Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 12 (1982).
Review of Research Faculty of Science-University of Novi Sad, Volume 12(1982)

POWER REGULAR SEMIGROUPS

Stojan Bogdanović
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića br.4, Jugoslavija

ABSTRACT

In the present paper power intra-regular and power (left, right) regular semigroups are considered and in this way the theory of R.Croisot, |7| is generalized.

R.Croisot considered in |7| the semigroups which are unions of simple semigroups (so called intra-regular semigroups) i.e. semigroups with the property that each element is in a simple subsemigroup. He also considered left regular, right regular and regular semigroups.

Let P be one of the following properties defined below: power intra-regular, power left regular, power regular, power inverse or power orthodox. In this paper we shall consider the semigroups with the property P.

For undefined notions and notations we refer to |6|, |10| and |13|.

1. POWER INTRA-REGULAR SEMIGROUPS

DEFINITION 1.1. *A semigroup* S *is power intra-regular if for every* $a \in S$ *there exists* $m \in N$ *such that* $a^m \in Sa^{2m}S$.

LEMMA 1.1. S *is power intra-regular if and only if for every* $a \in S$ *there exists* $m \in N$ *such that* $a^m \mathcal{J} a^{2m}$ (1).

(1) $a\mathcal{J}b \Longleftrightarrow J(a) = J(b)$

THEOREM 1.1. *S is a power intra-regular semigroup if and only if some power of each element of S lies in a simple subsemigroup.*

P r o o f. Let S be a power intra-regular semigroup. Then for an arbitrary $a \in S$ there exists $m \in N$ such that $J(a^m) = Sa^mS$ and it is clear that

(1) $$Sa^mS = Sa^{2m}S, \quad \text{(Lemma 1.1.)}$$

Assume that $b,c \in J_{a^m}^{(2)}$. Then cJa^m and c^2Ja^{2m}, so by (1) we have that cJc^2. Similarly, bJb^2. From this it follows that

(2) $$J(c) = Sc^kS, \quad J(b) = Sb^rS$$

for some $k,r \in N$. Since

$$Sc^kSSc^kS \subseteq Sc^kS$$
$$Sc^kS = Ss_1(c^k)^{2p}s_2S, \text{ for some } p \in N \text{ and } s_1,s_2 \in S$$
$$= Ss_1s_3(c^{kp})^{2h}s_4c^{kp}s_2S \subseteq Sc^kSSc^kS$$
$$= Ss_1s_3(c^{kp})^{2h}s_4c^{kp}s_2S \subseteq Sc^kSSc^kS$$

it follows that

(3) $$J(c) = J^2(c) .$$

Similarly

(4) $$J(b) = J^2(b) .$$

From (3) and (4) we have

(5) $$J(c) = J(c)J(c) = J(c)J(b) = Sc^kSSb^rS ,$$

(since cJb).

From $S(c^kSSb^r)^2S \subseteq Sc^kSSb^rS = S(S(c^kSSb^r)^{2h}S)S \subseteq S(c^kSSb^r)^2S$ we have

(6) $$S(c^kSSb^r)^2S = Sc^kSSb^rS .$$

By (5) and (6) it follows that

(7) $$Sc^kSSb^rS = S(c^kSSb^r)(c^kSSb^r)S \subseteq Sb^rc^kS$$

---

(2) J_{a^m} is a $\mathcal{J}$-class of element a^m.

(8) $Sb^rc^kS \subseteq S(b^rc^k)^{2t}S \subseteq \ldots \subseteq Sc^kSSb^rS$.

Hence, by (7) and (8) we have

(9) $Sc^kSSb^rS = Sb^r c_k S$.

Now by (5) and (9) we have $J(c) = J(c)J(b) = Sb^rc^kS \subseteq SbcS \subseteq J(bc)$ and since $J(bc) \subseteq J(b)J(c)$ we have $J(bc) = J(c)$. Hence, J_{a^m} is a subsemigroup of S. We shall show that J_{a^m} is simple. If $b,d \in J_{a^m}$, then $b, bd^3b \in J_{a^m}$ and

$$b = xbd^3by .$$

From this it follows that $b = x^2bd^3bybd^3y$. Put $u = xbd$. Then $b = xud^2by$. Hence bJu. Similarly we have that $dby = vJb$. Therefore, $b = (xbd)d(dby)$, $(xbd, dby \in J_{a^m})$, i.e. J_{a^m} is a simple subsemigroup of S.

