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ANOTHER CONSTRUCTION OF RANK 4 PAVING MATROIDS ON 8 ELEMENTS (I)

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ABSTRACT

All 950 non-isomorphic simple matroids on 8 elements were constructed in |2| by the use of a computer programme. By using elementary methods, without computer aid, we give another construction for probably the most interesting (when non-isomorphisms are considered) subclass P of 322 rank 4 paving matroids on 8 elements. The class P is partitioned into three disjoint subclasses. The construction of the first two is given in this paper, while the construction of the third subclass of P is given in a sequel paper |1| (these two papers make a whole). We study in greater detail the ways in which the non-isomorphic possibilities arise. Our main tool in the construction of P are three auxiliary classes of graphs (these graphs can be bijected to some of the paving matroids) and some properties of the Steiner system S(3,4,8).

PRELIMINARIES

We emphasize that the whole understanding of the construction itself can be gained without knowing what a matroid is, although some familiarity with elements of matroid theory is desirable. A few basic graph-theoretical definitions are necessary; all can be found in any general text on graphs (e.g., |3|). Non-defined notions on matroids (which appear only in several commentaries of the construction) can be found in |4|.

We shall mainly consider the sets which are subsets of the set $S = \{1,2,3,4,5,6,7,8\}$. If there is no possibility of confusion, then the fixed subsets of S are denoted without brackets and commas (e.g., "1234" instead of "{1,2,3,4}"). We shall often speak about, e.g., "1234" instead of "the set 1234". An n-set (n-intersection, etc,) is a set (intersection, etc.) of cardinality n. In particular, an n-path of a graph is a path of n (distinct) edges. The cardinality of a set X is denoted by $|X|$.

By a "graph" we shall always mean a non-oriented graph without loops and multiple edges.

A ground-set of a family of sets is the union of all its sets.

Two families of sets F_1 and F_2, with the ground-sets G_1 and G_2 respectively, are isomorphic if there is a bijection $\alpha : G_1 \to G_2$ such that

$$X \in F_1 \iff \alpha(X) \in F_2$$

The addition of a set X to a family F, where X does not belong to F, is the operation which gives the family $F \cup \{X\}$. The set X is said to be added to F.

If the addition of several different sets to the same family F gives the isomorphic families with one set more, then we choose one of these sets. Its addition to F gives a representative of a class of isomorphic larger families.

A P-family is a family F of distinct subsets of S satisfying:

(a) $X \in F \Rightarrow |X| \geq 4$

(b) $(X_1, X_2 \in F \wedge X_1 \neq X_2) \Rightarrow |X_1 \cap X_2| \leq 2$

(c) $S \notin F$

An A-family is a P-family F which additionally satisfies:

(d) $(\exists X)(X \in F \wedge |X| \geq 5)$

A B-family is a family F of distinct 4-subsets of S satisfying:

(e) $(X_1, X_2 \in F \wedge X_1 \neq X_2) \Rightarrow |X_1 \cap X_2| \in \{0,2\}$

A C-family is a P-family, which is neither an A-family nor a B-family.

INTRODUCTION

There is an obvious bijection between non-isomorphic P-families and non-isomorphic rank 4 paving matroids on 8 elements (P-families correspond exactly to the families of non-trivial hyperplanes of such matroids). On the other hand, it is clear that non-isomorphic P-families may be partitioned into non-isomorphic A-, B- and C-families.

In this paper we shall give separate constructions for all non-isomorphic A- and B-families. There are, respectively, 52 and 86 families in these classes. The construction of all non-isomorphic C-families is given in |1|.

Our main tool for dealing with the lion´s share of non-isomorphic A- and C-families are three classes of auxiliary graphs. We establish the theorems which provide the necessary correspodence in each case. On the other hand, the construction of all non-isomorphic subfamilies of the Steiner system S(3, 4,8) yields all the non-isomorphic B-families.

The non-isomorphism of any two constructed matroids (i.e., of any two constructed P-families) should be obvious from our construction. This is not so in catalogue |2|. For example, there are 63 non-isomorphic rank 4 paving matroids on 8 elements, which are given in |2| by families of seven 4-hyperplanes.

