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ANOTHER CONSTRUCTION OF RANK 4 PAVING MATROIDS ON 8 ELEMENTS (II)

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ABSTRACT

In this paper, which is a sequel to |1|, we give the construction of all non-isomorphic C-families. This completes the construction of all non-isomorphic rank 4 paving matroids on 8 elements.

PRELIMINARIES

The reader should primarily read the preliminaries and the introduction to paper |1|; these will be used without reference.

A C-family is a family F of distinct 4-subsets of $S = \{1,2,3,4,5,6,7,8\}$ satisfying:

(a) $(X_1, X_2 \in F \wedge X_1 \neq X_2) \Rightarrow |X_1 \cap X_2| \leq 2$

(b) $(\exists X_1)\ (\exists X_2)\ (X_1 \in F \wedge X_2 \in F \wedge |X_1 \cap X_2| = 1)$

A D-family is a C-family F in which the condition **(b)** can be replaced by the stronger

(c) $(\forall X_1)\ (X_1 \in F \Rightarrow (\exists X_2)\ (X_2 \in F \wedge |X_1 \cap X_2| = 1))$

We shall give the construction of all 184 non-isomorphic C-families. We shall primarily construct 36 non-isomoprhic D-families, after which the construction of the remaining non-isomorphic C-families is considerably simplified.

FURTHER DEFINITIONS

A C-family F arises from a D-family F´ if F´ is the (unique) maximal D-subfamily of F.

A 4-set X is permitted for a C-family F, which does not contain X, if $F \cup \{X\}$ is a C-family.

A 4-set X is 2-permitted for a D-family F, which does not contain X, if X is permitted for F, but $F \cup \{X\}$ is NOT a D-family.

A D-graph of a D-family F is a graph G such that

(i) the vertices of G are 4-sets of F

(ii) there is an edge {X,Y} in G if and only if $|X \cap Y| = 1$

We define a binary relation, denoted by "→", on the set of edges of a D-graph G, as follows:

If $a = \{A_1, A_2\}$ and $b = \{B_1, B_2\}$ are two edges of G, then

$$a \to b \overset{def}{\Longleftrightarrow} A_1 \cap A_2 = S \setminus (B_1 \cup B_2)$$

A marked D-graph is a graph G with a given binary relation ϕ on its edges, such that there exists a D-family with the D-graph G_1, which satisfies the following condition:

"There is an isomorphism of G onto G_1, which maps ϕ to →". The relation ϕ determines a markation of G.

Let F be a C-family, which satisfies the following condition: "The D-graph of the (unique) maximal D-subfamily of F is a star". Let the central vertex of the star (if the star is just an edge, then an arbitrary vertex) be the 4-set denoted as {a,b,c,d}.

We define the C-graph G of F as follows:

(i) the vertices of G are a,b,c,d

(ii) there is an edge {x,y} in G if and only if there is a 4-set in F, which contains {x,y} and which is different from {a,b,c,d}.

We define a 2-colouring of the defined C-graph G as follows: A vertex x of G is black if and only if there exists a 4-set X in F such that $X \cap \{a,b,c,d\} = \{x\}$. The vertex x is white otherwise.

A coloured C-graph is a graph G on four vertices, each of which is coloured either white or black, satisfying the condition that there exists at least one black vertex. The colours of the vertices determine a colouring of G.

In is easy to prove that the coloured C-graphs are actually C-graphs of some C-families.

CONSTRUCTION OF NON-ISOMORPHIC D-FAMILIES

We begin with a number of easy lemmas describing some properties of D-graphs and marked D-graphs. Some of them will be used in our construction of D-families. They may offer a better insight into the D-graphs which we construct and may be used for another approach to the construction as well. These lemmas are denoted by apostrophes, for they do not relate to any particular theorem by this time.

LEMMA 1´. *Each element of S corresponds to at most one edge of a D-graph, in the sense that it is the intersection of the two end-vertices of that edge.*

REMARK: In the figures of D-graphs the edges will be denoted by the corresponding intersection elements.

P r o o f. Suppose that the same element of S corresponds to two edges of a D-graph. It is easy to check that at least 9 different elements are needed for their vertices, regardless of whether these two edges have a common vertex or not.

LEMMA 2´. *A D-graph cannot have more than 8 edges.*

P r o o f. Immediate consequence of Lemma 1´.

LEMMA 3´. *The maximal degree of a vertex of a D-graph is 4.*

P r o o f. Immediate consequence of Lemma 1´.

LEMMA 4´. *Each element of S corresponds to at most one edge of a D-graph, in the sense that it is the complement (with respect to S) of the union of the two end-vertices of that edge.*

P r o o f. Seven elements are not sufficient for the end-vertices of two edges in any case.

LEMMA 5′. *For each edge x of a D-graph there exists at most one edge y such that $x \to y$ and at most one edge z such that $z \to x$.*

P r o o f. The two statements follow immediately from Lemmas 4 and 1 respectively.

LEMMA 6′. *If $A_1A_2A_3A_4$ is a 3-path of a D-graph, then it holds that at least one of $A_1A_2 \to A_3A_4$ and $A_3A_4 \to A_1A_2$ is true.*

P r o o f. Suppose that $A_1A_2 \to A_3A_4$ is not true, that is, $S \setminus (A_3 \cup A_4) \neq A_1 \cap A_2$. $A_1 \cap A_2 \not\subset A_3$ by Lemma 1′ and so we have $A_1 \cap A_2 \subset A_4$. This implies that $S \setminus (A_1 \cup A_2) \subset A_4$ (otherwise A_4 would have a 3-intersection with some of A_1, A_2). Since $|A_3 \cap A_2| = 1$ and $|A_3 \cap A_1| \leq 2$, we have also $S \setminus (A_1 \cup A_2) \subset A_3$, which gives $A_3A_4 \to A_1A_2$

LEMMA 7′. *No two vertices of a D-graph may have the empty intersection.*

P r o o f. Otherwise one of them would have a 3-intersection with any vertex incident to the other.

LEMMA 8′. *If K and L are two incident vertices of a D-graph and $K \cap L = \{p\}$, $S \setminus (K \cup L) = \{q\}$, then $L = \{p\} + ((S \setminus K) \cup \{q\})$.*

P r o o f. Trivial.

LEMMA 9′. *If $x = A_1A_2$ and $y = B_1B_2$ are two edges of a D-graph such that $x \to y$, then there exists an edge adjacent to both x and y.*

P r o o f. $B_1 \cap B_2$ does not belong to both A_1 and A_2 by Lemma 1′. Suppose that $B_1 \cap B_2 \not\subset A_1$ (similarly if $B_1 \cap B_2 \not\subset A_2$). $B_1 \cup B_2 \not\supset A_1 \cap A_2$ by the assumption. Lemma 7′ combined with $|A_1 \setminus A_2| = 3$ gives that either $|A_1 \cap B_1| = 1$ or $|A_1 \cap B_2| = 1$, i.e., there exists one of the edges A_1B_1 and A_1B_2.

LEMMA 10′. *If a D-graph contains the edges A_1A_2 and B_1B_2, such that $A_1A_2 \to B_1B_2$, and does not contain any of the edges A_1B_1 and A_1B_2, then $A_1 \supset B_1 \cap B_2$.*

P r o o f. Immediately follows from the previous proof.

LEMMA 11′. *There are no odd cycles in any D-graph.*

P r o o f. We have to exclude just the 3-, 5- and 7-cycles, by Lemma 2´. 3-cycles: A 3-cycle $A_1A_2A_3$ of a D-graph yields

$$A_3\cap(A_1\cap A_2) = \emptyset \text{ and } |A_3\cap(A_1\setminus A_2)| = |A_3\cap(A_2\setminus A_1)| = 1,$$

which implies $|A_3|\leqslant 3$, a contradiction.

