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ON COMMON FIXED POINT THEOREMS IN 2-METRIC SPACES

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ABSTRACT

In |3| S.Gähler introduced the notion of 2-metric space and many fixed point theorems in such spaces are proved in |8|,|10|,|11|,|12|,|13|, |14|,|15|.

Here we shall prove a result similar to Theorem 2 |7| for bounded and complete 2-metric spaces. Also two common fixed point theorems for a family $\{A_j\}_{j\in N}$, S and T are obtained.

First, we shall give some definitions from |3|. Let X be an arbitrary set and d a real valued function on $X \times X \times X$ satisfying the following conditions:

1. To each pair of point $(x,y) \in X \times X$ with $x \neq y$ there is $z \in X$ such that $d(x,y,z) \neq 0$.
2. $d(x,y,z) = 0$ only when at least two of tree points are equal.
3. For every $x,y,z \in X$:
$$d(x,y,z) = d(x,z,y) = d(y,z,x).$$
4. For every $x,y,z,u \in X$:
$$d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z).$$

Then (X,d) is said to be a 2-metric space. It is easy to see that d is non-negative functional.

The convergence in a 2-metric space (X,d) is introduced in the following way |3|.

Let $\{x_n\}_{n\in N}$ be a sequence from X and $x \in X$. We say that the sequence $\{x_n\}_{n\in N}$ converges to x if and only if for every $a \in X$, $\lim_{n\to\infty} d(x_n,x,a) = 0$. The sequence $\{x_n\}_{n\in N}$ is a Cauchy sequence if $\lim_{m,n\to\infty} d(x_m,x_n,a) = 0$, for all $a \in X$. If every Cauchy sequence in X converges to a point in X we say that (X,d) is a complete 2-metric space.

THEOREM 1. *Let (X,d) be a bounded, complete 2-metric space, S and T be one to one continuous mapping from X into X , $A : X \to SX \cap TX$ be continuous and T and S be commutative with A. If for every $x \in X$ there exists $n(x) \in N$ so that for every $y \in X$ and every $a \in X$:*

$$d(A^{n(x)}x, A^{n(x)}y, a) \leq q \min\{d(Sx,Ty,a), d(Tx,Sy,a)\}$$

where $q \in (0,1)$, then there exists one and only one element $z \in X$ such that $z = Az = Sz = Tz$.

P r o o f. The proof is similar to the proof of Theorem 2 from |7|. Let $x_o \in X$. Since $AX \subseteq SX \cap TX$ we can define the sequence $\{x_n\}_{n\in N}$ from X in the following way:

$$Tx_{2k-1} = A^{n(x_{2k-2})}x_{2k-2}, \quad k \in N$$

$$Sx_{2k} = A^{n(x_{2k-1})}x_{2k-1}, \quad k \in N.$$

Further let $y_n = \begin{cases} Tx_{2k-1}, & n = 2k-1 \\ Sx_{2k}, & n = 2k \end{cases}$.

We shall prove that for every $a \in X$:

$$d(y_n, y_{n-1}, a) \leq q^{n-2} D(X)$$

for every $n \in N\setminus\{1\}$, where $D(X) = \sup_{x,y,z \in X} d(x,y,z)$. Let $n = 2k$. Then:

$$d(y_n,y_{n-1},a) = d(Sx_{2k},Tx_{2k-1},a) =$$

$$= d(A^{n(x_{2k-1})}x_{2k-1},A^{n(x_{2k-2})}x_{2k-2},a) = d(A^{n(x_{2k-1})}T^{-1}Tx_{2k-1},$$

$$A^{n(x_{2k-2})}x_{2k-2},a) = d(A^{n(x_{2k-1})}T^{-1}A^{n(x_{2k-2})}x_{2k-2},A^{n(x_{2k-2})}x_{2k-1},a)$$

$$\leq qd(Sx_{2k-2},A^{n(x_{2k-1})}x_{2k-2},a) \leq \ldots \leq q^{2k-3}d(Sx_2,A^{n(x_{2k-1})}x_2,a)$$

$$\leq q^{2k-2}d(Tx_1,A^{n(x_{2k-1})}x_1,a) \leq q^{n-2}D(X) .$$

Similarly for $n = 2k-1$:

$$d(y_n,y_{n-1},a) = d(Tx_{2k-1},Sx_{2k-2},a) \leq qd(Tx_{2k-3},A^{n(x_{2k-2})}x_{2k-3})$$

$$\leq \ldots \leq q^{n-2}d(Tx_1,A^{n(x_{2k-2})}x_1,a) \leq q^{n-2}D(X) .$$

Now, we shall prove that $\{y_n\}_{n \in N}$ is a Cauchy sequence. Since:

$$d(y_n,y_{n+m},a) \leq d(y_n,y_{n+1},y_{n+m}) + d(y_n,y_{n+1},a) +$$

$$+ d(y_{n+1},y_{n+2},y_{n+m}) + d(y_{n+1},y_{n+2},a) + \ldots +$$

$$+ d(y_{n+m-2},y_{n+m-1},y_{n+m}) + d(y_{n+m-1},y_{n+m},a) \leq$$

$$\leq \sum_{k=n}^{n+m-2} q^{k-1}D(X) + \sum_{k=n}^{n+m-1} q^{k-1}D(X)$$

for every $n,m \in N$ and $q \in (0,1)$ it follows that $\{y_n\}_{n\in N}$ is a Cauchy sequence. So there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$ which means that for every $a \in X$: $\lim_{n\to\infty} d(y_n,z,a) = 0$.

Since $\{Sx_{2k}\}_{k\in N}$ and $\{Tx_{2k-1}\}_{k\in N}$ are subsequences of the sequence $\{y_n\}_{n\in N}$ it follows that for every $a \in X$:

(1) $$\lim_{k\to\infty} d(Sx_{2k},z,a) = \lim_{k\to\infty} (Tx_{2k-1},z,a) = 0.$$

Let us prove that $\lim_{k\to\infty} Ax_{2k} = \lim_{k\to\infty} A^2x_{2k} = z$. As in |7|:

$$d(Sx_{2k},Ax_{2k},a) = d(A^{n(x_{2k-1})}x_{2k-1},AS^{-1}Sx_{2k},a) =$$

$$= d(A^{n(x_{2k-1})}x_{2k-1},AS^{-1}A^{n(x_{2k-1})}x_{2k-1},a) \leq qd(Tx_{2k-1},Ax_{2k-1},a)$$

$$= qd(A^{n(x_{2k-2})}x_{2k-2},AT^{-1}Tx_{2k-1},a) = qd(A^{n(x_{2k-2})}x_{2k-2},$$

$$AT^{-1}A^{n(x_{2k-2})}x_{2k-2},a) \leq q^2\, d(Sx_{2k-2},Ax_{2k-2},a) \leq$$

$$\leq \dots \leq q^{2k-1}d(Tx_1,Ax_1,a) \leq q^{2k-1}D(X)$$

and so:

$$d(Ax_{2k},z,a) \leq d(Ax_{2k},z,Sx_{2k}) + d(Ax_{2k},Sx_{2k},a) + d(Sx_{2k},z,a) \leq$$

$$\leq 2q^{2k-1}D(X) + d(Sx_{2k},z,a) .$$

Using (1) we conclude that $\lim_{k\to\infty} d(Ax_{2k},z,a) = 0$. Further:

$$d(Sx_{2k},A^2x_{2k},a) = d(A^{n(x_{2k-1})}x_{2k-1},A^2S^{-1}Sx_{2k},a) \leq$$

$$\leq qd(Tx_{2k-1},A^2x_{2k-1}) \leq \dots \leq q^{2k-1}d(Tx_1,A^2x_1,a)$$

$$\leq q^{2k-1}D(X)$$

which implies that $\lim_{k\to\infty} d(A^2x_{2k},z,a) = 0$.

Then from:

$$d(Az,Sz,a) \leq d(Az,Sz,ASx_{2k}) + d(Az,ASx_{2k},a) + d(ASx_{2k},Sz,a) =$$

$$= d(Az,ASx_{2k},Sz) + d(Az,ASx_{2k},a) + d(SAx_{2k},Sz,a)$$

since A and S are continuous, it follows that:

$$d(Az,Sz,a) = 0, \quad \text{for every } a \in X.$$

Using 1. we conclude that $Az = Sz$. Similarly we can prove that $Az = Tz$. From the continuity of A it is easy to obtain that $Az = z$. The uniqueness of the common fixed point z follows as in |7|.