The exhaustion of all non-isomorphic possibilities immediately follows from |2|. We have made efforts, however, to provide a self-consistent proof of this exhaustion, although a full understanding of such a proof sometimes still requires small case analyses.

We would by no means recommend our construction instead of that of |2|:, in fact, our construction could be hardly completed without the help of the first one. We simply hope that this construction could help a better understanding of the ways in which non-isomorphic matroids arise and, perhaps, be used for some further research.

FURTHER DEFINITIONS AND DENOTATIONS

Given an A-family F, which has exactly one 5-set, denoted by X, we define the A-graph G of F as follows:

(i) vertices of G are elements of X

(ii) there is an edge {a,b} in G if and only if there is a 4-set containing {a,b} in F.

Two (non-incident) edges of the just defined A-graph G are said to be similar if the corresponding two 4-sets of F contain the same 2-subset of S\X.

A denoted A-graph is a graph G on five vertices, with at most six edges, such that

(i) some disjoint pairs of non-incident edges may be distinguished

(ii) if G has k edges ($4 \le k \le 6$), then there are at least k-3 pairs of distinguished edges .

REMARK: (ii) implies that G has not a vertex of degree 4.

The pairs of distinguished edges determine a denotation of G.

We stress that it should be proved that denoted A-graphs are A-graphs of some A-families. In fact, the definition of denoted A-graphs includes the characterization of those graphs, which can be represented as A-graphs of some A-families.

The family of blocks of the Steiner system S(3,4,8) is the family ϕ, where:

$$\phi = \begin{Bmatrix} 1234, & 1256, & 1278, & 1357, & 1368, & 1458, & 1467 \\ 5678, & 3478, & 3456, & 2468, & 2457, & 2367, & 2358 \end{Bmatrix}$$

(The blocks are for convenience written in two lines).

The complement of a block X of ϕ is the block S\X.

The complementary subfamily of a subfamily F of ϕ is the family $\bar{F}$ such that $F \cup \bar{F} = \phi$; $F \cap \bar{F} = \emptyset$.

The family ϕ_1 is the family of those blocks of ϕ, which contain the element 1 (the first row in the entry above). The blocks of ϕ without 8 are called heptahedron hyperplanes.

A complement of a subfamily F of ϕ is a block, which is not in F, but the complement of which is in F. In some cases we say only "the complement", when F is clear from the context.

We shall sometimes speak about some elements or some subsets of S sharing the isomorphic position in (with respect to) a subfamily F of ϕ. We found it difficult, and perhaps unnecessary in this context, to make this rather vague notion quite precise. It means that the corresponding subsets of S should be equally treated when constructing non-isomorphic subfamilies of ϕ, which contain F. Such subsets are easily recognized in each particular case. For example, a necessary condition for two elements of S to share the isomorphic position in F is for them to appear the same number of times in the blocks of F. In most cases this condition appears to be sufficient. A similar conclusion holds for 2-subsets of S.

A subset of S has a special position in a subfamily of ϕ if it does not share the isomorphic position (with respect to that subfamily) with another subset of S.

CONSTRUCTION OF NON-ISOMORPHIC A-FAMILIES

We differentiate four cases for an A-family F:

Case 1: F has a 7-set X.

The rules (a) and (b) give $F = \{X\}$.

Case 2: F has a 6-set X.

It is easily seen that all the other sets of F can be just the 4-sets which contain $S \setminus X$ and no two have a common element in X. There are therefore just four non-isomorphic

A-families with a 6-set, having from none up to three 4-sets.

Case 3: F has two 5-sets X_1 and X_2.

We have $|X_1 \cap X_2| = 2$. An easy argument shows that all the other sets of F can be just the 4-sets having two elements in $X_1 \setminus X_2$ and the other two in $X_2 \setminus X_1$. At most three such 4-sets may appear simultaneously and each two of them have one common element in $X_1 \setminus X_2$ and another one in $X_2 \setminus X_1$. So again we have four non-isomorphic possibilities depending on the number of 4-sets.

Case 4: F has exactly one 5-set X.

If F has a 4-set containing $S \setminus X$, then we denote this 4-set by Y.

