to the conditions* $(C_1)$ *and* $(C_2)$. *Then* (E) *has a nonoscillatory solution of the minimal type iff*

$$(C_4) \quad \sum_{n=0}^{\infty} p(n)\,|f(\alpha\rho(n))| < \infty\,, \qquad \text{for some constant } \alpha \neq 0,$$

*holds.*

P r o o f. First, we assume that $(C_4)$ holds. Then, there exists $N \geq 0$ such that

$$\sum_{n=N} p(n) f(\alpha\rho(n)) \leq \frac{\alpha}{2} \quad .$$

Let S be the set of all nonincreasing sequences x such that

$$\frac{\alpha}{2}\,\rho(n) \leq x(n) \leq \alpha\rho(n)\,, \quad n \geq N \;.$$

S is considered endowed with the usual point-wise ordering $\leq$ as in the proof of Theorem 3.1. Moreover, we consider the mapping G on S defined in the following way:

$$(Gx)(n) = \frac{\alpha}{2}\,\rho(n) + \sum_{i=n}^{\infty} \frac{1}{r(i)} \sum_{j=N}^{i-1} p(j+1) f(x(j+1))\,, \quad n=N+1, \ldots \;.$$

Obviously, all conditions of Theorem 2.1. are satisfied, which implies that G has a fixed point z i.e. $Gz = z$ and it is clear that z is required solution of (E).

Next, suppose that y is a nonoscillatory solution of the minimal type. Taking into account Lemma 2.1. we conclude that y is a nonincreasing sequence for which there exists $M \geq 0$ such that

$$n \geq M \implies y(n) \geq \alpha\rho(n) \quad \text{and} \quad \Delta y(n) \leq 0 \quad \text{where } \alpha > 0 \text{ is constant.}$$

Now, (E) implies

$$r(n)\Delta y(n) + \sum_{i=N}^{n-1} p(i+1) f(y(i+1)) = r(N)\Delta y(N) \quad \text{for} \quad n > N.$$

On the other hand, in |7| the following identity was obtained

$$(2)\quad y(k+1) = y(N) - r(M)\Delta y(M)\sum_{i=k}^{N}\frac{1}{r(i)} + \left(\sum_{i=k}^{N}\frac{1}{r(i)}\right)\sum_{i=M-1}^{k} p(i+1)f(y(i+1)) + \sum_{n=k-1}^{N}\left(\sum_{i=n}^{N}\frac{1}{r(i)}\right)p(n+1)f(y(n+1)),\quad N>k>M,$$

which immediately implies

$$y(k+1) \geq \left(\sum_{i=k}^{N}\frac{1}{r(i)}\right)\sum_{i=M-1}^{k} p(i+1)f(y(i+1)),\quad N>k>M$$

and so

$$(3)\quad y(k+1) \geq \rho(k)\sum_{i=M-1}^{k} p(i+1)f(y(i+1)) \geq \rho(k+1)\sum_{i=M-1}^{k} p(i+1)f(y(i+1)) .$$

Hence, we get

$$\sum_{i=M}^{\infty} p(i)f(\alpha\rho(i)) \leq \sum_{i=M}^{\infty} p(i)f(y(i)) \leq \lim_{k\to\infty}\frac{y(k)}{\rho(k)} < \infty ,$$

which completes the proof of the theorem.

## 4. OSCILLATION THEOREMS

In this section we shall present some oscillation results for (E) which are the complements to the results of the preceding section.

THEOREM 4.1. *Consider the difference equation* (E) *subject to condition* $(C_1)$, $(C_2)$,

$$(C_5)\qquad \sum_{n=o}^{\infty} \rho(n)p(n) = \infty ,$$

*and*

$$(C_6)\qquad \sum_{m=o}^{\infty}\frac{1}{\rho(m)r(m)}\sum_{i=o}^{m}\rho(i)p(i) = \infty .$$

*If the following condition holds*

$$(C_7) \qquad \int_0^{\pm\delta} \frac{du}{f(u)} < \infty , \qquad \delta > 0,$$