Conversely, if for an arbitrary $a \in S$ there exists $m \in N$ such that a^m is in a simple subsemigroup P of S, then $a^m \in Pa^{2m}P \subseteq Sa^{2m}S$.

A semigroup S is intra-regular if $a \in Sa^2S$ for every $a \in S$ |6|.

COROLLARY 1.1. *(|6|, |7|). S is intra-regular if and only if S is a union of the simple subsemigroups of S .*

THEOREM 1.2. *Every principal left ideal of S is a simple subsemigroup of S if and only if $a \in SabSa$ for every $a,b \in S$.*

P r o o f. If every principal left ideal of S is simple and L(a) is an arbitrary principal left ideal of S, then $L(a) = L(a)xL(a)$ for every $x \in L(a)$. For $b \in S$ we have that $ba \in L(a)$ and $L(a) = L(a)baL(a) \subseteq L(a)bL(a) \subseteq L(a)$, i.e. $L(a) = L(a)bL(a)$ for every $b \in S$. Hence, $a = aba$ or $a \in Saba$ or $a \in abSa$ or $a \in SabSa$ for every $b \in S$. From this if follows that $a \in SabSa$ for every $a,b \in S$.

Conversely, if $x,y \in L(a)$, then

$$x = a\ ,\quad y = a \Longrightarrow x = \alpha aa\beta a \in L(a)yL(a)$$
$$x = za\ ,\quad y = a \Longrightarrow x = \alpha(za)a\beta(za) \in L(a)yL(a)$$
$$x = a,\quad y = ua \Longrightarrow x = \alpha a(ua)\beta a \in L(a)yL(a)$$
$$x = za,\quad y = ua \Longrightarrow x = \alpha(za)(ua)\beta(za) \in L(a)yL(a).$$

Hence, $L(a)$ is a simple subsemigroup of S.

PROBLEM 1.1. Do the following formulas

$$(\forall a\in S)(\exists m\in N)(a^m\in Sa^{m+1}S),\quad (\forall a\in S)(\exists m\in N)(a^m\in Sa^{2m}S)$$

define the same class of semigroups?

PROBLEM 1.2. Describe the class of semigroups with the property that each proper left ideal is a power intra-regular (intra-regular, simple) semigroup.

REMARK. Semigroups in which every proper left ideal is a completely simple semigroup are described in |4|.

2. POWER LEFT REGULAR SEMIGROUPS

DEFINITION 2.1. *A semigroup* S *is power left regular if for every* $a\in S$ *there exists* $m\in N$ *such that* $a^m\in Sa^{m+1}$.

Analogously we define a power right regular semigroup.

THEOREM 2.1. *The following conditions are equivalent on a semigroup* S *:*

(i) S *is power left regular*

(ii) *For every* $a\in S$ *there exists* $m\in N$ *such that* $a^m\mathcal{L}a^{m+1}$

(iii) *Some power of each element of* S *lies in a left simple subsemigroup of* S *;*

(iv) *Every principal left ideal of* S *is power left regular.*

P r o o f. (i) <=> (ii) <=> (iv) and (iii) => (i) follow immediately. (i) => (iii). Let S be a power left regular semigroup. Then by (i) and (ii) we have that for $a\in S$ there exists $m\in N$ such that $a^m\mathcal{L}a^{2m}$. If $b,c\in L_{a^m}$ then $\mathcal{L}\ bc\mathcal{L}c^2$ (since $\mathcal{L}$ is a right congruence) and from $c\mathcal{L}a^m$ it follows that $c^2\mathcal{L}a^{2m}$, so $c^2\mathcal{L}a^m$. Hence $c^2\mathcal{L}c$. Therefore, $bc\mathcal{L}c$, i.e. L_{a^m} is a subsemigroup of S. If $b,d\in L_{a^m}$, then $b,bd\in L_d$, so $b=xbd$ for some $x\in S^1$. From

this we have $b = x^k b d^k$ for every $k \in N$. Let us put $c = xb$. It is sufficient to show that $c \in L_b$. Since S is power left regular we have that $x^k = yx^{k+1}$ for some $y \in S$ and $b = xbd = x^k bd^k =$
$= yx^{k+1}bd = yxx^k bd = yxb = yc$.