5-cycles: Suppose that there exists a 5-cycle $A_1A_2A_3A_4A_5$ (with edges A_iA_{i+1}, $1\leqslant i\leqslant 5$, $A_{5+1} = A_1$) in a D-graph. We may assume by Lemmas 6´and 5´, without a loss of generality, that $A_1A_2\to A_3A_4$ and $A_4A_5\to A_1A_2$. The first assumption gives $A_4\not\supset A_1\cap A_2$, while the second, combined with Lemma 10´ and the non-existence of edges A_1A_4 and A_2A_4 gives $A_4\supset A_1\cap A_2$, a contradiction.

7-cycles: Let $A_1A_2A_3A_4A_5A_6A_7$ (edges A_iA_{i+1}, $1\leqslant i\leqslant 7$, $A_{7+1} = A_1$) be a 7-cycle in a D-graph. Similarly as in the previous case, assume that $A_1A_2\to A_3A_4$ and $A_6A_7\to A_1A_2$. Then $A_4\not\supset A_1\cap A_2$, while Lemma 10´ gives $A_6\supset A_1\cap A_2$. Now $A_5\supset A_1\cap A_2$ contradicts Lemma 1´, while $A_5\not\supset A_1\cap A_2$ contradicts Lemma 4´.

LEMMA 12´. *If $A_1A_2A_3A_4$ is a 3-path of a D-graph G and both $A_1A_2\to A_3A_4$ and $A_3A_4\to A_1A_2$ are satisfied, then G contains the edge A_1A_4 and both $A_1A_4\to A_2A_3$ and $A_2A_3\to A_1A_4$ also hold.*

P r o o f. We have $a = A_1\cap A_2 = S\setminus(A_3\cup A_4)$ and $b=A_3\cap A_4 = S\setminus(A_1\cup A_2)$ by the assumption. Lemmas 7´ and 11´ provide that $|A_1\cap A_3| = |A_2\cap A_4| = 2$. As $a\notin A_3\cup A_4$ and $b\notin A_1\cup A_2$, we have that none of the sets $A_1\cap A_3$, $A_2\cap A_4$ contains a or b. The assumption combined with Lemma 1´ gives that the sets $A_1\cap A_3$, $A_2\cap A_4$, $A_2\cap A_3$ and $A_1\cap A_4$ are mutually disjoint. It follows that

$A_1\cap A_4 = S\setminus(\{a,b\} + (A_1\cap A_3) + (A_2\cap A_4) + (A_2\cap A_3))$, which gives $|A_1\cap A_4| = 1$. Since the sum in the brackets coincides with $A_2\cup A_3$, we have $A_1A_4\to A_2A_3$. The relation $A_2A_3\to A_1A_4$ follows from $A_1\cup A_4 = (A_1\cap A_4) + \{a,b\} + (A_1\cap A_3) + (A_2\cap A_4)$

Our construction of non-isomorphic D-families will be somewhat similar to that of B-families(|1|). We shall start with examples of "small" D-families and in each case look for the non-isomorphic possibilities for the addition of the permitted, but not 2-permitted 4-sets (that is, those 4-sets, the addition of which gives a new D-family). We must, however, also take care about 2-permitted 4-sets for a D-family, because they may be vertices of a new connected component of an augmented D-graph. These 4-sets necessarily appear in the complementary pairs, which are denoted by a short line between their two 4-sets.

The main novelty in the construction of non-isomorphic D-families will be in that we shall mostly look for non-isomorphic possibilities for augmentation of the corresponding marked D-graphs, instead of the D-families themselves. Such an approach will be justified by the later Theorem 2.

The constructed non-isomorphic D-families will be given by means of their marked D-graphs in the table which follows the construction. The types of marked D-graphs given in the table will be denoted by the numbers in brackets. (We shall mention all these numbers during the construction). The same denotation will be often used for the types of non-isomorphic D-families, which correspond to the examples of the marked D-graphs given in the table.

CONSTRUCTION OF ALL THE NON-ISOMOPRHIC MARKED D-GRAPHS AND ONE MORE

We start with an example of the simplest D-graph,

$$1234 \underset{1}{\bullet\!\!-\!\!-\!\!-\!\!\bullet} 1567$$

We may assume, without any loss of generality, that each example (=representative), of the non-isomorphic marked D-graphs that we construct, contains this edge.

There are (with respect to this edge) 36 permitted 4-sets: 9 contain {1,8},9 are contained in {2,3,4,5,6,7},9 have 1-intersections with {1,2,3,4} and 9 with {1,5,6,7}.

REMARK: An alternative proof that D-graphs have not 3-cycles follows from the fact that none of the permitted 4-sets has 1-intersections with both of 1234 and 1567.

Adding any of the last 18 permitted 4-sets we obtain a D-graph of type (2). Adding any two 4-sets, such that they have exactly one element in common, we obtain a D-graph of type (3).

The following theorem refers to the remaining (types of) D-graphs:

THEOREM 1. *Each D-graph with at least three edges belongs to exactly one of the following four classes:*

a) graphs with at least one 4-cycle

b) graphs with at least one vertex of degree ≥ 3

c) graphs with at least one 3-path , which do not belong to any of the classes a) and b)

d) disconnected graphs without 3-paths

P r o o f. Will be implicit in what follows.

We give separate constructions for each of these four classes of D-graphs: (all the here mentioned graphs are D-graphs, even if it is not explicitly stated)

Class a) A 4-cycle contains a subgraph of type (2). Starting with the family F_o = {1234,1567,2568}, it is easy to check that the only 4-set, which gives rise to a 4-cycle (type (4)),is 3478. The only permitted 4-sets for the family F_1 = {1234, 1567,2568,3478} are 1278,1358,1368,1458,1468 and their complements. None of them has a 1-intersection with any set of F_1,i.e., a 4-cycle in a D-graph is always a connected component. However, some pairs of the permitted 4-sets have 1-intersections. We choose 1358,2457 and so obtain (a D-graph of) type (5) with the permitted 4-sets just 1278-3456,1468, 2367. Adding further one (respectively both) of 1468,2367 we obtain type (6) (respectively type (7)).

Class b) Adding any of the 4-sets 3578,3678,4578,4678 to the family F_o = {1234,1567,2568}, we obtain a star (type (8)), with the centre 1234. Suppose that 3578 is chosen. The permitted 4-sets of the family F = {1234,1567,2568,3578} are 4678,1468,1478,2467,2478,3467,3468,1278-3456,1368-2457,1458-2367.
As there are no 1-intersections among the last six permitted (= 2-permitted) 4-sets, we conclude that all the graphs in class b) are connected.

The only possibility to obtain a star with four legs (type (9)) is to add 4678 to F. It is a maximal D-graph, for none of its permitted 4-sets (the last six of the list above) has 1-intersections with any of 1234,1567,2568,3578,4678.

Adding any of 1468,1478,2467,2478,3467,3468 to F, we obtain a D-graph of type (10). If 1468 is chosen, then the remaining permitted 4-sets are 2367,2457,2467,2478,3467,1278-3456. If we add 2467 to the family G = {1234,1567,2568,3578, 1468}, then we obtain a D-graph of type (11), which is easily seen to be a maximal one. If we add 2478 or 3467 to G, then we obtain two isomorphic (marked) D-graphs of type (12). However, if we add 2367, respectively 2457, to G, then we obtain two isomoprhic graphs, which have a different markation (types (13) and (14)).

In the next step we explore the possibilities for the augmentation of D-graphs corresponding to the families $G_1 = G \cup \{2478\}$, $G_2 = G \cup \{2367\}$, $G_3 = G \cup \{2457\}$, with the families of the permitted 4-sets {2367,3456,3467}, {2457,2478,1278-3456} and {2367,3467,1278-3456} respectively.