*then* (E) *is oscillatory.*

P r o o f. Let x be a nonoscillatory solution of (E). Then, x is of constant sign eventually, say for $n \geq N$. Let

$$w(n) = \frac{r(n)\Delta x(n)}{f(x(n))}\rho(n), \quad n \geq N,$$

then we have

$$\Delta w(n) = \rho(n+1)\Delta\left(\frac{r(n)\Delta x(n)}{f(x(n))}\right) + \frac{r(n)\Delta x(n)}{f(x(n))}\Delta\rho(n) =$$

$$= \rho(n+1)\left[\frac{\Delta(r(n)\Delta x(n))}{f(x(n+1))} - \frac{r(n)\Delta x(n)\Delta f(x(n))}{f(x(n))f(x(n+1))}\right] - \frac{\Delta x(n)}{f(x(n))} =$$

$$= -\rho(n+1)p(n+1) - \rho(n+1)\frac{r(n)x(n)\Delta f(x(n))}{f(x(n))f(x(n+1))} - \frac{\Delta x(n)}{f(x(n))},$$

which implies

$$\Delta w(n) \leq -\rho(n+1)p(n+1) - \frac{\Delta x(n)}{f(x(n))}, \qquad n > N,$$

and by summation from N to m-1, m > N, we get

$$(4) \qquad w(m) \leq w(N) - \sum_{n=N}^{m-1}\rho(n+1)p(n+1) - \sum_{n=N}^{m-1}\frac{\Delta x(n)}{f(x(n))}, \; m > N.$$

Now, taking into account conditions $(C_5)$ and $(C_7)$, we conclude that $\lim_{m\to\infty} w(m) = -\infty$ and consequently there exists $M > N$ such that $x(n)\Delta x(n) \leq 0$ for every $n \geq M$ and so $\lim_{n\to\infty}|x(n)| = \delta$, $\delta \geq 0$.

Hence, using $(C_5)$ and $(C_7)$, (4) implies

$$w(m) \leq -\frac{1}{2}\sum_{n=\tilde{N}+1}^{m}\rho(n)p(n) \qquad \text{for} \quad m \geq \tilde{N}$$

where $\tilde{N}$ is chosen appropriately. Thus, we have

$$\frac{\Delta x(m)}{f(x(m))} \leq -\frac{1}{2}\frac{1}{\rho(m)r(m)}\sum_{n=\tilde{N}+1}^{m}\rho(n)p(n), \quad m \geq \tilde{N},$$

which gives

$$\sum_{m=\tilde N}^{k} \frac{\Delta x(m)}{f(x(m))} \le -\frac{1}{2} \sum_{m=\tilde N}^{k} \frac{1}{\rho(m)r(m)} \sum_{n=N+1}^{m} \rho(n)p(n), \; k > \tilde N.$$

Taking $k\to\infty$ and using conditions $(C_6)$ and $(C_7)$ we obtain an immediate contradiction, which completes the proof the theorem.

THEOREM 4.2. *Consider the difference equation* (E) *subject to conditions* $(C_1)$, $(C_2)$ *and*

$$(C_8) \qquad \sum_{n=0}^{\infty} \rho(n)p(n)|f(\alpha\rho(n))| = \infty, \text{ for all } \alpha \ne 0.$$

*Then* (E) *is oscillatory.*

P r o o f. Let $y$ be a nonoscillatory solution of (E). Then, $y$ is of constant sign eventually and estimate (1) holds, which means that there exists $N \ge 0$ such that

$$n \ge N \Rightarrow a\rho(n) \le |y(n)| \le A \text{ and } \Delta y(n) \text{ is of a constant sign.}$$

WLOG we may suppose that $y > 0$. Thus, by $(C_8)$, we conclude

$$(5) \qquad \sum_{n=0}^{\infty} \rho(n)p(n)f(y(n)) = \sum_{n=0}^{\infty} \rho(n)p(n) = \infty ,$$

and summing (E) from N to n-1, we obtain

$$r(n)\Delta y(n) - r(N)\Delta y(N) + \sum_{i=N}^{n-1} p(i+1)f(y(i+1)) = 0,$$

which by (5) implies $\lim_{n\to\infty} r(n)\Delta y(n) = -\infty$. So, $\Delta y(n) \le 0$ eventually. WLOG we may suppose that $\Delta y(n) \le 0$ for $n \ge N$. Introducing the sequence $U(n)$ as in the proof of Theorem 3.1., we obtain the relation

$$U(k) + y(k) \le C - \sum_{n=N}^{k-1} \rho(n+1)p(n+1)f(y(n+1)) ,$$

where $C = U(N) + y(N)$.