THEOREM 1.*) *There is a bijection between non-isomorphic:*

a)	*denoted A-graphs*	*and*	*A-families which have exactly one 5-set X and have not Y*
b)	*denoted A-graphs which have at least one isolated vertex*	*and*	*A-families which have exactly one 5-set X and which have Y* .

These bijections are realized by establishing such isomorphisms between the denoted A-graphs and the A-graphs of the A-families, which map pairs of distinguished edges to pairs of similar edges and conversely.

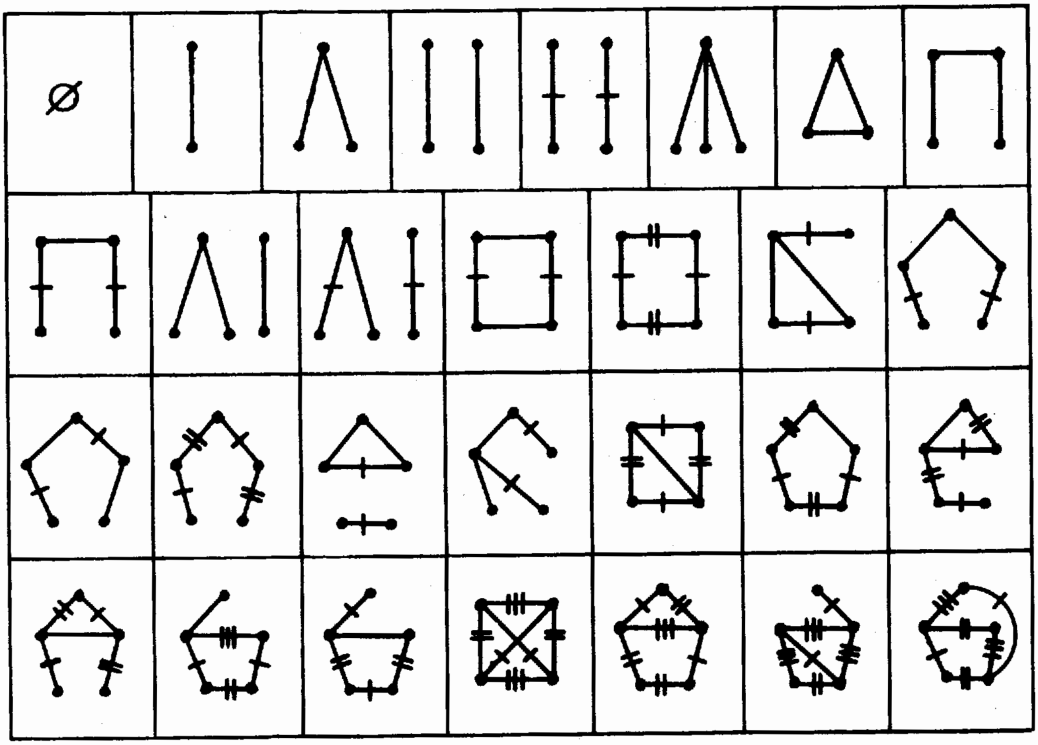
P r o o f. (Sketch) An A-family F of Case 4.has the unique A-graph G with fixed pairs of similar edges. It is easily shown that G satisfies all the conditions to be a denoted A-graph, when the similar edges are interpreted as distinguished. On the other hand, a denoted A-graph uniquely determines, up to an obvious isomorphism, the corresponding A-family. As for b), we observe that the element $X \cap Y$ cannot appear in another 4-set of F.

We give the table of all non-isomorphic denoted A-graphs. The graphs are given without isolated vertices, if

*) Two denoted A-graphs G_1 and G_2 are isomorphic if there is a graph-isomorphism which maps G_1 onto G_2 and preserves the denotation.

there are any. Two distinguished edges in the same pair are always crossed by the some number of short lines. The order of this denotation is unimportant.

THE TABLE OF NON-ISOMORPHIC DENOTED A-GRAPHS



It is well-known (see,e.g., the Appendix of |2|) that the above list includes (when the isolated vertices are returned) all non-isomorphic graphs on five vertices, which have at most six edges and which have not a vertex of degree 4. The exhaustion of non-isomorphic possibilities for denotation, however, should be checked by case analysis. Thus, for example, the last denoted A-graph in the table has two vertices of degree 3 connected by three disjoint paths of length 2. Any edge has just two non-incident edges and it is easy to see that there are only two isomorphic possibilities for denotation.