Adding 3467 to G_1 we obtain a maximal D-graph of type (15). Adding 2457 to G_2 (identically, 2367 to G_3), we obtain a maximal D-graph of type (16). Adding 2367 to G_1 or 2478 to G_2, we obtain two isomorphic marked D-graphs of type (17). Adding 3456 to G_1 or 3467 to G_3, we obtain two isomorphic marked D-graphs of type (18) (isomorphic to the graphs of type (17), but with a different markation). Finally, adding (the only possible) 3456 to $G_1 \cup \{2367\}$ (identically, 2367 to $G_1 \cup \{3456\}$), we obtain a maximal D-graph of type (19).

Class c) We start for the third time from the family $\overline{F_o}$ = {1234,1567,2568}. This time we add some of the remaining permitted 4-sets having 1-intersections with 1567 or 2568, i.e., some of 1378,1478,2378,2478,3457,3458,3467,3468 and obtain a D-graph of type (20). If we choose the family H = {1234,1567,2568,1378}, then the permitted 4-sets are 2457,2467,2478,3456,3457,3458,3467,3468,4578,4678,1458-2367, 1468-2357. The addition of any of 3457,3467,4578,4678 to H gives rise to a D-graph of type (10). The addition of 2478, respectively 3456, (to H) yields marked D-graphs of types (21), respectively (22), while the addition of any of 2457,2467,3458, 3468 gives a 4-path marked in a third style (type (23)). We have here also one (up to an isomoprhism) possibility to obtain an edge of another component of a D-graph, by the addition of both of, for example, 1458,2357 to H. Thus we obtain the type (24).

We denote the families H∪{2478}, H∪{3456}, H∪{2457} by H_1,H_2,H_3 respectively. Their families of permitted 4-sets are

{3456,3457,3458,3467,3468,1458-2367,1468-2357},
{2457,2467,2478,4578,4678,1458-2367,1468-2357} and
{1468,3456,3458,3467,3468,4678,1458-2367} respectively.

We omit all those possibilities for the augmentation of the D-graphs of types (21)-(23), in which a vertex of degree 3 is produced.

If we add 3456 to H_1 (identically, 2478 to H_2), then we obtain the marked D-graph of type (25). Note, however, that we can here also obtain three types of disconnected marked D-graphs (types (26), (27) and (28)) by adding, for example, 1458 and 2357 to H_1,H_2 and H_1∪{3456} respectively. If we add 3468 to H_3, then we obtain a maximal marked D-graph of type (29), which differs only by markation from the D-graph of type (25). If we add 1468 or 3458 to H_3, then we obtain a D-graph of type (30). The only permitted 4-sets for the family H_3∪{1468} are 2367,3458,3456,3467. We neglect the last two, because they yield a vertex of degree 3. If any of 2367,3458 is added to H_3∪{1468}, then we obtain a D-graph of type (31); if both are added, then we have a maximal D-graph of type (32).

Class d). If we start again with F_o, then we should add only some of those permitted 4-sets, which have no 1-intersections with any of the first three, that is, some of

1278-3456, 1358-2467, 1368-2457, 1458-2367, 1468-2357

Note that 1278 and 3456 have not 1-intersections with any of these 4-sets, so they cannot be the vertices of a

D-graph. Adding any of {1358,2367}, {1368,2357}, {1458,2467}, {1468,2457}, respectively any of {1358,2457},{1368,2467}, {1458,2357}, {1468,2367}, to the family F_o, we obtain D-graphs of types (33), respectively (34). The D-graphs of these two types are isomorphic and without markation, but observe the following difference between them: The intersection element, which corresponds to the one-edge component, is contained in the vertex of degree 2 with the type (33), which is not the case with the type (34).

The only permitted 4-sets, apart from 1278-3456, for $F_o \cup$ {1358,2367}, respectively $F_o \cup${1358,2457}, which do not augment the component of F_o, are 1468,2457, respectively 1468,2367. By adding any, but just one, of these 4-sets, we obtain a D-graph of type (35), while the addition of both permitted 4-sets in both cases yields a 4-cycle (type (6)).

The only remaining case is when each component of a D-graph has just one edge. We start with the D-family J = {1234,1567,1258,2367} (corresponding to the D-graph of type (3)). The only permitted 4-sets, which do not give rise to a component with (at least) two edges, are 1368-2457 and 1378-2456. We obtain a D-graph of type (36) by the adding of any of {1368,2456}, {1378,2457} to J.

We give (on the next two pages) the table of (examples of) D-graphs corresponding to non-isomorphic D-families. This last notion coincides with "non-isomorphic marked D-graphs",*) except for the types (33) and (34), where two isomorphic (marked) D-graphs correspond to non-isomorphic D-families.

Apart from the explained designations, we give, beside the number of the type, for each D-graph of the table, the number of non-isomorphic C-families having the maximal D-subfamily of the corresponding type. The production of these numbers will be explained in the last section.

The markation (the edges related by →) will be denoted only in the cases when it is necessary for distinguishing two isomorphic graphs corresponding to two non-isomorphic D-families.