Now it is easy to show that $U(k) + y(k) \ge 0$ eventually, in which case the last inequality implies the required conclusion.

Actually, multiplying (E) by $\sum_{i=n+1}^{k} \frac{1}{r(i)}$ and then summing from m-1 to k-1 we get

$$\sum_{n=m-1}^{k-1} \Big( \sum_{i=n+1}^{k} \frac{1}{r(i)} \Big) \Delta(r(n)\Delta y(n)) +$$

$$+ \sum_{n=m-1}^{k-1} \Big( \sum_{i=n+1}^{k} \frac{1}{r(i)} \Big) p(n+1) f(y(n+1)) = 0,$$

which by the summation by parts formula, implies

$$(6) \quad y(m) + \Big( \sum_{i=m}^{k} \frac{1}{r(i)} \Big) r(m)\Delta y(m) = y(k+1) +$$

$$+ \sum_{n=m-1}^{k-1} \Big( \sum_{i=n+1}^{k} \frac{1}{r(i)} \Big) p(n+1) f(y(n+1)) \geq 0 .$$

Thus, we obtain

$$y(m) + \rho(m) r(m) \Delta y(m) \geq 0,$$

which completes the proof of the theorem.

COROLLARY 4.1. *Consider the difference equation* (E) *subject to conditions* $(C_1)$, $(C_2)$ *and* $(C_7)$. *Then,* (E) *is oscillatory iff condition* $(C_5)$ *holds.*

P r o o f. It follows immediately from Theorem 3.1. and 4.1., taking into account the fact that $(C_2)$ and $(C_5)$ imply $(C_5)$ and $(C_6)$.

REMARK 4.1. Theorem 4.1. can be considered a partial discrete analogue of a result of Kulenović and Grammatikopoulos [5] pertaining to the corresponding differential equation. Furthermore, Theorem 4.2. is a dicsrete analogue of a result of Kulenović [6].

## 5. APPLICATION TO THE EMDEN-FOWLER EQUATION

In this section we shall consider the Emden-Fowler difference equation

(EF) $\Delta(r(n)\Delta y(n)) + p(n+1)|y(n+1)|^{\nu}\operatorname{sgn} y(n+1) = 0, \ \nu > 0,$
$n=0,1,\dots,$ .

and obtain some on the global asymptotic behaviour of its oscillatory and nonoscillatory solutions.

THEOREM 5.1. *Consider the difference equation* (EF) *subject to the conditions* ($C_1$) *and* ($C_2$). *If* ($C_3$) *holds and* $\nu \geq 1$, *then every nonoscillatory solution of* (EF) *is either of the minimal or of the maximal type.*

P r o o f. Since $\nu \geq 1$, condition ($C_3$) implies ($C_4$) and by Theorem 3.1. and 3.2. the equation (EF) has both types of solutions. Now, we shall prove that (EF) has no other types of solutions.

Let $y$ be an arbitrary nonoscillatory solution of (EF). WLOG we can suppose that $y > 0$ eventually, which implies that $\Delta y(n)$ is eventually of constant sign. If $\Delta y(n) \geq 0$ eventually, we have the conclusion of the theorem.Otherwise $\Delta y(n) \leq 0$ eventually and we can suppose that $\lim_{n\to\infty} y(n) = 0$. Thus there exists $N \geq 0$ such that

$$n \geq N \Longrightarrow y(n) > 0 \text{ and } \Delta y(n) \leq 0 \text{ and the estimate (1) holds.}$$

Starting from relation (6) and taking $k \to \infty$, we obtain

$$y(m) = \sum_{n=m-1}^{\infty} \rho(n+1)p(n+1)(y(n+1))^{\nu} - \rho(m)r(m)\Delta y(m), \quad m \geq N,$$

which by estimate (1) and condition ($C_3$) yields

$$y(m) \leq y(m) \sum_{n=m}^{\infty} \rho(n)p(n)(y(n))^{\nu-1} - \rho(m)r(m)\Delta y(m) \leq$$

$$\leq A^{\nu-1}y(m) \sum_{n=N}^{\infty} \rho(n)p(n) - \rho(m)r(m)\Delta y(m) \leq$$

$$\leq \varepsilon y(m) - \rho(m)r(m)\Delta y(m), \quad m \geq N,$$

if N is large enough to provide $A^{\nu-1}\sum_{n=N}^{\infty}\rho(n)p(n)\le\varepsilon$, for an arbitrary $\varepsilon>0$.