REMARK. From the proof of the Theorem it is easy to see that we can suppose instead of boundedness of the space X that for every $a \in X$: $\sup_{\substack{y \in O_A(a) \\ z \in X}} d(Ta,y,z) \leq M_a$, $O_A(a) = \{A^n a,\ n \in N \cup \{0\}\}$

The following theorem is a common fixed point theorem for $\{A_j\}_{j\in N}$, S and T.

THEOREM 2. *Let* (X,d) *be a complete* 2*-metric space* ,$S,T: X \to X$ d *be continuous functional* , $A_j: X \to SX \cap TX$ $(j \in N)$, S *and* T *be continuous mapping,* A_j *commutes with* S *and* T $(j \in N)$ *and for every* i, $j \in N$ $(i \neq j)$*:*

$$d(A_i x, A_j y, a) \leq q d(Sx, Ty, a) \qquad \textit{for every} \quad x,y,a \in X$$

where $q \in [0,1)$. *Then there exists a unique common fixed point for the family* $\{A_j\}_{j\in N}$, S *and* T.

P r o o f. It is easy to see that there exists a sequence $\{x_n\}_{n\in N}$ in X such that:

$$Tx_{2n+1} = A_{2n+1}x_{2n}, \quad n \in N \cup \{0\}$$

$$Sx_{2n} = A_{2n}x_{2n-1}, \quad n \in N.$$

Let $y_n = \begin{cases} Tx_{2k+1}, & \text{for } n = 2k+1 \\ Sx_{2k}, & \text{for } n = 2k \end{cases}$. We shall prove that the sequence $\{y_n\}_{n\in N}$ is a Cauchy sequence. Since for $n = 2k$

$$d(y_{n-1},y_n,a) = d(Tx_{2k-1},Sx_{2k},a) = d(A_{2k-1}x_{2k-2},A_{2k}x_{2k-1},a) \leq$$

$$\leq qd(Sx_{2k-2},Tx_{2k-1},a) = qd(A_{2k-2}x_{2k-3},A_{2k-1}x_{2k-2},a) \leq$$

$$\leq q^2 d(Tx_{2k-3},Sx_{2k-2},a) \leq \dots \leq q^{n-2} d(Sx_2,Tx_1,a) = q^{n-2} d(y_2,y_1,a)$$

and similarly for $n = 2k+1$:

$$d(y_{n-1},y_n,a) \le q^{n-2} d(y_2,y_1,a) \quad , \text{ for every } a \in X$$

it is easy to prove that $\{y_n\}_{n\in N}$ is a Cauchy sequence.

Namely, by standard arguments we have for every $m \in N$ that:

(2) $$d(y_1,y_2,y_m) = 0 \ .$$

This can be easily proved by induction. Further:

$$d(y_{m+1},y_m,y_n) \le q^{m-1} d(y_2,y_1,y_n)$$

for every $m,n \in N$ and so from (2) it follows that:

$d(y_{m+1},y_m,y_n) = 0$, for every $m,n \in N$. It is well known that this implies $\lim_{m,n\to\infty} d(y_m,y_n,a) = 0$. From the completness of the space X it follows that there exists $z \in X$ such that:

$z = \lim_{n\to\infty} y_n$. Then $\lim_{n\to\infty} Tx_{2n-1} = \lim_{n\to\infty} Sx_{2n} = z$. Further:

$$d(Tz,TA_{2n}x_{2n-1},a) = d(Tz,A_{2n}Tx_{2n-1},a) = d(Tz,A_{2n}A_{2n-1}x_{2n-2},a)$$

tends to zero when $n \to \infty$ since T is continuous. So $Tz = \lim_{n\to\infty} A_{2n}A_{2n-1}x_{2n-1}$ and similarly $Sz = \lim_{n\to\infty} A_{2n+1}A_{2n}x_{2n-1}$.

