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A COMBINATORIAL IDENTITY AND ITS APPLICATIONS

Ratko Tošić
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul.dr Ilije Djuričića br.4, Jugoslavija
Rade Doroslovački
Fakultet tehničkih nauka. Institut za primenjene osnovne discipline, 21000 Novi Sad, Veljka Vlahovića br.3, Jugoslavija

ABSTRACT

An identity which has some interesting combinatorial interpretations is proved. A bijection is established between a set of strings over the alphabet B = {0,1,2,3} and the set of all symmetric monotone functions of n variables over the three-valued logic algebra. As a consequence, a simple formula for the number of such functions is obtained. A different proof of this formula is given in |2|.

1. DEFINITIONS AND NOTATION

Let X denote a finite and nonempty set of symbols; X is called an alphabet. By X^n we shall denote the set of all strings of the length n over the alphabet X, i.e. $X^n = \{x_1x_2 \ldots x_n | x_1, x_2, \ldots, x_n \in X\}$, the only element of X^o being the empty string λ, i.e. the string of length 0. The set of all finite strings over the alphabet X is $X^* = \bigcup_{i \geq o} X^i$.

We shall also use some special notations:

$A = \{0,1\}$;

$B = \{0,1,2,3\}$;

$C = \{1,2\}$;

$\ell_j(a)$ – the number of j´s in the string $a \in A^*$, for $j \in A$;

$\ell_j(b)$ – the number of j´s in the string $b \in B^*$, for $j \in B$;

$\ell_j(c)$ – the number of j´s in the string $c \in C^*$, for $j \in C$;

$$K_A(n,s) = \{a | a \in A^{2n},\ \ell_1(a) - \ell_o(a) = 2s\} = \{a | a \in A^{2n},\ \ell_1(a) = n+s,\ \ell_o(a) = n-s\} \quad ;$$

$K_B(n,s) = \bigcup_{s\le i\le n} K_B^i(n,s)$, where, for each i such that $s \le i \le n$ and $i - s \equiv 0 \pmod 2$

$$K_B^i(n,s) = \{b \mid b \in B^n,\ \ell_3(b)+\ell_o(b) = i,\ \ell_3(b)-\ell_o(b) = s\} =$$

$$= \{b \mid b \in B^n,\ \ell_3(b) = \frac{i+s}{2},\ \ell_o(b) = \frac{i-s}{2}\} ;$$

$H_B(n) = \bigcup_{i=o}^{n} H_B^i(n)$, where, for each i such that $0 \le i \le n$

$$H_B^i(n) = \{b_1b_2\ldots b_n \mid b_1b_2\ldots b_n \in B^n,\ \ell_1(b_1b_2\ldots b_n) + \ell_2(b_1b_2\ldots b_n) = i,\ \ell_2(b_1b_2\ldots b_k) \ge \ell_1(b_1b_2\ldots b_k) \text{ for each } k \le n\} ;$$

$H_C(i) = \bigcup_{j=o}^{\lfloor \frac{i}{2} \rfloor} H_C^j(i)$, where, for each j such that $0 \le j \le \lfloor \frac{i}{2} \rfloor$

$$H_C^j(i) = \{c_1c_2\ldots c_i \mid c_1c_2\ldots c_i \in C^i,\ \ell_1(c_1c_2\ldots c_i) = j,\ \ell_2(c_1c_2\ldots c_k) \ge \ell_1(c_1c_2\ldots c_k) \text{ for each } k \le i\} .$$

It is obvious, that for $i_1 \ne i_2$:

$$K_B^{i_1}(n,s) \cap K_B^{i_2}(n,s) = \emptyset ,$$

$$H_B^{i_1}(n) \cap H_B^{i_2}(n) = \emptyset, \text{ and for } j_1 \ne j_2 :$$

$$H_C^{j_1}(i) \cap H_C^{j_2}(i) = \emptyset .$$

If S is a set, then $|S|$ is the cardinality of S. By $\lceil x \rceil$ and $\lfloor x \rfloor$ we denote the smallest integer $\ge x$ and the greatest integer $\le x$, respectively.