Thus, we obtain

$$-\rho(m)r(m)\Delta y(m)\ \ge\ (1-\varepsilon)y(m),\quad m\ge N.$$

Now, taking into account this relation, estimate (1) and ($C_3$), we get

$$(1-\varepsilon)\left[r(N)\Delta y(N)-r(n)\Delta y(n)\right]=(1-\varepsilon)\sum_{m=N}^{n-1}p(m+1)(y(m+1))^{\nu}\le$$

$$\le-\sum_{m=N}^{n-1}p(m+1)(y(m+1))^{\nu-1}\rho(m+1)r(m+1)\Delta y(m+1)\le$$

$$\le-A^{\nu-1}r(n)\Delta y(n)\sum_{m=N}^{n}\rho(m)p(m)\le$$

$$\le-A^{\nu-1}r(n)\Delta y(n)\sum_{m=N}^{\infty}\rho(m)p(m)\le-\varepsilon r(n)\Delta y(n)\ .$$

Thus, we conclude

$$(1-\varepsilon)\left[r(N)\Delta y(N)-r(n)\Delta y(n)\right]\le-\varepsilon r(n)\Delta y(n)\ ,$$

which implies

$$r(n)\Delta y(n)\ge\frac{1-\varepsilon}{1-2\varepsilon}r(N)\Delta y(N),$$

and so $\lim_{n\to\infty}r(n)\Delta y(n)=\delta\in(-\infty,0]$, which by the discrete L'Hospital rule gives

$$\lim_{n\to\infty}\frac{y(n)}{\rho(n)}=-\lim_{n\to\infty}r(n)\Delta y(n)=-\delta\ ,$$

which completes the proof of the theorem.

Similar arguments lead to the corresponding result for the sublinear ($\nu\le 1$) difference equation (EF).

THEOREM 5.2. *Consider the difference equation* (EF) *subject to conditions* ($C_1$) *and* ($C_2$). *If* $\nu\le 1$ *and condition* ($C_4$) *holds, then every nonoscillatory solution of* (EF) *is either of the minimal or of the maximal type.*

Now, we shall discuss the solutions of intermediate types and we shall get some sharper estimates than (1).

THEOREM 5.3. *Consider the difference equation* (EF) *subject to conditions* $(C_1)$ *and* $(C_2)$. *If the following condition holds*

$$(C_9) \qquad \liminf_{n\to\infty} (\rho(n))^{\lambda} \sum_{k=0}^{n} p(k) > 0$$

*then, for every nonoscillatory solution of* (EF) *the following estimate holds*

$$a(\rho(n))^{\frac{1-\lambda}{1-\nu}} \le |y(n)| \le A, \quad \textit{eventually}, \quad 0 < \nu < \lambda \le 1,$$

*and, if the additional condition*

$$(C_{10}) \qquad \limsup_{n\to\infty} (\rho(n))^{\lambda} \sum_{k=0}^{n} p(k) = \infty$$

*holds, then the following estimate holds:*

$$(7) \qquad a\rho(n) \le |y(n)| \le A(\rho(n))^{\frac{\lambda-1}{\nu-1}}, \quad \textit{eventually}, \quad 1 < \lambda \le \nu,$$

*for some appropriate constants* a *and* A *depending on the solution* y .

P r o o f. Assume that y is a nonoscillatory solution of (EF). WLOG we suppose that $y > 0$ eventually. Then, it is easy to conclude that $\Delta y(n) \le 0$ eventually.