Then from the inequality :

$$d(A_{2n}A_{2n-1}x_{2n-2},A_{2n+1}A_{2n}x_{2n-1},a) \le qd(S(Sx_{2n}),T(Tx_{2n-1}),a)$$

and continuity of S,T and d it follows that:

$d(Tz,Sz,a) \le qd(Sz,Tz,a)$, for every $a \in X$. This implies that $Tz = Sz$. Further from $(2m \neq n)$:

$$d(TSx_{2m},A_n z,a) = d(A_{2m}(Tx_{2m-1}),A_n z,a) \le qd(S(Tx_{2m-1}),Tz,a)$$

when $m \to \infty$ we conclude that $Tz = A_n z = Sz$, for every $n \in N$. Let us prove that $A_n z = A_n A_n z$, for every $n \in N$. We have:

$$d(A_nz,A_nA_nz,a) \le d(A_nz,A_nA_nz,A_{2m}Tx_{2m-1}) +$$

$$+ d(A_nz,A_{2m}Tx_{2m-1},a) + d(A_{2m}Tx_{2m-1},A_nA_nz,a) \le$$

$$\le d(A_nz,A_nA_nz,T(A_{2m}x_{2m-1})) + d(A_nz,T(A_{2m}x_{2m-1}),a) +$$

$$+ qd(S(Tx_{2m-1}),TA_nz,a)$$

and when $m \to \infty$ we obtain:

$$d(A_nz,A_nA_nz,a) \le d(A_nz,A_nA_nz,Tz) + d(A_nz,Tz,a) +$$

$$+ qd(Sz,TA_nz,a) = qd(A_nz,A_nA_nz,a) .$$

From this we conclude that $A_nz = A_nA_nz$, for every $n \in N$ and so $A_nz = Tz = Sz$ is the common fixed point for the family $\{A_n\}_{n\in N}$ S and T.

Let $w \in X$ be such that $A_nw = Tw = Sw$ and $u = Tz = Sz = A_nz$. Then for $i \neq j$ we have:

$$d(w,u,a) = d(A_iw,A_ju,a) \le q\, d(Sw,Tu,a) = d(w,u,a) .$$

This implies that $u = w$.

Now, we shall prove a fixed point theorem in 2-Banach spaces. Let L be a linear space of dimension greater than one and $\|\cdot,\cdot\|$ a real function on $L \times L$ which satisfies the following conditions:

1. $\|a,b\| = 0$ if and only if a and b are linearly dependent.
2. $\|a,b\| = \|b,a\|$, for every a,b in L.
3. $\|ta,b\| = |t|\,\|a,b\|$, for every a,b in L and real number t.
4. $\|a+b,c\| \le \|a,c\| + \|b,c\|$, for every a,b,c in L.

Then $\|\,,\,\|$ is called a 2-norm on L and $(L,\|\,,\,\|)$ is called a 2-normed space.

Every 2-normed space is a 2-metric space with 2-metric d defined by $d(a,b,c) = \|b-a, c-a\|$.

In |3| a natural topology in 2-metric space and so in 2-normed space is introduced. For a 2-normed space $(L, \|\ ,\ \|)$ this is a locally convex T_2- topology in which the fundamental system of neighbourhoods of zero in L is given by the family:

$$W_\Sigma(0) = \{a, a \in L,\ \|a, a_j\| < \varepsilon_j,\ \text{ for every } j \in \{1,2,\ldots,m\}\}$$

where

$$\Sigma = \{(a_1,\varepsilon_1),\ldots,(a_m,\varepsilon_m)\}(a_j \in L,\ j \in \{1,2,\ldots,m\},\ \varepsilon_j > 0,\ j \in \{1,2,\ldots,m\}).$$

A subset M of L is bounded if and only if every semi-norm ($b \in L$) μ_b defined by $\mu_b(a) = \|a,b\|$ ($a \in L$) is bounded on M. Every 2-normed space $(L, \|\ ,\ \|)$ is uniformisable since it is a topological vector space. For the base of this uniformity we take the family:

$$\{(a,b)\,|\,b-a \in W_\Sigma(0)\}\ ,\quad \Sigma = \{(a_1,\varepsilon_1),\ldots,(a_m,\varepsilon_m)\}$$
$$(a_j \in L, \varepsilon_j > 0 \quad \text{for every} \quad j \in \{1,2,\ldots,m\}).$$

So a Moore-Smith sequence $\{b_i\}_{i \in I}$ from L is Cauchy sequence if and only if for every $b \in L$ and every $\varepsilon > 0$ there exists a $j \in I$ so that $\|b_i - b_{i'}, b\| < \varepsilon$ for every $i, i' \in I$ so that $i, i' \geq j$.