Hence, $b = yc$, $c = xb$, i.e. $c \in L_b$. Therefore, $L_{a^m} = L_b$ is a left simple subsemigroup of S.

Theorem 2.1. is a generalization of Theorem 4.2. |6|.

COROLLARY 2.1. *Every proper subsemigroup of* S *is power left regular if and only if* S *is periodic.*

LEMMA 2.1. |5| *Every proper left ideal of* L *is minimal (left simple) if and only if* L *contains exactly one minimal left ideal or* S *sontains exactly two minimal left ideals* L_1 *and* L_2 *and* $S = L_1 \cup L_2$.

The following lemma is known.

LEMMA 2.2. *The union of all minimal left ideals of* S *is a (twosided) ideal of* S, *and is a kernel of* S.

LEMMA 2.3. |5|. *Let* I *be a left (twosided) ideal of* S. *If* K *is a left simple (simple) subsemigroup of* S *and* $K \cap I \neq \emptyset$, *then* $K \subseteq I$.

LEMMA 2.4. |5|. *Let* I *be a proper twosided ideal of* S *which is not contained as a proper subset in a left ideal* $L \neq S$. *Then* $S \setminus I$ *is a left simple semigroup or* $S \setminus I = \{a\}, a^2 \in I$.

THEOREM 2.2. *Every proper left ideal of a semigroup* S *is minimal if and only if one of the following conditions holds:*

1^o S *has a kernel* K *which is a left simple subsemigroup of* S *and* $S \setminus K$ *is a left simple subsemigroup of* S ;

2^o S *has a kernel* K *which is a left simple semigroup and* $S \setminus K = \{a\}, a^2 \in K$;

3^o S *contains exactly two minimal left ideals* L_1 *and* L_2 *and* $S = L_1 \cup L_2$.

P r o o f. If all proper left ideals of S are minimal, then by Lemma 2.1. we have two cases. Assume that S has exactly one minimal left ideal K. Then by Lemma 2.2. K is a twosided ideal of S and it is the kernel of S. By Lemma 2.4. we have that $S\setminus K$ is a left simple subsemigroup of S or $S\setminus K = \{a\}$, $a^2 \in K$. If S contains exactly two minimal left ideals, then by Lemma 2.1. we have the case 3°.

Conversely, suppose that 1° holds. Let K be a kernel of S and let K be left simple. If L is a proper left ideal of S, then $K\cap L \neq \emptyset$, so $K\subseteq L$. If $K\neq L$, then $L\cap(S\setminus K)\neq\emptyset$ and by Lemma 2.3. we have that $S\setminus K\subseteq L$. Hence, $S = K\cup(S\setminus K)\subseteq L$, which is not possible. Therefore, $K = L$, i.e. K is the unique proper left ideal od S. The case 2°. If L is a proper left ideal of S and $L\neq K$, then from $K\subseteq L$ it follows that $L = K\cup\{a\} = S$ which is not possible.

A subsemigroup B of S is a <u>bi-ideal</u> of S if $BSB\subseteq B$, |6|.

COROLLARY 2.2. *Every proper bi-ideal of S is minimal if and only if one the following condition holds:*

1° S *has a kernel* G *which is a group and* $S\setminus G$ *is a group;*
2° S *has a kernel* G *which is a group and* $S\setminus G = \{a\}, a^2\in G$ *;*
3° S *is a left group* $I\times G$ $(|I|=2)$ *or* S *is a right group* $G\times J$ $(|J|=2)$.

REMARK. The semigroups described in Corollary 2.2 are, in fact, the F-semigroups considered by Schwarz, |17|.

A semigroup S is a <u>band</u> Y <u>of left</u> (<u>right</u>) <u>ideals</u> L_i $(i\in Y)$ if

$$S = \bigcup_{i\in Y} L_i,\quad L_i\cap L_j = \emptyset,\ (i\neq j),\ |3| .$$

PROPOSITION 2.1. |3|. S *is a left (right) zero band of semigroups from the class* $\mathcal{K}$ *if and only if* S *is a band of right (left) ideals from* $\mathcal{K}$.