Actually, if $\Delta y(n) \ge 0$, in the first case, then we have the required conclusion. In the second case, using $(C_{10})$ we have

$$\sum_{k=N}^{n} \rho(k)p(k) \ge \sum_{k=N}^{n} (\rho(k))^{\lambda} p(k) \ge (\rho(n))^{\lambda} \sum_{k=N}^{n} p(k) \to \infty ,$$

which by Theorem 3.1. implies that $\lim_{n\to\infty} y(n) = 0$ and so $\Delta y(n) \le 0$ eventually.

Let $M \ge 0$ be such that

$$n \ge M \Rightarrow y(n) > 0 \quad \text{and} \quad \Delta y(n) \le 0.$$

Starting from identity (2), we get inequality (3) which implies

$$y(k+1) \geq \rho(k+1) \sum_{i=M-1}^{k} p(i+1)(y(i+1))^{\nu} \geq$$

$$\geq \rho(k+1)(y(k+1))^{\nu} \sum_{i=M}^{k+1} p(i) ,$$

and by condition $(C_9)$ we get

$$1 \geq \rho(k+1)(y(k+1))^{\nu-1} \sum_{i=M}^{k+1} p(i) =$$

$$= \frac{(y(k+1))^{\nu-1}}{(\rho(k+1))^{\lambda-1}} (\rho(k+1))^{\lambda} \sum_{i=M}^{k+1} p(k) \geq C \frac{(y(k+1))^{\nu-1}}{(\rho(k+1))^{\lambda-1}} ,$$

for some $C > 0$, which immediately implies the required estimate.

REMARK 5.1. Since condition $(C_{10})$ implies condition $(C_5)$, the second part of Theorem 5.3. can be understood as follows: if there exists a nonoscillatory solution of (EF) then estimate (7) holds.

COROLLARY 5.1. *Consider the difference equation* (EF) *subject to conditions* $(C_1)$ *and* $(C_2)$. *If* $\nu > 1$, *then* (EF) *is oscillatory iff the following condition holds*

$(C_{11})$ $$\sum_{n=0}^{\infty} p(n)(\rho(n))^{\nu} = \infty.$$

P r o o f. The necessery part follows from Theorem 3.2., while the sufficient part can be obtained following similar steps as in the proofs of Theorems 4.1. and 4.2..

Now, we shall give the result concerning the global behaviour of oscillatory solutions of (E).

THEOREM 5.4. *Consider the difference equation* (E) *subject to conditions* $(C_1)$, $(C_2)$ *and*

$$(C_{12}) \qquad \sum_{n=0}^{\infty} \frac{1}{r(n)} \sum_{k=n+1}^{\infty} p(k) < \infty \quad .$$

*If f is weakly sublinear i.e.* $\lim_{|u|\to\infty} \sup \frac{f(u)}{u} < \infty$, *then all oscillatory solutions of* (E) *tend to zero.*

P r o o f. If our assertion is not true, then there exist sequences of natural numbers $\{t(n)\}_0^\infty$ and $\{\pi(n)\}_0^\infty$ and the oscillatory solution x, such that $\lim_{n\to\infty} t(n) = \lim_{n\to\infty} \pi(n) = \infty$, $t(n+1) > t(n) + 1$ and $\pi(n) \in (t(n-1), t(n))$ with the properties $x(t(n))x(\pi(n)) \le 0$ and $|x(t(n))| > \delta > 0$. WLOG we can suppose that $x(\pi(n)) \le 0$ and $x(t(n)) = \max\{x(m) : t(n-1) < m < t(m+1)\}$ .

Now, summing (E) from k to $t(n) - 1$, we obtain

$$r(t(n))\Delta x(t(n)) = r(k)\Delta x(k) - \sum_{i=k}^{t(n)-1} p(i+1)f(x(i+1)),$$

$n = 1, 2, \ldots$ .