As there are 29 non-isomorphic denoted A-graphs, 14 of which have isolated vertices, Theorem 1 gives that there are 43 non-isomorphic A-families arising in Case 4. We point out, however, that the Table above does not give only the number of these A-families, but an immediate construction of any of them as well.

REMARK: We can immediately construct 47 non-paving rank 4 simple matroids on 8 elements by taking duals of the paving matroids determined by A-families of Cases 3. and 4.

(2) CONSTRUCTION OF NON ISOMORPHIC B-FAMILIES

We shall primarily give three theorems, which will be used in the construction.

THEOREM 2. *Each B-family F can be embedded into* Φ .

P r o o f. As the assertion obviously holds for the empty family, the family with exactly one 4-set and the family with exactly two complementary 4-sets, we may assume that the family F contains two intersecting 4-sets $X_1=\{a,b,c,d\}$ and $X_2=\{a,b,e,f\}$. It is easy to check that the only remaining 4-subsets of $S=\{a,b,c,d,e,f,g,h\}$, which have not 1- or 3-intersections with either of X_1, X_2, are the sets of $I \cup J \cup K$, where

$$I=\{\{a,b,g,h\},\ \{c,d,e,f\},\ \{c,d,g,h\},\ \{e,f,g,h\}\}$$

$$J=\left\{\begin{matrix}\{a,c,e,g\}, & \{a,c,f,h\}, & \{a,d,e,h\}, & \{a,d,f,g\} \\ \{b,d,f,h\}, & \{b,d,e,g\}, & \{b,c,f,g\}, & \{b,c,e,h\}\end{matrix}\right\}$$

$$K=\left\{\begin{matrix}\{a,c,e,h\}, & \{a,c,f,g\}, & \{a,d,e,g\}, & \{a,d,f,h\} \\ \{b,d,f,g\}, & \{b,d,e,h\}, & \{b,c,f,h\}, & \{b,c,e,g\}\end{matrix}\right\} .$$

There are no "forbidden" intersections among the sets of $I \cup J$ or $I \cup K$, while each set of J has either 1- or 3-intersection with each set of K. Thus the only two maximal B-families containing $X=\{X_1,X_2\}$ are $X+I+J = \Phi_2$ and $X+I+K = \Phi_3$. We have, however, that $\Phi_2 \simeq \Phi$ and $\Phi_3 \simeq \Phi$. The isomorphisms are established by the permutations, which map $\{a,b,c,d,e,f,g,h\}$ to $\{1,2,3,4,5,6,7,8\}$ and $\{1,2,3,4,5,6,8,7\}$ respectively.

THEOREM 3. *If two subfamilies of Φ are isomorphic, then their complementary subfamilies are also isomorphic. In other words, there always exists such an isomorphism between two isomorphic subfamilies of Φ, which can be extended to an automorphism of Φ.*

LEMMA 1. *If a permutation α of S satisfies the condition: "$\alpha(\Phi)\cap\Phi$ contains some three blocks with exactly one common element", then $\alpha(\Phi) = \Phi$*

P r o o f. As $\alpha(\Phi)$ is isomorphic to Φ, it suffices to prove that $\alpha(\Phi)$ is uniquely determined by the given three blocks X_1,X_2,X_3. Denote $X_1\cap X_2\cap X_3 = \{z\}$. The other four blocks of $\alpha(\Phi)$, which contain z, must be:

$$(X_1\cap X_2)+(S\setminus(X_1\cup X_2)) \quad , \quad (X_1\cap X_3)+(S\setminus(X_1\cup X_3))$$

$$(X_2\cap X_3)+(S\setminus(X_2\cup X_3)) \quad , \quad \{z\}+(X_1\setminus(X_2\cup X_3))+(X_2\setminus(X_1\cup X_3))+(X_3\setminus(X_1\cup X_2))$$

where "+" denotes the union of disjoint sets.

