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A SEQUENTIAL WEIGHING PROCEDURE

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ABSTRACT

The paper is concerned with a sequential search on a finite set $X = F_1 \cup F_2 \cup \ldots \cup F_n$ of coins, such that $|F_i| = 5$; $i=1,2,\ldots,n$; and $F_i \cap F_j = \emptyset$, for $i \neq j$. There is exactly one counterfeit (heavier) coin in each F_i and we want to identify all of them. A weighing procedure is constructed for an arbitrary n, which implies an upper bound close enough to the theoretical lower bound.

Let $X = F_1 \cup F_2 \cup \ldots \cup F_n$ ($F_i \cap F_j = \emptyset$ if $i \neq j$) be a set of 5n coins, where $F_i = \{c_1^i, c_2^i, c_3^i, c_4^i, c_5^i\}$ is a set containing 5 coins ($i=1,2,\ldots,n$). All coins are indistinguishable except that exactly n of them are slightly heavier than the rest and each of the sets F_i contains exactly one heavier coin. Given a balance scale, we want to find an optimal weighing procedure i.e. a procedure which minimizes the maximum number of steps (weighings) which are required to identify all the heavier coins.

We suppose that all the heavier coins are of equal weight, and so are all the light coins. If w is the weight of a light coin, then the weight of a heavy coin is less than $\frac{n+1}{n}$ w, so that the larger of the two numerically unequal subsets of X is always the heavier. This means that no information is gained by balancing two numerically unequal sets. We also suppose that the scale reveals which, if either, of the subsets of X is heavier but not by how much.

Some other problems of this type are discussed in greater detail in |1|,|2|,|3|,|4|,|5| and |6|.

Consider a pair (A,B) of numerically equal disjoint subsets of X. Step (A,B) will mean the balancing of A against B. If $A=\{a\}$ and $B=\{b\}$, we write simply (a,b). The possible outcomes are:

(a) The sets balance, symbolized by $A=B$;

(b) The sets do not balance, in which case we use the notation $A>B$, $B>A$, where $>$ between two sets means "is heavier than".

We denote with $\ell_5(n)$ the maximum number of weighings in an optimal weighing procedure for the set X. By information-theoretical arguments, we have the following lower bound for $\ell_5(n)$:

(1) $$\ell_5(n) \geq \lceil n \log_3 5 \rceil$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$.

Now, we are going to construct a weighing procedure S_n for an arbitrary n, which implies an upper bound close enough to the theoretical lower bound, i.e. we are going to prove the following statement:

THEOREM. $\ell_5(n) \leq \left\lceil \frac{3n}{2} \right\rceil$ (2)

P r o o f. We shall construct a weighing procedure S_n for the set X, with the maximum number of weighings $\left\lceil \frac{3n}{2} \right\rceil$. This procedure will be sequential i.e. the next pair of subsets to be compared is dependent on the answers to previous weighings.

For $n=1$, the procedure S_1 consists of two independent weighings:

1. (c_1^1, c_2^1) 2. (c_3^1, c_4^1)

It can be easily checked that the heavy coin is determined by the answers to the weighings 1. and 2. according to the given TABLE.

TABLE

Answers to the weighings 1.	2.	Heavy coin
<	<	outcomes impossible
<	=	c_2^1
<	>	outcomes impossible
=	<	c_4^1
=	=	c_5^1
=	>	c_1^1
>	<	outcomes impossible
>	=	c_3^1
>	>	outcomes impossible

For $n=2$, we construct a sequential weighing procedure S_2 as follows:

STEP 1. (A,B), where $A=\{a_1^1,a_2^1,a_3^1\}$, $B=\{a_4^1,a_1^2,a_2^2\}$.

(i) If $A>B$, this means that each of the sets A and $B'=\{a_3^2,a_4^2,a_5^2\}$ contains exactly one heavy coin and to identify them we can use two independent weighings, (a_1^1,a_2^1) and (a_3^2,a_4^2), respectively.

(ii) If $A=B$, we go to Step 2.

STEP 2. (a_1^2,a_2^2), and now

(1) If $a_1^2>a_2^2$, the heavy coins are a_1^2 and one coin from the set A, which can be determined by one weighing e.g. (a_1^1,a_2^1);

(2) If $a_1^2=a_2^2$, the heavy coins are a_5^1 and one coin from the set B', which can be determined by one weighing e.g. (a_3^2, a_4^2);

(3) If $a_1^2 < a_2^2$, the heavy coins are a_2^2 and one coin from the set A, and we continue as in case (1).

(iii) If $A < B$, we also go to Step 2, and

(1) If $a_1^2 > a_2^2$, the heavy coins are a_1^2 and one of the coins a_4^1 and a_5^1, which can be determined by the balancing of a_4^1 against a_5^1 ;

(2) If $a_1^2 = a_2^2$, the heavy coins are a_4^1 and one coin from the set B′, and we continue as in case (2) for $A = B$;

(3) If $a_1^2 < a_2^2$, the heavy coins are a_2^2 and one of the coins a_4^1 and a_5^1, and we continue as in case (1).

It follows from (1), that the constructed procedures S_1 and S_2 are optimal.

Let $n > 2$. Now, if n is even, we partition the set X into the subsets $X_i = F_{2i-1} \cup F_{2i}$; $i=1,2,\ldots,\lfloor \frac{n}{2} \rfloor$. If n is odd, we partition the set X into the sets $X_i = F_{2i-1} \cup F_{2i}$; $i=1,2,\ldots$ $\ldots,\lfloor \frac{n}{2} \rfloor$, and the set $X_{\lfloor \frac{n}{2} \rfloor +1} = F_n$. Here, $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. We sort each subset X_i $(i=1,2,\ldots,\lfloor \frac{n}{2} \rfloor)$ by applying the optimal procedure S_2 and $X_{\lfloor \frac{n}{2} \rfloor +1}$ by applying the procedure S_1. So, a procedure S_n, for which the maximum number of weighings is $\lceil \frac{3n}{2} \rceil$, is constructed. Hence follows (2).

The constructed procedure S_n is optimal at least for those n′s for which the upper bound $\lceil \frac{3n}{2} \rceil$ coincides with the information - theoretical lower bound. So, as a consequence of the Theorem, we have:

COROLLARY. *The procedure* S_n *is optimal at least for* $n = 1,2,3,4,5,6,7,8,9,10,11,12,13,14,16,18,20,22,24,26,28$.

We think that the procedure S_n is optimal for many other n′s too, but we do not known how to prove it.

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REZIME

JEDNA SEKVENCIJALNA PROCEDURA
MERENJA

U radu je konstruisana jedna sekvencijalna procedura za identifikaciju neispravnih (težih) elemenata u skupu $X = F_1 \cup F_2 \cup \dots \cup F_n$; $|F_i| = 5$, $i=1,2,\dots,n$, i $F_i \cap F_j = \emptyset$ za $i \neq j$, gde je n proizvoljan ceo pozitivan broj. Na taj način dobijena je jedna gornja granica za maksimalan broj koraka optimalne procedure, koja je vrlo bliska teorijskoj donjoj granici.