2. RESULTS AND DISCUSSION

THEOREM 1. *If* $0 \le s \le n$, *then*

$$(1) \qquad \sum_{i=s}^{n} \binom{n}{i} \binom{i}{\lfloor \frac{i-s}{2} \rfloor} 2^{n-i} = \binom{2n+1}{n-s} .$$

First, we shall prove the following lemma:

LEMMA 1. *If* $0 \le s \le n$, *then*

$$\sum_{\substack{s\le i\le n\\ i-s\equiv 0 \pmod 2}} \binom{n}{i}\binom{i}{\frac{i-s}{2}}2^{n-i} = \binom{2n}{n-s} \quad . \tag{2}$$

P r o o f. It is obvious that the sets A^{2n} and B^n are of the same cardinality $2^{2n}=4^n$. An obvious bijection between the sets A^{2n} and B^n is the function $f: A^{2n}\to B^n$ such that for $a=a_1a_2\dots a_{2n}\in A^{2n}$ and $b=b_1b_2\dots b_n\in B^n$, $f(a)=b$ iff for each $k=1,2,\dots,n$, $\phi(a_{2k-1}a_{2k})=b_k$, where ϕ is the bijection $\phi=\begin{pmatrix}00 & 01 & 10 & 11\\ 0 & 1 & 2 & 3\end{pmatrix}$ between the sets A^2 and B. It is easy to see that $a\in K_A(n,s)\subset A^{2n}$ iff $b\in K_B(n,s)\subset B^n$, where $b=f(a)$. It means that the restriction of f to $K_A(n,s)\subset A^{2n}$ is a bijection between the sets $K_A(n,s)$ and $K_B(n,s)$, i.e. $|K_A(n,s)| = |K_B(n,s)|$. On the other hand

$$|K_A(n,s)| = \binom{2n}{n-s} \quad \text{and} \tag{3}$$

$$|K_B(n,s)| = \sum_{\substack{s\le i\le n\\ i-s\equiv 0 \pmod 2}} \binom{n}{i}\binom{i}{\frac{i-s}{2}}2^{n-i} \quad . \tag{4}$$

The last equality follows from the fact that for each i such that $s\le i\le n$ and $i-s\equiv 0 \pmod 2$:

$$|K_B^i(n,s)| = \binom{n}{i}\binom{i}{\frac{i-s}{2}}2^{n-i} \quad ,$$

which can be easily proved.

Now, (2) follows from (3) and (4).

P r o o f of Theorem 1. Substituting s+1 instead of s in (2), we obtain

$$\sum_{\substack{s+1\le i\le n\\ i-s-1\equiv 0 \pmod 2}} \binom{n}{i}\binom{i}{\frac{i-s-1}{2}}2^{n-i} = \binom{2n}{n-s-1}$$

i.e.

$$(5)\qquad \sum_{\substack{s+1\le i\le n\\ i-s\equiv 1 \pmod 2}} \binom{n}{i}\binom{i}{\frac{i-s-1}{2}} 2^{n-1} = \binom{2n}{n-s-1} .$$

Summing up(2) and (5), we obtain (1).

A combinatorial interpretation of (1). The number of strings of the length 2n+1 over the alphabet {x,y}, for which the difference between the number of x's and the number of y's is 2s+1, equals the number of strings of the length n over the alphabet {x,y,z,u} for which the difference between the number of x's and the number of y's is either s or s+1.

For s=0, we have

COROLLARY.

$$(6)\qquad \sum_{i=0}^{n} \binom{n}{i}\binom{i}{\lfloor\frac{i}{2}\rfloor} 2^{n-i} = \binom{2n+1}{n} .$$

A combinatoral interpretation of (6). The number of strings of the length 2n+1, over the alphabet {x,y} with n+1 x's and n y's, equals the number of strings of the length n over the alphabet {x,y,z,u} with $\lceil\frac{i}{2}\rceil$ x's and $\lfloor\frac{i}{2}\rfloor$ y's, for some i=0,1,...,n.

3. APPLICATIONS

First, we use (6) to count the number of strings in the set $H_B(n)$.

THEOREM 2. $|H_B(n)| = \binom{2n+1}{n}$.

First, we shall prove the following lemma:

LEMMA 2. $|H_C(i)| = \binom{i}{\lfloor\frac{i}{2}\rfloor}$.