Now, taking into account the fact that $\Delta x(t(n)) = x(t(n)+1) - x(t(n)) \le 0$, the last relation implies

$$r(k)\Delta x(k) \le \sum_{i=k}^{t(n)-1} p(i+1)f(x(i+1), \quad n=1,2,\ldots,t(n-1) \le k < t(n)$$

which yields

$$\Delta x(k) \le \frac{1}{r(k)} \sum_{i=k}^{t(n)-1} p(i+1)f(x(i+1)), \quad n=1,2,\ldots,t(n-1) \le k < t(n)$$

and summing from $\pi(n)$ to $t(n)-1$, we have

$$x(t(n)) \le x(\pi(n)) + \sum_{k=\pi(n)}^{t(n)-1} \frac{1}{r(k)} \sum_{i=k}^{t(n)-1} p(i+1)f(x(i+1)) \le$$

$$\le f(x(t(n))) \sum_{k=\pi(n)}^{t(n)-1} \frac{1}{r(k)} \sum_{i=k}^{t(n)-1} p(i+1) \le$$

$$\le f(x(t(n))) \sum_{k=\pi(n)}^{\infty} \frac{1}{r(k)} \sum_{i=k+1}^{\infty} p(i) \quad .$$

Thus, we obtain

$$1 \le \frac{f(x(t(n)))}{x(t(n))} \sum_{k=\pi(n)}^{\infty} \frac{1}{r(k)} \sum_{i=k+1}^{\infty} p(i) ,$$

which is, by $(C_{12})$ and the conditions imposed on the function f, an immediate contradiction.

REMARK 5.2. Obviously condition $(C_{12})$ is implied by condition

$$(C'_{12}) \qquad \sum_{n=0}^{\infty} \frac{1}{r(n)} < \infty, \quad \sum_{n=0}^{\infty} p(n) < \infty ,$$

as well as by condition

$$(C''_{12}) \quad \sum_{n=0}^{\infty} \frac{1}{r(n)} = \infty , \quad \sum_{n=0}^{\infty} R(n)p(n) < \infty, \text{ where } R(n) = \sum_{i=0}^{n-1} \frac{1}{r(i)} .$$

Since $(C'_{12})$ implies conditions $(C_3)$ and $(C_4)$ we can get the following result.

COROLLARY 5.2. *Consider the difference equation* (EF) *subject to conditions* $(C_1)$ *and* $(C'_{12})$ . *If* $\nu \le 1$, *then every nonoscillatory solution is either of the minimal or of the maximal type and every oscillatory solution tends to zero. If* $\nu = 1$, *it means that all solutions of* (EF) *are stable.*

Also, in the light of Theorem 5.4., we can get the following result.

COROLLARY 5.3. *Consider the difference equation* (EF) *subject to conditions* $(C_1)$ *and* $(C''_{12})$. *If* $\nu \le 1$, *then every nonoscillatory solution is either asymptotic to nonzero constant or to* AR(n) $(A \ne 0)$ *and every oscillatory solution tends to zero.*

P r o o f. It follows immediately from Theorem 5.4. and |7|.

REMARK 5.3. Some further results on stability and asymptotic stability of the linear equation (EF) ($\nu = 1$) can be obtained by comparing the conditions of our results with the previous results of Hooker and Patula [3], where some stability results are given. We note that the linear second order difference equation

$$c(n+1)y(n+2) + c(n)y(n) = b(n+1)y(n+1), \; n=0,1,\ldots ,$$

where $c(n) > 0$, $n=0,1,\ldots$, can always be reduced to the equation (EF) with $\nu = 1$

$$\Delta(c(n)\Delta x(n)) + a(n+1)\; x(n+1) = 0, \; n=0,1,\ldots ,$$

where $a(n) = c(n) + c(n-1) - b(n)$, $n=1,2,\ldots$ (see [2] and [3]). Using this fact and the oscillation and nonoscillation results from [2] and [3] we can give some further results on the stability and asymptotic stability of (EF).

## 6. CONCLUDING REMARKS

REMARK 6.1. In the case of the Thomas-Fermi difference equation (E) i.e. in the case where condition $(C_1)$ is replaced by the following one

$(C_1')$ $p(n) \leq 0$ for $n=0,1,\ldots$ and $p(n)$ is not eventually zero.