Namely, the first three blocks are due to the fact that each of the 2-sets $X_1\cap X_2$, $X_1\cap X_3$, $X_2\cap X_3$ appears in exactly three blocks of $\alpha(\Phi)$. A similar argument applied to the 2-sets $\{z\}+(X_1\setminus(X_2\cup X_3))$, $\{z\}+(X_2\setminus(X_1\cup X_3))$, $\{z\}+(X_3\setminus(X_1\cup X_2))$ yields the fourth block.

The remaining seven blocks of $\alpha(\Phi)$ must be the complements of the first seven.

LEMMA 2. *If α is an isomorphism between two subfami-*

lies F_1 and F_2 of Φ, each of which contains (at least) three blocks with exactly one common element, then α can be extended to an automorphism of Φ.

P r o o f. As the cardinality c of the ground-set of F_1 is at least 7, the bijection α between the ground-sets of F_1 and F_2 can be extended to a unique permutation $\bar{\alpha}$ of S (if c=8, then $\bar{\alpha}=\alpha$). The proof is completed by applying Lemma 1.

P r o o f of Theorem 3. Due to Lemma 2., we have to consider only the case when none of two isomorphic subfamilies F_1 and F_2 of Φ has three blocks with exactly one common element. We differentiate three cases:

Case 1. F_1 has two intersecting blocks

$$X_1 = \{a,b,c,d\} \quad , \quad X_2 = \{a,b,e,f\}.$$

The 3-set $\{a,c,e\}$ is included in a block $X_3=\{a,c,e,g\}$ of Φ, for some $g\in S\setminus(X_1\cup X_2)$. Let α denote any isomorphism of F_1 onto F_2. Similarly there exists a block $X_4=\{\alpha(a),\alpha(c),\alpha(e),v\}$ of Φ, for some $v\in S\setminus(\alpha(X_1)\cup\alpha(X_2))$. Obviously $X_3\notin F_1, X_4\notin F_2$. We define a permutation α_1 of S as follows:

$$\alpha_1(x)=\alpha(x) \quad \text{for} \quad x\in X_1\cup X_2, \qquad \alpha_1(g)=v$$
$$\alpha_1(S\setminus(\{g\}\cup X_1\cup X_2)) = S\setminus(\{v\}\cup\alpha(X_1)\cup\alpha(X_2)).$$

The permutation α_1 estabilishes an isomorphism between the subfamilies $\{X_1,X_2,X_3\}$ and $\{\alpha(X_1),\alpha(X_2),X_4\}$ of Φ. Thus α_1 is an automorphism of Φ by Lemma 2.

The only blocks, which may arise in F_1 (by reason of the assumption), apart from X_1 and X_2, are:

$$(X_1\cup X_2)\setminus(X_1\cap X_2),\ (X_1\cap X_2)+(S\setminus(X_1\cup X_2)),\ S\setminus X_1,\ S\setminus X_2.$$

Since $\alpha_1(X_1)=\alpha(X_1)$ and $\alpha_1(X_2)=\alpha(X_2)$, the images of each of these blocks under α and α_1 coincide. This implies that α_1 is an extension of α.

Case 2. F_1 has a block $X_1=\{a,b,c,d\}$, but has not two intersecting blocks.

There exist two blocks $X_2=\{a,b,e,f\}$ and $X_3=\{a,c,e,g\}$

of Φ for some $e,f,g\in S\setminus X_1$. $X_2,X_3\notin \Phi_1$.

Let α denote an arbitrary isomorphism of F_1 onto F_2. Similarly, there exist blocks $X_4 = \{\alpha(a),\alpha(b), h,i\}$ and $X_5 = \{\alpha(a),\alpha(c),h,j\}$ of Φ for some $h,i,j\in S\setminus\alpha(X_1)$. Obviously $X_4,X_5\notin F_2$. We define the following permutation α_2 of S:

$$\alpha_2(x) = \alpha(x) \quad \text{for } x\in X_1$$

$$\alpha_2(e) = h \quad ; \quad \alpha_2(f) = i \quad ; \quad \alpha_2(g) = j$$

$$\alpha_2(S\setminus(X_1\cup X_2\cup X_3)) = S\setminus(\alpha(X_1)\cup X_4\cup X_5).$$

The permutation α_2 establishes an isomorphism between the subfamilies $\{X_1,X_2,X_3\}$ and $\{\alpha(X_1),X_4,X_5\}$ of Φ. α_2 is an automorphism of Φ by Lemma 2.