P r o o f. It is known (see |1|, pp. 65-66) that for $j \le \lfloor\frac{i}{2}\rfloor$:

$$|H_C^j(i)| = \binom{i}{i-j} - \binom{i}{i-j+1} .$$

Hence, it follows that

$$|H_C(i)| = \sum_{j=o}^{\lfloor\frac{i}{2}\rfloor} |H_C^j(i)| = \sum_{j=o}^{\lfloor\frac{i}{2}\rfloor} \left(\binom{i}{i-j} - \binom{i}{i-j+1}\right) = \binom{i}{\lfloor\frac{i}{2}\rfloor} .$$

P r o o f of Theorem 2. It is easy to see that

$$|H_B^i(n)| = \binom{n}{i} |H_C(i)| 2^{n-i} = \binom{n}{i} \binom{i}{\lfloor\frac{i}{2}\rfloor} 2^{n-i} .$$

Now, we have, by using Lemma 2 and Corollary:

$$|H_B(n)| = \sum_{i=o}^{n} |H_B^i(n)| = \sum_{i=o}^{n} \binom{n}{i} \binom{i}{\lfloor\frac{i}{2}\rfloor} 2^{n-i} = \binom{2n+1}{n} .$$

Now, we are going to determine the number of symmetric monotone functions of n variables over the three - valued logic algebra i.e. the number of functions $F : E_3^n \to E_3$, where $E_3 = \{0,1,2\}$, and the following two conditions are satisfied (|3|):

(i) $(x_1,x_2,\dots,x_n) \le (y_1,y_2,\dots,y_n)$ implies

$F(x_1,x_2,\dots,x_n) \le F(y_1,y_2,\dots,y_n)$,

under the assumption that $0 < 1 < 2$ and

$(x_1,x_2,\dots,x_n) \le (y_1,y_2,\dots,y_n)$ iff $x_i \le y_i$ for all $i=1,2,\dots,n$.

(ii) $F(x_{i_1},x_{i_2},\dots,x_{i_n}) = F(x_1,x_2,\dots,x_n)$, for each permutation

$(x_{i_1},x_{i_2},\dots,x_{i_n})$ of $(x_1,x_2,\dots,x_n)$.

Since F is a symmetric function, the set of all n-tuples with p 0's, q 2's and n-p-q 1's can be represented by the n-tuple $(\underbrace{0,\dots,0}_{p},1,\dots,1,\underbrace{2,\dots,2}_{q})$ which we denote, shortly, by (p,q). So there is a bijection between the set of all symmetric monotone functions $F : E_3^n \to E_3$ and the set of all monotone functions $F' : L_{n+1}^{(2)} \to E_3$, where $L_{n+1}^{(2)}$ is the set of all ordered pairs (p,q) of integers such that $p \geq 0$, $q \geq 0$, $p+q \leq n$, and $(p,q) \leq (p',q')$ iff $p \geq p'$ and $q \leq q'$. The set $L_{n+1}^{(2)}$ partially order by the relation $\leq$ can be represented as a lattice on the Cartesian plane.

Figure 1. is such a lattice for $n=6$.