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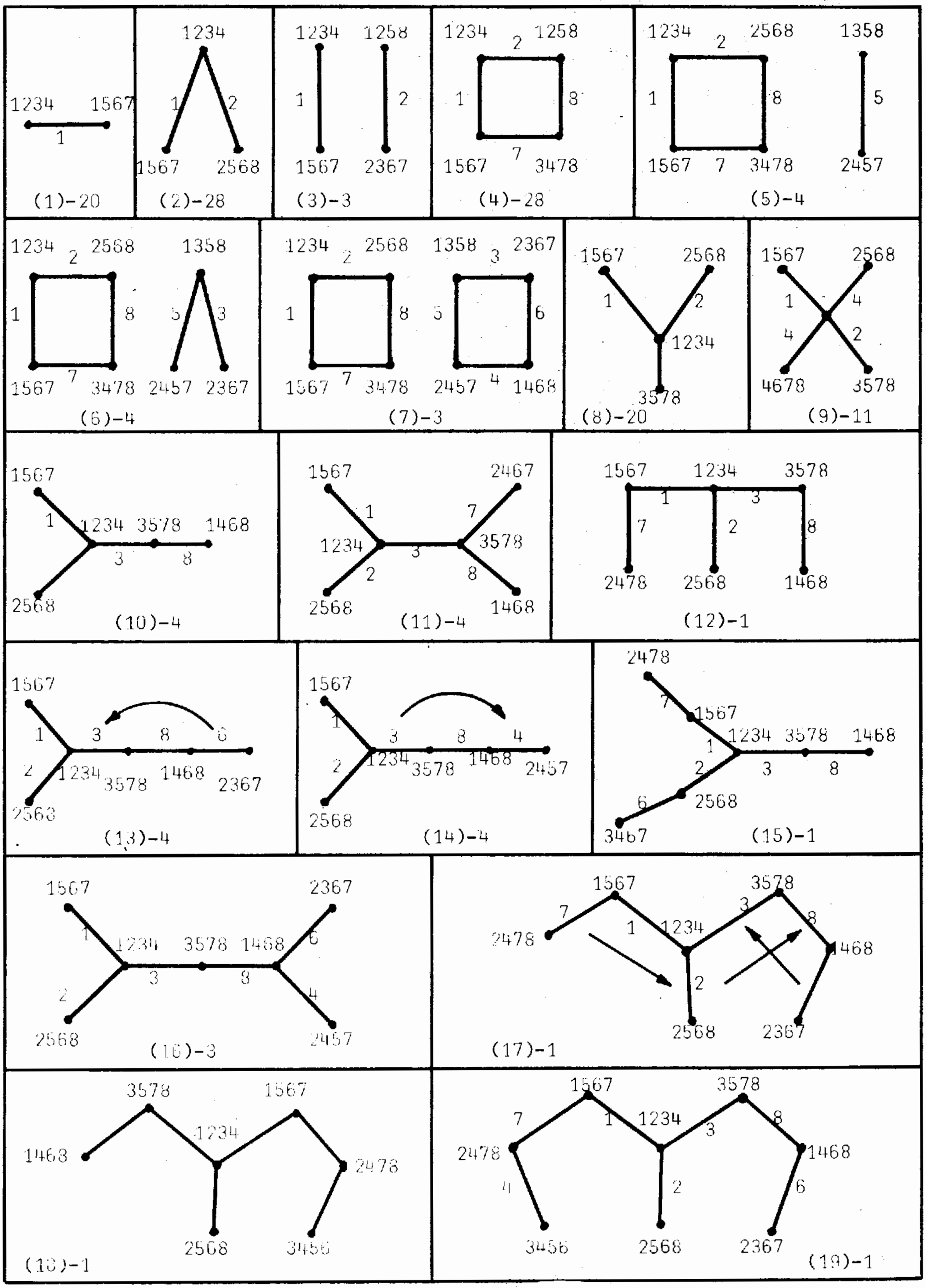
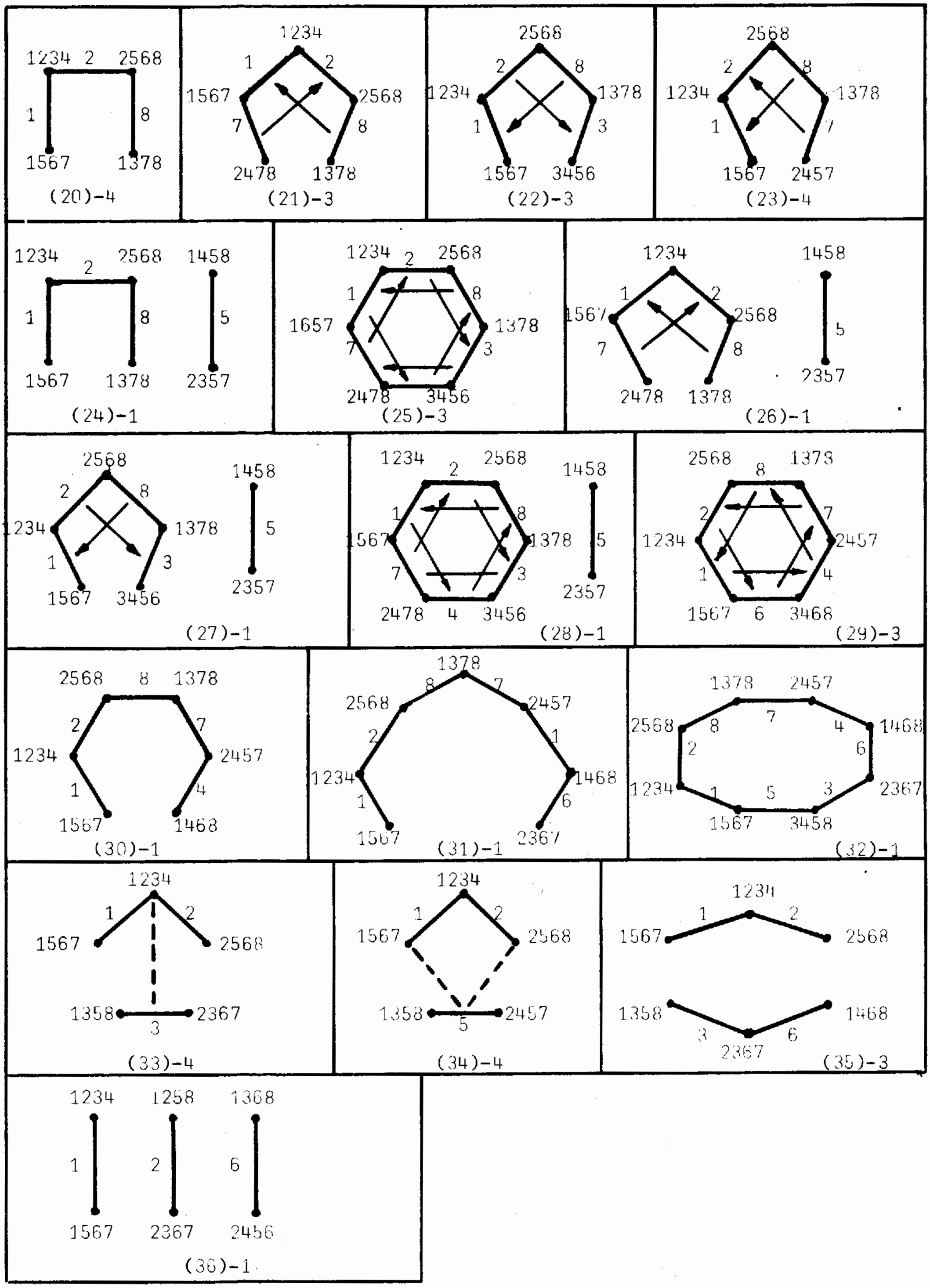
*) Two marked D-graphs G_1 and G_2 are isomorphic if there is a graph-isomorphism which maps G_1 onto G_2 and preserves the markation.

THE TABLE OF (EXAMPLES OF) D-GRAPHS CORRESPONDING TO NON-ISOMORPHIC D-FAMILIES



THE TABLE OF D-GRAPHS (CONTINUED)



It might be interesting to mention that the unique (up to an isomorphism) matroid with the corresponding D-graph of type (32) (8-cycle) is, in fact, Piff's cyclic (= with a cyclic automorphism group) matroid on 8 elements (|3|,p. 330). It is quite natural that such a matroid has the marked D-graph with the cyclic automorphism group C_8.

We observe that five pairs of non-isomoprhic D-families, which have isomoprhic D-graphs, give isomorphic D-families in the next step of augmentation. Thus (13) and (14) give (16), (17) and (18) give (19), (22) and (23) give (25) (although they could also give non-isomorphic (26) and (27)), (26) and (27) give (28), and (33) and (34) give (35). This process looks like some kind of symmetrization.

It is obvious that two isomorphic D-families must have isomorphic marked D-graphs (family isomorphisms preserve 1-intersections). However, we shall prove that the converse is also true, with one single exception. This justifies our construction of all non-isomorphic D-families by the exhaustion of possibilities for non-isomorphic marked D-graphs, because this does not allow some new non-isomorphic D-families to appear, in addition to the constructed.

THEOREM 2. *Any two D-families (having the marked D-graphs) of the same type ((1)-(36)) are isomorphic.*

This theorem may be reformulated as follows:

THEOREM 2'. *If two marked D-graphs are isomorphic and have not exactly three edges in two connected components each, then their corresponding D-families are also isomorphic. In other words, the types (33) and (34) are the only exception to the rule that non-isomorphic marked D-graphs correspond exactly to non-isomorphic D-families. Each of the types (33) and (34) of D-graphs uniquely (up to an isomorphism) determines the corresponding D-family.*

LEMMA 1. *If two marked D-graphs G_1 and G_2 are isomorphic and G_1 contains a 3-path, which is not included in a 4-cycle, then their corresponding D-families (F_1 and F_2 respectively)* *are also isomorphic.*

P r o o f of Lemma 1. Let $\gamma: G_1 \to G_2$ be the given isomorphism and let $X_1Y_1Z_1T_1$ be a 3-path in G_1, such that X_1T_1 is not an edge of G_1. The image of $X_1Y_1Z_1T_1$ under γ (in G_2) is denoted by $X_2Y_2Z_2T_2$. Lemmas 6´ and 12´ give that either $X_1Y_1 \to Z_1T_1$ or $Z_1T_1 \to X_1Y_1$. We suppose the first, the other case is treated quite similarly. We have $X_2Y_2 \to Z_2T_2$ and denote for $i \in \{1,2\}$:

$X_i \cap Y_i = a_i$, $Y_i \cap Z_i = b_i$, $Z_i \cap T_i = c_i$, $S \setminus (X_i \cup Y_i) = d_i$,
$S \setminus (Y_i \cup Z_i) = e_i$, $(X_i \cap Z_i) \setminus T_i = f_i$, $Y_i \cap T_i = \{g_i, h_i\}$

We claim that at least one of the permutations

$$\alpha = \binom{a_1b_1c_1d_1e_1f_1g_1h_1}{a_2b_2c_2d_2e_2f_2g_2h_2} \qquad \beta = \binom{a_1b_1c_1d_1e_1f_1g_1h_1}{a_2b_2c_2d_2e_2f_2h_2g_2}$$

induces an isomorphism of F_1 onto F_2, which would complete the proof.