The only block, which may arise in F_1, apart from X_1, is $S\setminus X_1$ and

$$\alpha_2(S\setminus X_1) = S\setminus\alpha_2(X_1) = S\setminus\alpha(X_1) = \alpha(S\setminus X_1).$$

proves that α_2 is an extension of α.

Case 3. F_1 has no blocks

$\Phi\setminus F_1 = \Phi\setminus F_2 = \Phi$, which completes the proof of Theorem 3.

Consequence: Non-isomorphic subfamilies of Φ, which have more than seven blocks, are uniquely determined (up to an isomorphism) as the complementary subfamilies of those non-isomorphic subfamilies of Φ, which have at most six blocks.

THEOREM 4. *The assertion of Theorem 3 still holds, when Φ is replaced by Φ_1.*

P r o o f. Omitted as similar to and easier than the previous one. (Deleting 1 from all sets of Φ_1, we have the lines of the Fano plane, and we should use the families of three non-concurrent lines as the "key").

Theorem 2 and the Consequence of Theorem 3 reduce the construction of all non-isomorphic B-families to the construction of all non-isomorphic subfamilies of Φ which have at most seven blocks. This last construction will be done step- by-step, beginning with the smaller subfamilies. Given a subfamily of k blocks, we should naturally look for its non-isomorphic superfamilies of k+1 blocks and choose one representative from each class of isomorphic families for the next step of generation. A great defect of this procedure, however, is that non-isomorphic families may have isomorphic superfamilies. In order to lessen its effect as much as possible, we develop the following approach:

We shall favorize the element 1 of S in our construction in the following sense:

No element of {2,3,4,5,6,7,8} appears in more blocks than 1 does and no 2-subset of S appears in three blocks of Φ if a 2-subset containing 1 does not,

in any subfamily of Φ which we construct (to be the representative of a class of isomorphic subfamilies).

This condition still preserves the generality, because all the elements, respectively 2-subsets, of S are in isomorphic positions in Φ. What is more, we may assume that the 4-sets containing 1, of any subfamily of Φ that we construct, necessarily form one of the <u>fixed</u> (representatives of) non-isomorphic subfamilies of Φ_1.

We primarily construct non-isomorphic subfamilies of Φ_1 and add thereafter some 4-sets from $\Phi \setminus \Phi_1$ to them in order to obtain other subfamilies of Φ. For each number of blocks we primarily list the representatives of the corresponding non-isomorphic subfamilies (separated by commas). This list is, except for the trivial cases, followed by short explanations of the construction.

NON-ISOMORPHIC SUBFAMILIES OF Φ_1

0 blocks: $\emptyset$

1 block: A = {1234}

2 blocks: B = {1234,1256}. Each two blocks of Φ_1 have a 2-intersection.

3 blocks: C = {1234,1256,1278} and D = {1234,1256,1357}.

The three blocks have 1 as a common element. They may have one more common element, but need not.

Theorem 4 provides that there are just two non-isomorphic subfamilies of Φ_1 having four blocks and just one for each number of blocks between five and seven. We prefer to choose lexicographically the first representatives than to take the complementary subfamilies.

4 blocks: E = {1234,1256,1357,1467} and
F = {1234,1256,1278,1357}

The common intersection of the three missing 4-sets of Φ_1 may be of cardinality 2 or 1

5 blocks: G = {1234,1256,1278,1357,1368}
6 blocks: H = {1234,1256,1278,1357,1368,1458}
7 blocks: Φ_1.

NON-ISOMORPHIC SUBFAMILIES OF Φ WHICH HAVE AT MOST SEVEN BLOCKS

0 blocks: $\emptyset$
1 block: A
2 blocks: B, A∪{5678}. The two blocks may be disjoint.
3 blocks: C, D, B∪{3456}, B∪{3478}. None of the blocks containing 2 may be added to B, otherwise 2 would appear more frequently than 1. For a similar reason it is impossible to choose the subfamily A from Φ_1. The block from $\Phi\setminus\Phi_1$ may either intersect both blocks of B or just one of them.