A 2-normed space $(L, \|\ ,\ \|)$ is complete if every Cauchy sequence is convergent and such 2-normed space is called a 2-Banach space.

From the uniforme structure point of view the class of Banach space of dimension greater than 1 coincide with the class of strong locally bounded 2-Banach spaces |4| .

THEOREM 3. *Let* $(X, \|\ ,\ \|)$ *be 2-Banach space,* S *and* T *linear continuous mappings from* X *into* X, *for every* $j \in N$, $A_j : X \to SX \cap TX$ *continuous mapping,* A_j *commutes with* S *and* T *and for every* $j \in N$, A_jX *be bounded. Further, assume that the following conditions are satisfied:*

1. *For every* $i,j \in N$ $(i \neq j)$, *every* $x,y \in X$ *and every* $a \in X$ *:*

$$\| A_i x - A_j y, a \| \leq \| Sx - Ty, a \|$$

2. *There exist* $j_o \in N$, $m_o \in N$ *and* $k_o \in N$ *so that the set* $A_{j_o}^{m_o} M$ *is relatively sequentially compact where:*

$$M = [0,1]\ A_{k_o} X.$$

Then there exists $x \in X$ *so that*

$$x = A_j x = Sx = Tx\ ,\ \textit{for every}\ j \in N\ .$$

P r o o f. Let $\{r_n\}_{n\in N}$ be a sequence of real numbers from $(0,1)$ so that $\lim_{n\to\infty} r_n = 1$ and $A_{j,n}x = r_n A_j x$, $x \in X$, $(j,n) \in N \times N$. Then:

$$\| A_{j,n}x - A_{k,n}y, a \| = r_n \| A_j x - A_k y, a \| \leq r_n \| Sx - Ty, a \|$$

for $j \neq k$, every $n \in N$ and every $(x,y,a) \in X \times X \times X$. It is obvious that $A_{j,n}$ is commutative with S and T and so for every $n \in N$ there exists $x_n \in X$ so that:

$$x_n = A_{j,n} x_n = Sx_n = Tx_n,\ \text{for every}\ n \in N\ \text{every}\ j \in N.$$

Let us prove that $\lim_{n\to\infty} \| x_n - A_j x_n, a \| = 0$, for every $a \in X, j \in N$. Since $A_j X$ is bounded for every $j \in N$ there exists $M(j,a)$ so that $\| A_j x_n, a \| \leq M(j,a)$, for every $n \in N$. So we have that:

$$\| x_n - A_j x_n, a \| = \| r_n A_j x_n - A_j x_n, a \| =$$

$$= \| (r_n - 1) A_j x_n, a \| \leq |r_n - 1|\ M(j,a)$$

and since $\lim_{n\to\infty} r_n = 1$ it follows that $\lim_{n\to\infty} \| x_n - A_j x_n, a \| = 0$. Further, let us prove that for every $(j,i,n,k) \in N \times N \times N \times N$ $(j \neq i)$:

$$\text{(3)} \qquad \| A_j^{k+1} x_n - A_i^k x_n, a \| \leq \| A_j x_n - x_n, a \| \quad \text{for every}\ a \in X.$$

The proof will be given by induction. For $k = 1$ we have:

$$\| A_j^2 x_n - A_i x_n, a\| \le \| S(A_j x_n) - Tx_n, a\| = \| A_j(Sx_n) - Tx_n, a\| = \\ = \| A_j x_n - x_n, a\| .$$