For $i \in \{1,2\}$ we have:

$X_i = \{a_i, c_i, e_i, f_i\}$; $Y_i = \{a_i, b_i, g_i, h_i\}$; $Z_i = \{b_i, c_i, d_i, f_i,\}$;
$T_i = \{c_i, e_i, g_i, h_i\}$

Namely, X_i obviously contains a_i and f_i, X_i contains e_i by Lemma 4´, and contains c_i, for we have not $Z_iT_i \to X_iY_i$. The other three pairs of sets are easily deduced by applying Lemma 8´.

Thus both α and β map the sets X_1, Y_1, Z_1, T_1 to the sets X_2, Y_2, Z_2, T_2 respectively.

The only 4-sets, which are permitted for X_i, Y_i, Z_i, T_i,, are ($i=1,2$):

$A_i^1 = \{a_i, c_i, d_i, g_i\}$; $A_i^3 = \{b_i, e_i, f_i, g_i\}$; $A_i^5 = \{a_i, b_i, d_i, e_i,\}$;
$A_i^6 = \{d_i, f_i, g_i, h_i\}$
$A_i^2 = \{a_i, c_i, d_i, h_i\}$; $A_i^4 = \{b_i, e_i, f_i . h_i\}$
$A_i^7 = \{a_i, d_i, f_i, g_i\}$; $A_i^9 = \{b_i, d_i, e_i, g_i\}$; $A_i^{11} = \{a_i, d_i, e_i, g_i\}$;
$A_i^{13} = \{d_i, e_i, f_i, g_i\}$
$A_i^8 = \{a_i, d_i, f_i, h_i\}$; $A_i^{10} = \{b_i, d_i, e_i, h_i\}$; $A_i^{12} = \{a_i, d_i, e_i, h_i\}$;
$A_i^{14} = \{d_i, e_i, f_i, h_i\}$

Let W_1 belong to $\{A_1^1,\dots,A_1^{14}\}\cap F_1$. We shall primarily prove that either $\alpha(W_1)\in F_2$ or $\beta(W_1)\in F_2$ or both (equivalently, that holds at least one of $\alpha(W_1)=\gamma(W_1)$, $\beta(W_1) = \gamma(W_1)$).

If $W_1\in\{A_1^{13}, A_1^{14}\}$, then there exists the edge Y_1W_1 in G_1, which implies the existence of the edge $Y_2\,\gamma(W_1)$ in G_2 and gives $\gamma(W_1)\in\{A_2^{13}, A_2^{14}\}$, providing that either $\gamma(W_1) = \alpha(W_1)$ or $\gamma(W_1) = \beta(W_1)$. We similarly derive, by keeping the incidence with Z_i, X_i, T_i respectively, that

$$W_1\in\{A_1^{11}, A_1^{12}\}\Rightarrow\gamma(W_1)\in\{A_2^{11}, A_2^{12}\}$$

$$W_1\in\{A_1^6,A_1^9,A_1^{10}\} \Longrightarrow \gamma(W_1)\in\{A_2^6,A_2^9,A_2^{10}\}$$

$$W_1\in\{A_1^5,A_1^7,A_1^8\} \Longrightarrow \gamma(W_1)\in\{A_2^5,A_2^7,A_2^8\}$$

As the inverse argument also holds, we have

$$W_1\in\{A_1^1,A_1^2,A_1^3,A_1^4\} \Longrightarrow \gamma(W_1)\in\{A_2^1,A_2^2,A_2^3,A_2^4\}$$

However, since γ preserves the markation, we cannot have, for example, $\gamma(A_1^6) = A_2^9$, otherwise the subgraph of G_1 with vertices $A_1^6, X_1, Y_1, Z_1, T_1$ would not have the same markation as its image under γ. By a similar argument we derive:

$$W_1 = A_1^5 \Rightarrow \gamma(W_1) = A_2^5 \quad ; \quad W_1 = A_1^6\Rightarrow\gamma(W_1) = A_2^6$$

$$W_1\in\{A_1^7,A_1^8\} \Rightarrow \gamma(W_1)\in\{A_2^7,A_2^8\}$$

$$W_1\in\{A_1^9,A_1^{10}\} \Rightarrow \gamma(W_1)\in\{A_2^9,A_2^{10}\}$$

Thus $\gamma(W_1)$, which is a set if F_2, equals at least one of $\alpha(W_1)$, $\beta(W_1)$ for $W_1\in\{A_1^5,\dots,A_1^{14}\}$.

If the set $W_1\in\{A_1^1,A_1^2,A_1^3,A_1^4\}$, then the vertex W_1 is not incident (in G_1) to any of the vertices X_1,Y_1,Z_1,T_1. Since G_1 does not contain isolated vertices, W_1 is joined to another vertex V_1 of G_1. It is easy to check that ordered pair (W_1,V_1) must be one of the following twelve

$(A_1^1,A_1^3), (A_1^3,A_1^1), (A_1^2,A_1^4), (A_1^4,A_1^2)$ ......, $(A_1^1,A_1^{10}), (A_1^1,A_1^{14})$.

$(A_1^2,A_1^9), (A_1^2,A_1^{13}), (A_1^3,A_1^8), (A_1^3,A_1^{12}), (A_1^4,A_1^7), (A_1^4,A_1^{11})$

while $\gamma(W_1,V_1)$ must be one of the ordered pairs obtained from the above pairs by replacing the lower indices 1 by 2.

In the first four cases, we use that γ preserves the edges and maps $\{A_1^1,\dots,A_1^4\}$ to $\{A_2^1,\dots,A_2^4\}$. This gives, for example, that

either $\gamma(\{A_1^1,A_1^3\}) = \{A_2^1,A_2^3\} = \alpha(\{A_1^1,A_1^3\})$

or $\gamma(\{A_1^1,A_1^3\}) = \{A_2^2,A_4^4\} = \beta(\{A_1^1,A_1^3\})$

Since $\gamma(A_1^{10}) \in \{A_2^9,A_2^{10}\}$, we have that

either $\gamma(\{A_1^1,A_1^{10}\}) = \{A_2^1,A_2^{10}\} = \alpha(\{A_1^1,A_1^{10}\})$

or $\gamma(\{A_1^1,A_1^{10}\}) = \{A_2^2,A_2^9\} = \beta(\{A_1^1,A_1^{10}\})$

The remaining seven cases are treated similarly.

Now we know that each set of F_1 is mapped by at least one of α,β to a set of F_2. The only kind of counterexample to our claim, which could possibly arise, should be of the following form:

There exist two sets P_1,Q_1 in F_1 satisfying

$$\alpha(P_1) \in F_2 \;;\; \beta(P_1) \notin F_2 \;;\; \alpha(Q_1) \notin F_2 \;;\; \beta(Q_1) \in F_2$$

Since P_1 and Q_1 have different images under α and β, it follows that

$$|P_1 \cap \{g_1,h_1\}| = |Q_1 \cap \{g_1,h_1\}| = 1$$

In any of the four possible cases we have

$$\big|\,|P_1 \cap Q_1| - |\alpha(P_1) \cap \beta(Q_1)|\,\big| = 1$$

The sets $\alpha(P_1)$ and $\beta(Q_1)$ are vertices of G_2, which are images of P_1,Q_1 respectively under the D-graph isomorphism γ. So we must have

$$|P_1 \cap Q_1| = 1 \Leftrightarrow |\alpha(P_1) \cap \beta(Q_1)| = 1$$

Since both $|P_1 \cap Q_1|$ and $|\alpha(P_1) \cap \beta(Q_1)|$ belong to $\{0,1,2\}$, we have a contradiction, which proves Lemma 1.