4 blocks: E, F, C∪{3456}, D∪{2358}, D∪{2367}
D∪{2468}, B∪{3478,5678}. None of the blocks with 2 may be added to C. Each of the remaining blocks (in $\Phi\setminus\Phi_1$) is the complement of a block of C. Similarly none of the blocks containing 2, or 3456, may be added to B. Each of 2367,2457, 3456, constitutes with D up to an isomorphism unique subfamily of four heptahedron hyperplanes, without an element common for

all the four (i.e.,which is not isomorphic to E). The set 2358 is the only one from $\Phi\setminus\Phi_1$, which satisfies the following two conditions: it is not included in the ground-set of D and has a non-empty intersection with all sets of D. All complements (of sets) of D share the isomorphic position with respect to D.

5 blocks: G, E∪{2358}, E∪{2367}, F∪{2358}, F∪{2468},
F∪{3456}, C∪{3456,3478}, D∪{2468,3456}, D∪{2468,3478}.

The elements 2,3,4,5,6,7 share the isomorphic position in E and the same conclusion holds for the sets of E. Adding one of the complements to E we obtain therefore four isomorphic families, while by the adding of one of 2367,2457, 3456 we obtain the (up to an isomorphism unique) family of five heptahedron hyperplanes. Note that the set 1357 has a special position in F (with respect to the other three blocks). The additions of 2468 and one of the remaining three complements to F give rise consequently to two non-isomorphic families. The blocks 2358,2367 and 2457 share the isomorphic position with respect to F. As for C, two of the blocks 3456, 3478,5678 must be chosen, which gives three isomorphic possibilities. When D is considered, notice that none of the elements 2,3,5 may appear twice in the blocks of $\Phi\setminus\Phi_1$. We must therefore choose at least one block from each of the families {3456,3478,5678}, {2457,2468,5678} and {2367,2468, 3478}. Obviously one of the "intersection blocks" 2468,3478 and 5678 must be chosen, but the third family may be represented by the block, which does not exist in the other two. Since $(5\cdot4):8>2$, some elements of S must occur thrice in five blocks and the subfamily B of Φ_1 must not be chosen.

6 blocks: H, G∪{2358}, G∪{2457}, G∪{5678}, E∪{2367,2457},
E∪{2358,2468}, F∪{2358,3456}, F∪{2367,3456}, F∪{2468,3456},
F∪{3456,3478}, C∪{3456,3478,5678}, D∪{2468,3478,5678}

$\Phi_1\setminus G$ = {1458,1467}, so the element 4 has a special position in G (besides 1). This implies that there are three non-isomorphic possibilities to add the sixth block to G; it either is the complement (of a block of G) or is not; in the first case we must make a difference between 5678 and the complements containing 4. There is, up to an isomorphism, just one family of six blocks with the ground-set of cardinality 7. It arises from E. The elements 2,3,4,5,6,7 share the isomorphic position in E and we still have two non-isomorphic possibilities; to add one or two complements to E. We ignore the first one, for it necessarily includes a 2-subset of S, which does not contain 1 and which appears in three blocks (such a situation would contradict our favorizing 1) and is isomorphic to the case F∪{2367,3456}. The family F contains 1 four times and 2 three times; one of the added blocks must be therefore any of 3456,3478,5678. Notice that the elements 4,6,8 share the isomorphic position in F and so do 3,5,7. If the sixth block is not a complement of F, then there are two possibilities; its intersection with the only complement may be included in {3,5,7} or not. If both the blocks added to F are the complements of F, then we have two non-isomorphic cases depending on whether the "special" 2468 is included or not.

None of the blocks with 2 may be added to C; this leaves just one possibility. As 1 appears in three blocks of D, we have (because of $(6\cdot 4):8 = 3$) that each element of S must occur in three blocks of the corresponding family of six blocks. Thus three blocks containing 8 should be added to D, but it is easy to see that 2358 must not be added.

7 blocks: Φ_1, H∪{2358}, H∪{2367}, G∪{2358,2367}
G∪{2358,2457}, G∪{2358,5678}, G∪{2457,2468}, G∪{2457,3456},
G∪{2457,5678}, E∪{2367,2457,3456}, E∪{2358,2468,3478},
F∪{2468,3456,3478}, F∪{3456,3478,5678}, F∪{2358,3478,5678}.