THEOREM 2.3. *The following conditions are equivalent on a semigroup* S :

(i) *Every principal left ideal of* S *is a left simple subsemigroup of* S ;

(ii) S *is a right zero band of left simple semigroups;*

(iii) $a \in Sba$ *for every* $a,b \in S$.

P r o o f. (i) => (ii). If all principal left ideals of S are left simple, then the principal left ideals are minimal, so the principal left ideals are disjoint. From this and Proposition 2.1. it follows that S is a right zero band of left simple semigroups.

(ii) => (iii). If S is a right zero band Y of left simple semigroups S_α $(\alpha \in Y)$, then for $a \in S_\alpha$, $b \in S_\beta$ we have

$$ba \in S_\beta S_\alpha \subseteq S_{\beta\alpha} \subseteq S_\alpha \;, \quad \text{so} \quad a \in S_\alpha ba \subseteq Sba \;.$$

(iii) => (i). Let condition (iii) hold. Assume $a \in S$ and $x,y \in L(a)$. Then we have:

(a) $x = a$, $y = a$. Then $x = a \in Saa \subseteq L(a)y$. Hence,

(*) $\quad L(a) = L(a)y$ for every $y \in L(a)$

(b) $x = za$, $y = a$. Then $x = za \in zSaa \subseteq L(a)y$, i.e. condition (*) holds.

(c) $x = a$, $y = ua$. Then $x = a \in S(au)a \subseteq L(a)ua \subseteq L(a)y$ i.e. (*) holds.

(d) $x = za$, $y = ua$. Then $x = za \in zS(au)a \subseteq L(a)ua = L(a)y$ i.e. (*) holds. By (a),(b),(c) and (d) we have that L(a) is left simple.

3. POWER REGULAR SEMIGROUPS

DEFINITION 3.1. S *is power regular if for every* $a \in S$ *there exists* $m \in N$ *such that* $a^m \in a^m S a^m$.

PROPOSITION 3.1. *An element* $a \in S$ *is power regular if and only if there exists* $m \in N$ *and an idempotent* $e \in S$ *such that*

(1)′ $\qquad a^m S^1 = eS$.

P r o o f. If $a \in S$ is power regular, then $a^m = a^m x a^m$ for some $m \in N$ and $x \in S$ and $e = a^m x$ is an idempotent such that $ea^m = a^m$. Therefore, (1)' holds.

Conversely, if (1) holds, then $a^m = ex$ for some $x \in S^1$, so $ea^m = e^2x = ex = a^m$ and $e = a^m y$ for some $y \in S$. It follows from this that $a^m = ea^m = a^m y a^m$. Hence, a is power regular.

A semigroup S is <u>regular</u> if $a \in aSa$ for every $a \in S$ |6|

COROLLARY 3.1. |6|. *An element* $a \in S$ *is regular if and only if there exists an idempotent* $e \in S$ *such that* $aS^1 = eS$.

DEFINITION 3.2. S *is power completely regular if for every* $a \in S$ *there exists* $m \in N$ *and* $x \in S$ *such that* $a^m = a^m x a^m$, $a^m x = x a^m$.

PROPOSITION 3.2. *The following conditions are equivalent on a semigroup* S :

(i) S *is power completely regular* ;

(ii) *For every* $a \in S$ *there exists* $m \in N$ *such that* $a^m \in a^m S a^{m+1}$; *(Equivalently* $a^m \in a^{m+1} S a^m$*)*

(iii) *Every left ideal of* S *is power regular.*

P r o o f. (i) ⟹ (ii). This implication follows immediately. (ii) ⟹ (i). If for every $a \in S$ there exists $m \in N$ and $x \in S$ such that $a^m = a^m x a^{m+1}$, then $a^m \in Sa^{m+1}$, so $a^m \in Sa^{2m}$. This and Theorem 4.3. |6| imply that a^m lies in a subgroup of S. Therefore, S is a power completely regular semigroup.
(ii) ⟹ (iii). This implication follows immediately.
(iii) ⟹ (ii). For any $a \in S$ we have that $a^m \in a^m L(a^m) a^m = a^{3m} \cup a^m S a^{2m}$. Hence, condition (ii) holds.