P r o o f of Theorem 6. Due to Lemma 1, we just have to prove the uniqueness (up to an isomorphism) of D-families corresponding to D-graphs of the types (1)-(9) and (33)-(36) inclusive. We construct some of the possible isomorphisms between any two D-families (having D-graphs) of these types. When defining the corresponding permutations of S, we shall use some of the Lemmas 1´,4´,9´,12´ to prove that the required conditions are uncontradictory, but we shall not state this explicitly in most of the particular cases.

Stars (types (1),(2),(8),(9)): Given two D-graphs G_1 and G_2, which are stars of the same type, we denote their centres (arbitrarily chosen in case (1)) by X_1, X_2 respectively and other vertices of degree 1 by Y_1^i, Y_2^i respectively ($i \in N$; the upper bound for i varies from 1 to 4, depending on the type).

Any permutation α of S, which satisfies the conditions:

$\alpha(X_1)=X_2$; $\alpha(X_1 \cap Y_1^i)=X_2 \cap Y_2^i$; $\alpha(S \setminus (X_1 \cup Y_1^i))=S \setminus (X_2 \cup Y_2^i)$

for each i

establishes an isomorphism between the D-families F_1, F_2 corresponding to G_1, G_2 respectively. Namely, α maps the sets of F_1 onto the sets of F_2, because all non-central vertices are uniquely determined by Lemma 8′.

Types (3) and (36): Lemmas 1′ and 9′ give that each of some two disconnected edges of a D-graph has exactly one vertex "containing the intersection element of the other edge"

Let X_1Y_1 and Z_1T_1 be two (in the case of (36), arbitrary two) disconnected edges of G_1 and let X_2Y_2 and Z_2T_2 be the corresponding two edges of G_2, so that the vertices denoted by X and Z have the property described in the previous sentence.

Any of the permutations α_1, α_2 of S satisfying (for i=1,2)

$\alpha_i(X_1)=X_2$; $\alpha_i(Z_1)=Z_2$; $\alpha_i(X_1 \cap Y_1)=X_2 \cap Y_2$; $\alpha_i(Z_1 \cap T_1)=Z_2 \cap T_2$

$\alpha_i(S \setminus (X_1 \cup Y_1))=S \setminus (S_2 \cup Y_2)$; $\alpha_i(S \setminus (Z_1 \cup T_1))=S \setminus (Z_2 \cup T_2)$

maps $\{X_1, Y_1 Z_1, T_1\}$ to $\{X_2, Y_2, Z_2, T_2\}$, that is, it establishes an isomorphism between the corresponding D-families of type (3).

As for the case (36), we denote the remaining pair of edges in G_1 and G_2 by U_1V_1, respectively U_2V_2. It is easy to check that one of the permutations α_1, α_2 maps $\{U_1, V_1\}$ to $\{U_2, V_2\}$, that is, it establishes also an isomorphism between the corresponding D-families of tipe (36).

D-graphs with a 4-cycle (types (4), (5), (6), (7)):

Let $X_1Y_1Z_1T_1$ and $X_2Y_2Z_2T_2$ be the two (in the case of (7, arbitrary two) corresponding 4-cycles of the isomorphic D-graphs G_1 and G_2. Each of the permutations α of S, satisfying, for example:

$\alpha(X_1\cap Y_1)=X_2\cap Y_2$; $\alpha(Y_1\cap Z_1)=Y_2\cap Z_2$ (*)

$\alpha(Z_1\cap T_1)=Z_2\cap T_2$; $\alpha(T_1\cap X_1)=T_2\cap X_2$; $\alpha(X_1\cap Z_1)=X_2\cap Z_2$

maps $\{X_1,Y_1,Z_1,T_1\}$ to $\{X_2,Y_2,Z_2,T_2\}$

that is, it establishes an isomorphism between the corresponding D-families (in the case) of type (4).

As for the type (7), we denote the other two 4-cycles of G_1 and G_2 by $P_1Q_1R_1S_1$ and $P_2Q_2R_2S_2$ respectively. We require additionally

$$\alpha(P_1\cap Q_1) = P_2\cap Q_2 \quad ; \quad \alpha(Q_1\cap R_1) = Q_2\cap R_2$$

$$\alpha(R_1\cap S_1) = R_2\cap S_2 \quad ; \quad \alpha(S_1\cap T_1) = S_2\cap T_2$$

This is not a contradiction with the previous conditions on α. In particular, the following property is preserved with such α: 2-intersections of non-incident vertices of one 4-cycle exactly correspond to the (unordered) pairs of 1-intersections corresponding to the non-incident edges of the other.

The only permutation α, which satisfies all the above conditions, establishes an isomorphism between the correspon-ding D-families of type (7).

Type (5): If the remaining edges of G_1 and G_2 are denoted by U_1V_1 and U_2V_2 respectively, then we require, in addition to (*)

$\alpha(U_1\cap V_1) = U_2\cap V_2$ $\alpha(S\setminus(U_1\cup V_1)) = S\setminus(U_2\cup V_2)$ (**)

Observe that it might be necessary to change the initial mapping of the vertices of 4-cycles in order to make the vertices containing $U_1\cap V_1$ map to the vertices containing $U_2\cap V_2$. If none of the constructed two permutations α gives an isomorphism between the corresponding two D-families of type (5), then an isomorphism should be looked for among the two permutations satisfing (**) and

$\alpha(X_1\cap Y_1) = Y_2\cap Z_2$; $\alpha(Y_1\cap Z_1) = Z_2\cap T_2$ (***)

$\alpha(Z_1\cap T_1) = T_2\cap X_2$; $\alpha(T_1\cap X_1) = X_2\cap Y_2$; $\alpha(X_1\cap Z_1) = Y_2\cap T_2$

Type (6): We denote the additional edges of G_1 and G_2

by U_1V_1, U_1W_1 and U_2V_2, U_2W_2 respectively.

An isomorphism between the corresponding D-families of type (6) should be realized by the permutation α of S satisfying $\alpha(U_1) = U_2$ and either the conditions (*) or the conditions (***).

This permutation maps $U_1\setminus(W_1 \cup U_1)$ to $U_2\setminus(W_2 \cup V_2)$. The elements of the last two sets correspond to two pairs of pairs of opposite edges of the 4-cycles.

Types (33) and (34): We denote (in both cases) the edges of D-graphs G_i (i=1,2) by X_iY_i, X_iZ_i, U_iV_i.

An isomorphism between the corresponding D-families can be realized by a permutation α of S satisfying the following conditions:

$$\alpha(X_1 \cap Y_1) = X_2 \cap Y_2 \quad ; \quad \alpha(S\setminus(X_1 \cup Y_1)) = S\setminus(X_2 \cup Y_2)$$

$$\alpha(X_1 \cap Z_1) = X_2 \cap Z_2 \quad ; \quad \alpha(S\setminus(X_1 \cup Z_1)) = S\setminus(X_2 \cup Z_2)$$

$$\alpha(U_1 \cap V_1) = U_2 \cap V_2 \quad , \quad \alpha(S\setminus(U_1 \cup V_1)) = S\setminus(U_2 \cup V_2)$$

Type (35): We notice that each vertex of degree 2 contains exactly one intersection element, which corresponds to an edge of the other connected component.