There are two non-isomorphic possibilities to add the seventh block to H; it may be either 2358 or one of the complements. We have already noticed that the element 4 plays a special role in G. The block 5678 is the only complement of G without 4. We may have none, one or two complements among the two blocks added to G, but in the last two cases the possibilities which include 5678 are not isomorphic to those which do not. What is more, 2 and 3 are the only elements of S which appear in exactly three blocks of G. On account of that fact, in the case when two complements with 4 are chosen, we make a difference between the case when the same one and the different two of the elements 2,3 appear in these two complements. Since the element 1 appears only four times in E, (when adding three blocks to E) at least one block must be chosen from each of the families {3456,3478,5678},{2457,2468,5678},{2358,2367, 5678},{2367,2468,3478},{2358,2457,3478},{2358,2468,3456}. If the chosen three sets do not contain 8, then we have the family of heptahedron hyperplanes. The element 1 does not occur in the same 2-subset in three blocks of E. Any block from {2367,2457,3456} has a 2-intersection contained in a block of E, with any block from {2358,2468,3478,5678}. Thus the only possibility left is to add to E three of the four blocks containing 8.

As for the family F, it is obvious that at most one of the blocks containing 2 can be added. If all the three blocks added to F are the complements, then we differentiate the cases when the "special" 2468 is among them and when is not. If just two of the added blocks are the complements (none of them may be 2468), then there are two non-isomorphic cases depending on whether there exist two added hyperplanes with the intersection included in the "special" set {3,5,7} or not. The first of these cases, however, is isomorphic to the case $G\cup\{2457,5678\}$.

None of the subfamilies C,D of Φ_1 may be possibly used for the production of subfamilies of Φ with seven blocks. The average number of appearances of elements of S in seven blocks is 28:8>3, so there exists an element which appears in more than three blocks.

We conclude that there are

$$2 \cdot (1 + 1 + 2 + 4 + 7 + 9 + 12) + 14 = 86$$

non-isomorphic B-families.

There is a very simple routine to check that all the constructed B-families are really pairwise non-isomorphic. In fact, each two of the constructed subfamilies of ϕ (with the same number of blocks) differ in at least one of the following:

a)) the deck of incidence numbers of elements 1,2,...,8, without regard to order

b)) the number of 2-subsets of S appearing in three blocks.

Since each element (respectively 2-subset) of S appears in exactly seven (respectively three) blocks of Φ, these differences are preserved with the complementary subfamilies. This immediately gives, without the use of Theorem 3, the number of non-isomorphic subfamilies of Φ with more than seven blocks, which is the same to the number of those with less then seven blocks. We cannot prove in this way, however, that some new non-isomorphic possibilities do not arise with larger numbers of blocks or, equivalently, that a)) and b)) completely determine (up to an isomorphism) a subfamily of ϕ.

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REZIME

NOVA KONSTRUKCIJA PEJVING MATROIDA RANGA 4 NA SKUPU OD 8 ELEMENATA (I)

U radu |2| je pomoću kompjutera konstruisano svih 950 neizomorfnih prostih matroida na skupu od 8 elemenata.

Koristeći elementarne metode, bez pomoći kompjutera, mi izvodimo novu konstrukciju potklase P od 322 pejving matroida ranga 4 na skupu od 8 elemenata. (Pot)klasa P je po svojoj prilici najkomplikovanija, kad je u pitanju (ne)izomorfnost matroida, kod matroida na skupovima od najviše 8 elemenata.

Klasa P se razbija u tri disjunktne potklase. Konstrukcija prve dve je data u ovom radu, a konstrukcija treće potklase od P je data u narednom radu |1| (ova dva rada čine celinu).

Prilikom konstrukcije detaljno ispitujemo mogućnosti za javljanje neizomorfnih matroida. Pritom se u značajnoj meri služimo sa tri pomoćne klase grafova (koje su u obostrano jednoznačnoj korespodenciji sa odredjenim potklasama od P), kao i nekim osobinama Štajnerovog sistema S(3,4,8).