Following Drazin |8| we say that an element x of a semigroup S is <u>pseudo-invertible</u> in S iff there is an element $\bar{x} \in S$ such that

(i) $x\bar{x} = \bar{x}x$
(ii) $x^n = x^{n+1}\bar{x}$ for some $n \in N$
(iii) $\bar{x} = \bar{x}^2 x$.

Following Munn |12| an element $a \in S$ is said to be pseudo-invertible iff some power of a lies in a subgroup of S.

By Proposition 3.2., Theorem 2.1. and its dual and by the results of Munn we have the following:

THEOREM 3.1. *The following conditions are equivalent on a semigroup* S :

(i) *S is power completely regular;*

(ii) *For every* $a \in S$ *there exists* $m \in N$ *such that* $a^m \in a^m S a^{m+1}$;

(iii) *Every left ideal of* S *is power regular;*

(iv) *Every element of* S *is pseudo-invertible;*

(v) *Some power of each element of* S *lies in a subgroup of* S.

REMARK. Power completely regular semigroups are treated also in |11,14,16| .

COROLLARY. 3.2. |1|. *Every proper subsemigroup of* S *is regular if and only if* S *is a monogenic semigroup of the index 2 or* S *is a union of periodic groups.*

COROLLARY 3.3. *The following conditions are equivalent on a semigroup* S :

(i) S *is a union of groups;*

(ii) *Every left ideal of* S *is regular;*

(iii) *Every principal left ideal of* S *is regular.*

4. POWER INVERSE SEMIGROUPS

A semigroup S in <u>inverse</u> if for every $a \in S$ there exists a unique $x \in S$ such that $a = axa$, $x = xax$, |6|.

DEFINITION 4.1. |9|. *A semigroup* S *is power inverse if for every* $a \in S$ *there exists* $m \in N$ *and a unique* $x \in S$ *such that* $a^m = a^m x a^m$ *and* $x = x a^m x$.

THEOREM 4.1. *The following conditions are equivalent on a semigroup* S :

(i) S *is power regular and for every two idempotents* e *and* f *from* S *there exists* $n \in N$ *such that* $(ef)^n = (fe)^n$;

(ii) *For every* $a \in S$ *there exists* $m \in N$ *such that* $S^1 a^m$ *and* $a^m S^1$ *contain a unique idempotent generator;*

(iii) *S is power inverse.*

P r o o f. (i) $\Rightarrow$ (ii). By Proposition 3.1. there exists an idempotent e such that $a^m S^1 = eS$. If there exists an idempotent $f \in S$ such that $a^m S^1 = fS$, then $eS = fS$, so $efe = f$ and $fe = e$. Since $(ef)^n = (fe)^n$, for some $n \in N$ we have that $e = f$.

(ii) $\Rightarrow$ (iii). By Proposition 3.1. S is power regular. We have to prove the uniqueness of the inverse element of a^m. Let b and c be the inverses of a^m. Then $a^m bS = a^m S = a^m cS$, $Sba^m = Sa^m = Sca^m$ and since the idempotent is unique we have $a^m b = a^m c$, $ba^m = ca^m$, so $b = ba^m b = ba^m c = ca^m c = c$.

(iii) $\Rightarrow$ (i). This implication follows by Theorem 4.6. |9|.

DEFINITION 4.2. *S is a strongly power inverse semigroup if S is power regular and the idempotent elements commute.*

THEOREM 4.2. *S is a strongly power inverse semigroup if and only if S is power inverse and the product of any two idempotents of S is an idempotent.*

P r o o f. By Theorem 4.1.

5. POWER ORTHODOX SEMIGROUPS

DEFINITION 5.1. *A semigroup S is power orthodox if S is a power regular in which idempotents form a subsemigroup.*

The following theorem analogous to the theorem of Reilly and Scheiblich, |15| (see also |10|).