Using similar methods as in the proofs of Theorem 3.1. and 3.2., we can prove, under some appropriate conditions, the existence of solutions asymptotically equivalent to constant and to $\rho(n)$. Thus, we can prove the following

THEOREM 6.1. *Consider the difference equation* (E) *subject to condition* $(C_1')$ *and* $(C_2)$. *Then,* (E) *has a nonoscillatory solution of the minimal type iff the following condition holds*

$$\sum_{n=0}^{\infty} \frac{1}{r(n)} \sum_{k=0}^{n} p(k) < \infty .$$

(E) *has a solution of the maximal type iff condition* $(C_4)$ *holds.*

We note that in the case of the Thomas-Fermi equation the estimate related to (1) does not hold and so the terms minimal and maximal do not have the same sense as above.

Moreover, all the results of Section 2.-4. can be extended to the corresponding difference inequality

$$y(n)\left[\Delta(r(n)\Delta y(n))+p(n+1)f(y(n+1))\right]\leq 0,\ n=0,1,\dots ,$$

while Theorem 6.1. also holds for the following difference inequality

$$y(n)\left[\Delta(r(n)\Delta y(n))+p(n+1)f(y(n+1))\right]\geq 0,\ n=0,1,\dots .$$

REMARK 6.2. All the obtained results can be extended to the difference equations of the from

$$\Delta r(n)\Delta y(n))+p(n)f(y(n))=0,\ n=0,1,\dots,$$

and

$$\Delta(r(n)\Delta y(n))+p(n+2)f(y(n+2))=0,\ n=0,1,\dots ,$$

which are discrete approximations of the same differential equation $(ry')'+pf(y)=0$. In that case, the conditions are slightly different, while the methods are the same.

Finally, we note that the presented results of Section 2.-4. also hold if condition (0) on $f$ is replaced by the condition that $f$ is either sublinear ( $\frac{f(u)}{u}$ is nondecreasing for $u<0$ and nonincreasing for $u>0$) or superlinear ( $\frac{f(u)}{u}$ is nonincreasing for $u<0$ and nondecreasing for $u>0$).

## REFERENCES

|1| F.V.Atkinson, *Discrete and continuous boundary value problems*, *Academic Press, New York, 1964.*

|2| D.Hinton and R.T.Lewis, *Spectral analysis of second order difference equations, J.Math.Anal. 63(1978), 421-438.*

|3| J.W.Hooker and W.T.Patula, *Riccati type transformations for second-order linear difference equations, J.Math.Anal.Appl. 82 (1981), 451-462.*

|4| J.W.Hooker and W.T.Patula, *A second order nonlinear difference equation: oscillation and asymptotic behavior , J.Math.Anal.Appl. 91 (1983), 9-29.*

|5| M.R.Kulenović and M.K.Grammatikopoulos, *On the asymptotic behavior of second order differential inequalities with alternating coefficients*, Math.Nachr. 98(1980), 317-327.

|6| M.R.S.Kulenović, *Asymptotic properties of solutions of second order differential inequalities with general deviating argument* (to appear).

|7| M.R.S.Kulenović and M.Budinčević, *Asymptotic analysis of nonlinear second-order difference equation*, Analele ştiinţifice ale Univ."Al.I.Cuza" Sec.Mathematica IAŞI, (to appear).

|8| D.I.Martynyuk, *Lectures on qualitative theory of difference equations*, Naukova Dumka, Kiev, 1972, (in Russian).

|9| A.Tarski, *A lattice theoretical fixed point theorem and its applications*, Pacific J.Math. 5(1955), 285-309.

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REZIME

ASIMPTOTSKA ANALIZA NELINEARNE DIFERENTNE JEDNAČINE DRUGOG REDA (II)

U radu se izučava oscilatornost i asimptotsko ponašanje rešenja diferentne jednačine drugog reda.

(E) $\Delta(r(n)\Delta y(n)) + p(n+1)f(y(n+1)) = 0, \quad n=0,1,\dots .$

Posle uvodnih izlaganja, u Glavi 2 se daje korisna ocena za neoscilatorna rešenja od (E). U Glavi 3 dobijamo karakterizaciju minimalnog i maksimalnog neoscilatornog rešenja. U Glavi 4 dobijamo karakterizaciju oscilatornih rešenja. Glava 5 sadrži neke specifične rezultate za Emden-Fowlerovu jednačinu kao i neke rezultate vezane za stabilnost linearne jednačine. Poslednja glava sadrži neka uopštenja i proširenja prethodnih rezultata.