We denote the edges of D-graphs G_i (i=1,2) by $X_iY_i, X_iZ_i, U_iV_i, U_iW_i$ so that $U_i \supset X_i \cap Y_i$, $X_i \supset U_i \cap V_i$

The only permutation α of S, which satisfies the six conditions from the previous case, and, in addition,

$$\alpha(U_1 \cap W_1) = U_2 \cap W_2 \quad ; \quad \alpha(S\setminus(U_1 \cup W_1)) = S\setminus(U_2 \cup W_2)$$

induces an isomorphism between the corresponding D-families.

This completes the proof of Theorem 2.

Theorem 2 ensures that all non-isomorphic D-families are covered by our "example construction". It gives that each example F_o represents one class $\mathcal{F}_o$ of isomorphic D-families. The permitted 4-sets for F_o represent the permitted 4-sets for an arbitrary family of $\mathcal{F}_o$. The non-isomorphic possibilities for the augmentation of F_o represent the non-isomorphic possibilities for the augmentation of F_o, that is, all the non-isomorphic D-families, which contain a subfamily from F_o.

CONSTRUCTION OF THE REMAINING NON-ISOMORPHIC C-FAMILIES

Given a D-family, we construct all the non-isomorphic C-families having it as the (unique) maximal D-subfamily. We shall apply this construction to the already constructed examples ((1)-(36)) of D-families and so obtain all the non-isomorphic C-families, because these examples represent all the non-isomorphic D-families.

We shall primarily find the 2-permitted 4-sets for all 36 examples of D-families. The 2-permitted sets necessarily arise in complementary pairs (because the complement of a 2-permitted set is 2-permitted).

The D-families of types
(12),(15),(17),(18),(19),(24),(26),(27),(28),(30),(31),(32),(36)
have no 2-permitted 4-sets and they coincide with the only corresponding C-families.

We denote the complementary pairs

1258-3467,1268-3457,1278-3456		a,b,c	
1358-2467,1368-2457,1378-2456	by	d,e,f	respectively
1458-2367,1468-2357,1478-2356		g,h,i	

The complementary pairs of 2-permitted 4-sets for the given examples of the remaining types of D-families are given in the following list (the types are denoted in the upper row):

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
a,b,c,d,e,f,g,h,i	c,d,e,g,h	e,f	c,d,e,g,h	c	c	c	c,e,g

(9)	(10)	(11)	(13)	(14)	(16)	(20)	(21)	(22)	(23)	(25)	(29)	(33)	(34)	(3
c,e,g	c	c	c	c	c	g,h	g,h	g,h	g	g,h	g	c	c	c

THEOREM 3. *All the non-isomorphic C-families, which are not D-families, can be constructed from the given examples of D-families by the addition of some 2-permitted 4-sets from the pairs c,e and g.*

P r o o f. Due to the above list, it suffices to give proof for the types (1),(2),(3),(4),(20),(21),(22),(25).

We observe that no two 4-sets, which belong to two different complementary pairs, appearing in any of the following 3-sets of complementary pairs

{a,b,c},{d,e,f},{g,h,i},{a,d,g},{b,e,h},{c,f,i}

may be added to a D-family, otherwise either 3-intersections or new 1-intersections appear.

Equivalently, the only maximal sets of complementary pairs of 4-sets, the 4-sets from which may be added simultaneously to a D-family are

{a,e,i},{a,f,h},{b,d,i},{b,f,g},{c,d,h},{c,e,g}

Each of these six may appear with the D-family (1). Note, however, that the following permutations of S, denoted just by their non-trivial cycles

(57), (567), (576), (67), (56)

in this order, map the 4-sets appearing in {c,e,g} to the 4-sets in the corresponding one of the first five 3-sets of pairs. The elements 5,6,7 are in the equal position with respect to the given example of type (1) (all the three appear in the same set, 1567). Thus the addition of the 4-sets, contained in anyone of the first five 3-sets of pairs to 1234,1567 gives the isomorphic C-families with those obtained by addition to 1234,1567 just some of the 4-sets contained in c,e or g.

The transposition (56) maps {e,g} to {d,h} and {g} to {h}. The elements 5 and 6 always appear together in the sets of the given examples of D-families of types (2),(4),(20),(21),(22),(25). Consequently, the 2-permitted 4-sets from the families {d,h}, respectively {h}, can be always (when these six D-families are considered and when only non-isomorphic C-families are looked for) replaced by the 2-permitted 4-sets from the families {e,g}, respectively {g}.

Similarly, the 2-permitted 4-sets from {f}, with respect to the given example of D-family of type (3), can be always replaced by 2-permitted 4-sets from {e}, due to the transposition (67).Q.E.D.

The given examples of D-families of types (3),(5),(6),(7),(10),(11),(13),(14),(16),(20),(21),(22),(23),(25),(29),(33),(34),(35) have 2-permitted 4-sets in just one of the pairs c,e,g. There are at least three non-isomorphic C-families, corresponding to each of these D-families; the families obtained by addition of none, one and two 2-permitted 4-sets.

The only question which should be answered here is: "In which of these 18 cases does there exist the fourth non-isomorphic C-family?", or, equivalently
"In which of these 18 cases are the two 2-permitted 4-sets not in the equal position with respect to the D-family?"

It is not hard to check that the answer is:
"With D-families of types (5),(6),(10),(11),(13),(14),(20),(23),(33),(34)".

For example, we show that the 4-sets 1458 and 2367 are in the equal position with respect to the given example H_2 of the D-family of type (22), but not with respect to the given example H_3 of the D-family of type (23).

The permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 7 & 6 & 5 & 4 & 2 \end{pmatrix}$ maps $H_2 \cup \{1458\}$ to $H_2 \cup \{2367\}$.

The elements 1 and 7 have special, but different positions in the family H_3. Both are the intersection elements corresponding to the outer edges of the D-graph (4-path). However, 7 is missing in two 4-sets, which determine an edge, but 1 is not (equivalently, the elements 1 and 7 are in different positions with respect to the markation). Consequently, the 4-sets 1458 (containing 1) and 2367 (containing 7) are not in the equal position with respect to the family H_3.

THEOREM 4. *There is a bijection between the non-isomorphic C-families, arising from the given examples F_0 and F_1 of the D-families of types (2) and (4) respectively.*

P r o o f. As $3478 = (1234\setminus(1567\cup 2568)) + (((1567\cup 2568)\setminus(1567\cap 2568))\setminus 1234)$ and any isomorphism between two C-families arising from F_o preserves both 1234 and {1567,2568}, we conclude that it also preserves 3478, that is, it is also an isomorphism between the two C-families arising from F_1 and having the same 2-permitted 4-sets as the first two C-families (Note that the existence of the later two C-families is guaranteed).

Conversely, any isomorphism between two C-families arising from F_1 induces (as a restriction) an isomorphism between their C-subfamilies, obtained by deleting the same 4-set of F_1, from each of the first two C-families. These C-subfamilies have the maximal D-subfamilies of type (2) and hence each of them is isomorphic to a C-family arising from F_o.

The only non-isomorphic C-families, which are not covered by the discussion above, are those C-families, the maximal D-subfamilies of which have stars as the corresponding D-graphs (types (1),(2),(8),(9)).

The 4-set 1234 is the centre of the given examples of stars corresponding to D-families of types (2),(8),(9) and we adopt the convention that it is also the centre in the case of type (1). Each of the 4-sets from the pairs 1278-3456,1368-2457, 1458-2367 (that is, from the pairs of c,e,g) has a 2-intersection with 1234 and no two of them have the same 2-intersection with 1234.