THEOREM 5.1. *If S is a power regular semigroup, then the following statements are equivalent:*

(i) *S is power orthodox:*

(ii) *If for any $a,b \in S$ there exist $m,n \in N$ and $x,y \in S$ such that $a^m = a^m x a^m$, $x = x a^m x$, $b^n = b^n y b^n$, $y = y b^n y$ then yx is an inverse of $a^m b^n$.*

P r o o f. (i) $\Rightarrow$ (ii). Similarly as (A) $\Rightarrow$ (B) in Theorem IV 1.1. |10|. (ii) $\Rightarrow$ (i). If e and f are idempotents of S, then each is an inverse of itself and by (ii) we have that $ef = effeef = (ef)^2$.

COROLLARY 5.1. *If S is power orthodox, then every inverse of an idempotent is an idempotent.*

P r o o f. Let e be an idempotent and x its inverse, i.e. $e = exe$, $x = xex$. Then xe and ex are idempotents and so each is an inverse itself. By Theorem 5.1. we have that xex is an inverse of ex^2e. Hence, $x = x(ex^2e)x = xex\,xex = x^2$.

PROBLEM 5.1. Describe the power regular semigroups in which the inverse element of each idempotents is also idempotent.

PROBLEM 5.2. A more general approach than that of power orthodox semigroups is as follows: Let S have nonempty set R of regular elements. S is called an R-semigroup if R is a (regular) subsemigroup of S.

Examples are provided by: (1) A semigroup in which the identity $axabyb = abzab$ holds is a power regular R-semigroup; (2) A commutative power regular semigroup is an R-semigroup. The power regular R-semigroups remain to be described in the general case.

REFERENCES

|1| Bogdanović,S., *Sur les demi-groups dans lesquels tous les sous-demi-groupes propres sont idempotents*, Math. Sem.Notes, 9 (1981), 17-24.

|2| Bogdanović,S., *Sur les demi-groupes dans lesquels tous les sous-demi-groupes propres sont idempotents II*, Mat .Vesnik 5(18) (33), 1981, 239-243.

|3| Bogdanović,S., *Some characterizations of bands of power joined semi-groups*, Algebraic conference 1981, Novi Sad, 121-125.

|4| Bogdanović,S., and S.Gilezan, *Semigroups with completely simple kernel*, Zbornik radova PMF Novi Sad, Ser.Mat. 12(1982),429-445.

|5| Bogdanović,S., *Semigroups in which every proper left ideal is a left group* , K.Marx Univ.Economics, Dept.Math. Budapest, No.4(1982), 8-13.

|6| Clifford,A.H. and G.B.Preston, *The algebraic theory of semigroups I, Amer. Math. Soc. 1961.*

|7| Croisot,R., *Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples, Ann.Sci.École Norm.Sup. (3)70(1953), 361-379.*

|8| Drazin,M.P., *Pseudo-inverses in associative rings and semigroups, Amer. Math.Mon. 65(1958), 506-514.*

|9| Galbiati,J.L. e M.L.Veronesi, *Sui semigrouppi che sono un band di* t-*semigrouppi, Istituto Lombardo Rend,Sc.A-114-217-234 (1980).*

|10| Howie,J.M., *An introduction to semigroup theory,Acad.Press 1976.*

|11| Madison,B.L., T.K.Mukherjee and M.K.Sen, *Periodic properties of groupbound semigroups, Semigroup Forum 22(1981), 225-234.*

|12| Munn,W.D., *Pseudo-inverses in semigroups, Proc.Camb. Phil.Soc. 57(1961) 247-250.*

|13| Petrich,M., *Introduction to semigroups, Merill Publ.Comp. 1973.*

|14| Putcha,M.S., *Semigroups in which a power of each element lies in a subgroup, Semigroup Forum 5(1973), 354-361.*

|15| Reilly,N.R. and H.E.Scheiblich, *Congruences on regular semigroups, Pacific J.Math. 23(1967), 349-360.*

|16| Schein,B.M., *Semigroups for which every transitive representation by functions is a representation by reversible functions (Russian), Izv.Vyss.Uceb.Zaved., Matem. 7(1972),112-121.*

|17| Schwarz,S., *Semigroups in which every proper subideal is a group, Acta Sci. Math. 21(1969), 125-131.*

Received by the editors May 19,1983.

REZIME

STEPENO REGULARNE POLUGRUPE

U ovom radu razmatraju se polugrupe sa svojstvom P: stepeno intra-regularna, stepeno levo (desno) regularna, stepeno regularna, stepeno inverzna, stepena ortodoksna.