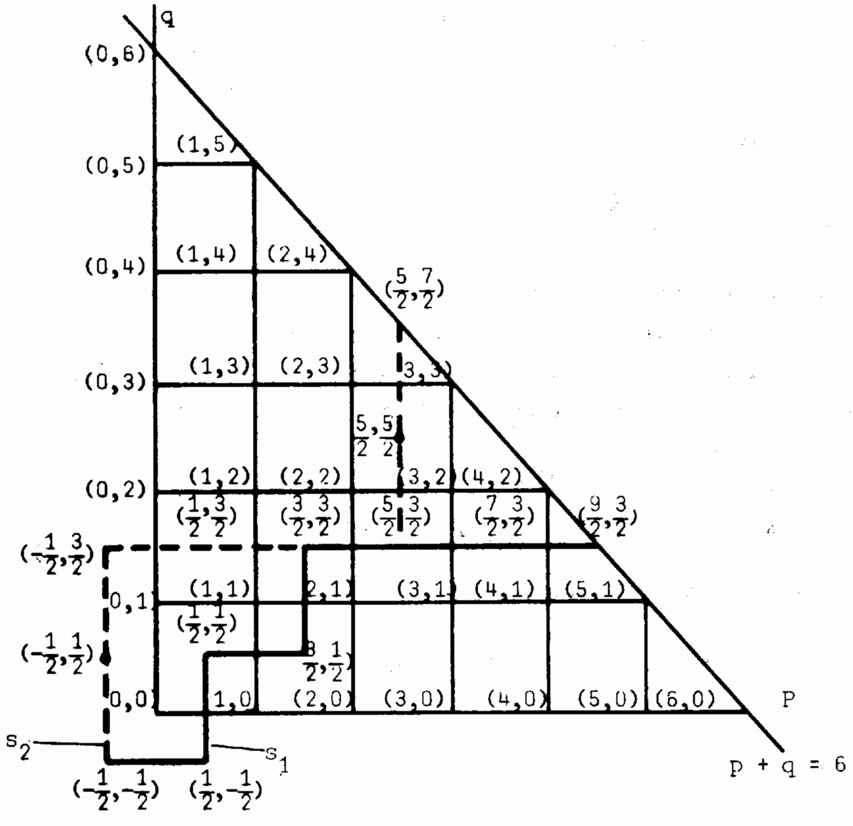

Figure 1.

Any function $F' : L_{n+1}^{(2)} \to E_3$ is completely determined by three sets

$$T_i = \{(p,q) \mid (p,q) \in L_{n+1}^{(2)},\ F'(p,q) = i\}, \quad \text{for } i=0,1,2.$$

We also consider the lattice of all points $(p-\frac{1}{2}, q-\frac{1}{2})$, such that $p \geq 0$, $q \geq 0$, $p+q \leq n+1$. An increasing path from $(-\frac{1}{2}, -\frac{1}{2})$ is a set of edges of this lattice which at each point increases in p or increases in q.

All increasing paths of $n+1$ edges which begin at $(-\frac{1}{2}, -\frac{1}{2})$ must end somewhere on the line $p+q=n$ (indicated in Figure 1 for $n=6$). Label each edge of such a path by 0 if it increases in p and by 1 if it increases in q. So, there is a bijection between the set of all increasing paths of $n+1$ edges which begin at $(-\frac{1}{2}, -\frac{1}{2})$ and the set A^{n+1}. In Figure 1, two such paths s_1 and s_2, for $n=6$, are drawn and corresponding strings are 0101000 and 1100011, respectively.

THEOREM 3. *There are* $\binom{2n+3}{n+1}$ *symmetric monotone functions of* n *variables over the three-valued logic algebra.*

P r o o f. It is sufficient to determine the number of all monotone functions $F': L^{(2)}_{n+1} \to E_3$. However, the sets T_0, T_1 and T_2 for such a monotone function are separated by two increasing paths s_1 and s_2 of $n+1$ edges beginning at $(-\frac{1}{2}, -\frac{1}{2})$ and such that none of the points of s_2 are below s_1. On the other hand, such two paths always determine a monotone function $F': L^{(2)}_{n+1} \to E_3$ by specifying corresponding sets T_0, T_1 and T_2.

So, there is a bijection between the set of all symmetric monotone functions $F: E_3^n \to E_3$ and the set of all pairs of strings $a_1a_2...a_{n+1}$, $a'_1a'_2...a'_{n+1} \in A^{n+1}$ such that $\ell_1(a_1a_2...a_k) \leq \ell_1(a'_1a'_2...a'_k)$ for each $k=1,2,...,n+1$. But, the number of such pairs of strings equals the number $|H_B(n+1)|$ of strings $b_1b_2...b_{n+1} \in B^{n+1}$ such that $\ell_1(b_1b_2...b_k) \leq \ell_2(b_1b_2...b_k)$ for each $k=1,2,...,n+1$; a corresponding bijection can be constructed by taking $b_i = 2a'_i + a_i$. From Theorem 2 it follows that this number is $\binom{2n+3}{n+1}$.

REFERENCES

|1| Ежов, И.И., Скороход, А.В., Ядренко, М.И., Элементы комбинаторики, Наука, Москва, 1977.

|2| Stojmenović, I., Tošić, R., *On the number of symmetric monotone functions over the three-valued logic algebra, Zbornik radova PMF 13(1983).*

|3| Яблонский, С.В., Функциональные построения в к-значной логике, Труды МИ АН СССР, т. 51, Москва, 1958, 5-142.

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REZIME

JEDAN KOMBINATORNI IDENTITET I NJEGOVE PRIMENE

U radu se dokazuje jedan identitet (formula (1)) koji ima interesantne kombinatorne interpretacije. Uspostavljenja je bijekcija izmedju jednog skupa reči nad četvoroelementnom azbukom i skupa svih simetričnih monotonih funkcija od n promenljivih troznačne logike. Kao posledica dobijena je formula za broj tih funkcija - dokazano je da ih ima $\binom{2n+3}{n+1}$.