THEOREM 5. *There is a bijection between the non-isomorphic C-families arising from the given examples of D-families of types (1),(2),(8),(9) and the non-isomorphic coloured C-graphs*(*) *having one, two, three, four black vertices respectively.*

P r o o f. We adjoin C-graphs to these C-families as follows:

The four vertices of the C-graphs are denoted by 1,2,3,4. The vertex 1 (resp.2,resp.3,resp.4) is black if and only if the corresponding C-family contains the set 1567 (resp.2568,resp. 3578,resp.4678). The edge 12 (resp.13,resp.14,resp.23,resp.34, resp.34) exists if and only if the corresponding C-family

(*) Two coloured C-graphs G_1 and G_2 are <u>isomorphic</u> if there is a graph-isomorphism which maps G_1 onto G_2 and preserves the colouring.

contains the 4-set 1278 (resp.1368,resp.1458,resp.2367,resp. 2457,resp.3456)

Conversely, given a coloured C-graph, the vertices of which are denoted by 1,2,3,4 so that the black vertices are denoted by smaller numbers, we simply read all the 4-sets of the corresponding C-family (except for 1234, the existence of which is assumed in advance), just by the use of the above mentioned correspondence.

(I) Suppose that two C-graphs G_1 and G_2 are isomorphic, that is, there exists a colour and incidence-preserving bijection between their vertices (which are denoted by 1,2,3,4).

We extend the corresponding permutation β of {1,2,3,4} to a permutation α of S by postulating:

$$\alpha(i) = \beta(i) \quad ; \quad \alpha(9-i) = 9-\beta(i) \quad , \quad \text{for } i=1,2,3,4$$

We claim that the permutation α estabilishes an isomorphism between the C-families F_1 and F_2, corresponding to the graphs G_1 and G_2 respectively.

Both of the families F_1 and F_2 have 1234 and the same sets from {1567,2568,3578,4678}, by the above given construction of the C-families, corresponding to C-graphs. The permutation α preserves 1234 and also preserves the later 4-sets (not necessarily identically), since it preserves the black vertices.

There is an edge {k,f} in G_1 if and only if there exists the edge $\{\alpha(k),\alpha(f)\}$ in G_2. The edges {k,f}, respectively $\{\alpha(k),\alpha(f)\}$, imply the existence of the 4-sets {k,f,9 -f,9 -k} respectively $\{\alpha(k),\alpha(f),9-\alpha(f),9-\alpha(k)\}$ in the families F_1, respectively F'_2.

(II) Conversely, suppose that two C-families K_1 and K_2, which arise from the same one of the given examples of D-families of types (1),(2),(8),(9), are isomorphic, that is, there is a permutation γ of S which maps the set of K_1 onto the sets of K_2. The set 1234 is necessarily preserved under γ in the case of types (2),(8),(9), because of its special position in the D-graph.

We claim that, in the case of types (2),(8),(9), the restriction of the permutation γ to the set {1,2,3,4} establishes an isomorphism between the corresponding C-graphs (with vertices denoted by 1,2,3,4)

Proof of the claim:

vertex i is black in $G_1 \Leftrightarrow K_1$ contains $\{i\} \cup (\{5,6,7,8\} \setminus \{9-i\}) \Leftrightarrow K_2$ contains $\{\gamma(i)\} \cup (\{5,6,7,8\} \setminus \{9-\gamma(i)\}) \Leftrightarrow$ vertex $\gamma(i)$ is black in G_2

vertices i,j are incident in $G_1 \Leftrightarrow$

$\Leftrightarrow K_1$ contains $\{i,j\} \cup (\{5,6,7,8\} - \{9-i, 9-j\}) \Leftrightarrow$

$\Leftrightarrow K_2$ contains $\{\gamma(i), \gamma(j)\} \cup (\{5,6,7,8\} - \{9-\gamma(i), 9-\gamma(j)\}) \Leftrightarrow$

$\Leftrightarrow$ vertices $\gamma(i), \gamma(j)$ are incident in G_2

REMARK: We used the fact that all the sets of {1567, 2568,3578,4678}, respectively of {1278,3456,1368,2457,1458, 2367}, can be represented in the first, respectively in the second, of the two forms above.

When the D-family {1234,1567} (of type (1)) is considered, it may happen that the given isomorphism γ between two corresponding C-families K_1, K_2 maps 1234 to 1567 and conversely. It necessarily has 1 and 8 as fixed points.

In this case we define another permutation δ of S by

$\delta(1) = 1$; $\delta(8) = 8$; $\delta(i) = 9-\gamma(i)$ for $2 \leq i \leq 7$

It is easy to check that

$\delta(X) = \gamma(X)$ for each $X \in \{1278, 3456, 1368, 2457, 1458, 2367\}$

and $\delta(1234) = 1234$ $\delta(1567) = 1567$

It follows that there exists an isomorphism between K_1 and K_2, which preserves 1234 and we may proceed as in the previous three cases.

This completes the proof of Theorem 5.

CONSEQUENCE 1. *There is a bijection between non-isomorphic C-families arising from (the given examples of) D-families of types (1) and (8).*

P r o o f. Follows immediately by interchanging the black and white colour in the vertices of the corresponding coloured C-graphs.

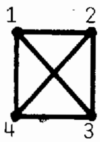
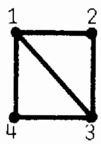
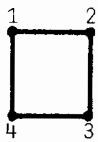
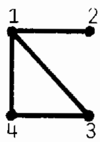
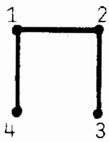
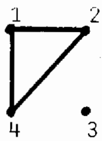
CONSEQUENCE 2. *There are 11 non-isomorphic C-families arising from (the given example of) D-family of type (9).*

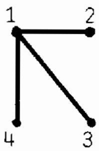
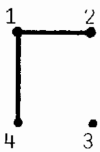
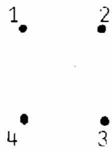
P r o o f. All the vertices in the corresponding coloured C-graphs are black. We simply use the well-known 11 non-isomorphic simple graphs on 4 vertices (e.g., in the Appendix of |2|)

In order to finish the construction of non-isomorphic C-families, we have just to find all the non-isomorphic 2-colouring of vertices of the above-mentioned 11 graphs, for the cases when exactly one and exactly two vertices are black.

We give the table of non-isomorphic coloured C-graphs with one or two black vertices, by stating their 1-sets and 2-sets of black vertices under the corresponding non-isomorphic graphs on four vertices. If several non-isomorphic colourings correspond to the same graph, then their denotations are separated by commas.

THE TABLE OF NON-ISOMORPHIC COLOURED C-GRAPHS
WITH 1 OR 2 BLACK VERTICES

						
One black vertex is	1	1,2	1	1,2,3	1,3	1,3
two black vertices are	12	12,13,14	12,13	12,13,23,34	12,13,23,34	12,13

					
one black vertex is	1,2	1,2,3	1	1,3	1
two black vertices are	12,23	12,13,23,24	12,13	12,13,34	12

We stress (somewhat similar as with the denoted A-graphs(|1|)) that the coloured C-graphs provide immediate constructions of the corresponding, up to isomorphisms determined, C-families with the maximal D-subfamilies of types (1),(2),(4),(8) and (9)

Summing up the numbers of non-isomorphic C-families, arising from all distrinct types of D-families, we obtain that there are 184 non-isomorphic C-families.

REFERENCE

|1| Acketa D.M., *Another construction of rank 4 paving matroids on 8 elements (I), Zbor.rad. Prir.mat.fak. Novi Sad, Ser.Mat. 12 (1982), 259-276.*

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REZIME

NOVA KONSTRUKCIJA PEJVING MATROIDA RANGA 4 NA SKUPU OD 8 elemenata (II)

U ovom radu, koji je nastavak rada {1}, dajemo konstrukciju svih neizomorfnih C-familija (familija 4-podskupova 8-skupa, kod kojih ne postoje 3-preseci, a postoje 1-preseci). Time se kompletira konstrukcija svih neizomorfnih pejving matroida ranga 4 na skupu od 8 elemenata.