Suppose that $\| A_j^k x_n - A_i^{k-1} x_n, a\| \le \| Ax_n - x_n, a\|$ for every $(i,j,n) \in N \times N \times N$ $(i \ne j)$. Then (3) follows from:

$$\| A_j^{k+1} x_n - A_i^k x_n, a\| \le \| S(A_j^k x_n) - T(A_i^{k-1} x_n), a\| =$$

$$= \| A_j^k Sx_n - A_i^{k-1} Tx_n, a\| = \| A_j^k x_n - A_i^{k-1} x_n, a\| \le \| A_j x_n - x_n, a\|$$

and so:

$$\| A_{j_o}^{m_o} x_n - x_n, a\| \le \sum_{s=j_o}^{j_o+m_o} \| A_s x_n - x_n, a\| . \tag{4}$$

From (4) it follows that $\lim_{n\to\infty} \| A_{j_o}^{m_o} x_n - x_n, a\| = 0$. Further, for every $n \in N$, $x_n = r_n A_{k_o} x_n$ and so $x_n \in M$ for every $n \in N$. Since the set $A_{j_o}^{m_o} M$ is sequentially compact it follows that there exists $x \in X$ so that for some subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ we have:

$$\lim_{k\to\infty} \| A_{j_o}^{m_o} x_{n_k} - x, a\| = 0. \tag{5}$$

The relation (5) implies that $\lim_{k\to\infty} \| x_{n_k} - x, a\| = 0$. From this it is easy to prove that x is such that:

$$x = A_j x = Sx = Tx, \quad \text{for every } j \in N.$$

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REZIME

TEOREME O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI U 2-METRIČKIM PROSTORIMA

U ovom radu je dokazana teorema o zajedničkoj nepokretnoj tački preslikavanja A,S i T u 2-metričkim prostorima.

TEOREMA 1. *Neka je* (X,d) *ograničen, kompletan 2-metrički prostor,* S *i* T *obostrano jednoznačna i neprekidna preslikavanja* X *u* X, $A: X \to SX \cap TX$ *neprekidno preslikavanje a* T *i* S *komutativna sa* A. *Ako za svako* $x \in X$ *postoji* $n(x) \in N$ *tako da je za svako* $y \in X$ *i svako* $a \in X$:

$$d(A^{n(x)}x, A^{n(x)}y, a) \leq q\min\{d(Sx,Ty,a), d(Tx,Sy,a)\}$$

gde je $q \in (0,1)$ *tada postoji jedan i samo jedan elemenat* $z \in X$ *takav da je* $z = Az = Sz = Tz$.

Takodje je dokazana sledeća teorema:

TEOREMA 2. *Neka je* (X,d) *kompletan 2-metrički prostor sa neprekidnom funkcionelom* d, $A_j: X \to SX \cap TX$ $(j \in N)$ *gde su* $S,T: X \to X$ *neprekidna preslikavanja komutativna sa* A *i za svako* $i,j \in N (i \neq j)$

$$d(A_i x, A_j y, a) \leq qd(Sx,Ty,a) \qquad \text{za svako} \quad x,y,a \in X$$

gde je $q \in (0,1)$. *Tada postoji jedinstvena zajednička nepokretna tačka familije* $\{A_j\}_{j\in N}$, S i T.

Primenom Teoreme 2 dokazana je sledeća teorema.

TEOREMA 3. *Neka je* $(X, \|\ ,\ \|)$ *2-Banachov prostor,* S *i* T *linearna neprekidna preslikavanja* X *u* X, *za svako* $j \in N$ *je* $A_j: X \to SX \cap TX$ *neprekidno preslikavanje,* A_j *je komutativno preslikavanje sa* S *i* T *i za svako* $j \in N$ *je* $A_j X$ *ograničen skup. Pretpostavimo, dalje, da su zadovoljeni sledeći uslovi:*

1. *Za svako* $i,j \in N (i \neq j)$, *za svako* $x,y \in X$ *i za svako* $a \in X$

$$\|A_i x - A_j y, a\| \leq \|Sx - Ty, a\|$$

2. *Postoje* $j_o \in N$, $m_o \in N$ *i* $k_o \in N$ *tako da je* $A_{j_o}^{m_o} M$ *relativno kompaktan gde je* $M = [0,1] A_{k_o} X$.

Tada postoji $x \in X$ *tako da je:* $x = A_j x = Sx = Tx$ *za svako* $j \in N$.