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AN EXPONENTIALLY FITTED SCHEME ON A NON-UNIFORM MESH*

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ABSTRACT

The discretisation (6) of problem (1) is construcеed on a non-uniform mesh. An exponentially fitted scheme is used and the linear convergence uniform on a small perturbation parameter is proved. In the case of a uniform mesh, our scheme reduces to the well known scheme of |1|. Numerical results are obtained on a special mesh, which is similar to the one in |3|.

1. INTRODUCTION

In this paper we shall consider the problem

$$(1)\qquad L_\varepsilon u(x) := -\varepsilon^2 u''(x) + b^2(x)u(x) = f(x), \quad x \in I = [0,1], \qquad u(0) = U_0, \quad u(1) = U_1,$$

where $b(x) \geq \beta > 0$, $x \in I$, $0 < \varepsilon \leq \varepsilon_0$, $\beta, \varepsilon, \varepsilon_0, U_0, U_1 \in \mathbb{R}$ and $b, f \in C^1(I)$. The parameter ε is a small perturbation parameter.

It is well known that there exists a unique solution $u_\varepsilon \in C^3(I)$ to problem (1). Furthermore, we have (|1|,|3|):

$$(2)\qquad |u^{(i)}(x)| \leq M\varepsilon^{-i}, \quad x \in I, \quad i=0,1,2,3 .$$

Here, and thereafter, M will denote each positive constant independent of ε . In section 3, M will be independent of discre-

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tisation mesh as well. The dependence on ε will always be denoted by writing ε as a subscript.

The solution u_ε has, in general, two boundary layers near the endpoints of the interval I. Our purpose is to solve (1) numerically by the difference method on a non-unifirm mesh. The reason for using a non-uniform mesh is our aim to obtain more mesh points in the region of boundary layers, those width is $O(\varepsilon)$, cf. |6|. We shall use an exponentially fitted scheme, which in the case of uniform mesh reduces to the well known schemes of |1|. For such a scheme we shall prove the convergence uniform in ε.

2. PROPERTIES OF THE EXACT SOLUTION

A special technique was used in |2| to obtain some differentiability properties of the solution to the boundary value problem with a small parameter, containing the first derivative. We shall here use a similar technique for our problem.

THEOREM 1. *For the solution* u_ε *of* (1) *we have*

(3) $$u_\varepsilon(x) = p_\varepsilon v_\varepsilon(x) + z_{0,\varepsilon}(x), \quad 0 \le x \le 1/2,$$

(4) $$u_\varepsilon(x) = q_\varepsilon w_\varepsilon(x) + z_{1,\varepsilon}(x), \quad 1/2 \le x \le 1,$$

where

$$v_\varepsilon(x) = \exp(-\frac{b(0)}{\varepsilon} x),$$

$$w_\varepsilon(x) = \exp(-\frac{b(1)}{\varepsilon}(1-x)),$$

$$|p_\varepsilon|, |q_\varepsilon| \le M \quad \textit{and}$$

(5) $$|z_{k,\varepsilon}^{(i)}(x)| \le M(1+\varepsilon^{1-i}), \; k=0,1, \quad i=0,1,2,3.$$

P r o o f. Let $x \in [0,1]$. Define

$$y_\varepsilon(x) = u_\varepsilon(x) - p_\varepsilon v_\varepsilon(x) - q_\varepsilon w_\varepsilon(x)$$

and take p_ε and q_ε so that $y_\varepsilon'(0)=0$, $y_\varepsilon'(1)=0$. It is easily obtained that such p_ε, q_ε exist and $|p_\varepsilon|, |q_\varepsilon| \le M$. Hence, we have

$$|y_\varepsilon(x)| \le M.$$

Then we get

$$L_\varepsilon y'_\varepsilon(x) = g'_\varepsilon(x) - (b^2(x))' y'_\varepsilon(x) =: G_\varepsilon(x) ,$$

where

$$g_\varepsilon(x) = f(x) - p_\varepsilon L_\varepsilon v_\varepsilon(x) - q_\varepsilon L_\varepsilon w_\varepsilon(x) .$$

Now it can be easily proved that, cf. |1|,

$$|g'_\varepsilon(x)| \le M,$$

which implies $|G_\varepsilon(x)| \le M$. Hence, $y'_\varepsilon(x)$ is a solution to the problem

$$L_\varepsilon y'_\varepsilon(x) = G_\varepsilon(x) , \quad x \in [0,1],$$

$$y'_\varepsilon(0) = y'_\varepsilon(1) = 0 ,$$

and because of (2), it follows:

$$|y_\varepsilon^{(i)}(x)| \le M\varepsilon^{1-i}, \quad i=1,2,3, \quad x \in [0,1].$$

Now we can take

$$z_{o,\varepsilon}(x) = y_\varepsilon(x) + p_\varepsilon w_\varepsilon(x), \quad x \in [0,1/2] ,$$

and since

$$|w_\varepsilon(x)| \le M, \quad x \in [0,1/2] ,$$

we obtain (3) and (5) for k=0. Similarly we take

$$z_{1,\varepsilon}(x) = y_\varepsilon(x) + q_\varepsilon w_\varepsilon(x), \quad x \in [1/2,1] ,$$

to complete the proof.

A similar result was obtained in |1|. For similar estimates of the derivatives of u_ε cf. |3|.

3. THE SCHEME

We use the discretisation mesh

$$I_h = \{x_i | x_o = 0, \ x_i = x_{i-1} + h_i, \quad i=1,2,\dots,n\} ,$$

where $n \in \mathbb{N}$, $h_i > 0$, $i=1,2,\dots,n$, $h_1+h_2+\dots+h_n = 1$. For simplicity we take $n = 2m$, $m \in \mathbb{N}\setminus\{1\}$, and $x_m = 1/2$. Let $h_i \le h_{i+1}$, $i = 1, 2,\dots,m-1$, and $h_i \ge h_{i+1}$, $i=m+1,\dots,n$, which is a natural assumption bacause of boundary layers. Let $h := \max\{h_m, h_{m+1}\} \le 1/4$.

Instead of 1/4 we can take any fixed constant in (0,1/2), so this inequality is not a real restriction.

On the mesh I_h we form the discretisation of the problem (1):

$$u_o = U_o ,$$

(6) $$L_\varepsilon^h u_i := -\varepsilon^2 \sigma_i D_h u_i + b^2(x_i) u_i = f(x_i), \quad i=1,2,\ldots,n-1 ,$$

$$u_n = U_1 ,$$

where

$$D_h u_i = \frac{2}{(h_i + h_{i+1}) h_i h_{i+1}} (h_{i+1} u_{i-1} - (h_i + h_{i+1}) u_i + h_i u_{i+1}) ,$$

$$i=1,2,\ldots,n-1 ,$$

and

$$\sigma_i = \begin{cases} \sigma_i^o(\rho_o) := \frac{1}{2} \rho_o^2 \dfrac{h_i h_{i+1} (h_i + h_{i+1})}{h_{i+1}(e^{\rho_o h_{i-1}}) + h_i (e^{-\rho_o h_{i+1}} - 1)} & i=1,2,\ldots,m-1 \\ 1, \quad i=m & \\ \sigma_i^1(\rho_n) := \frac{1}{2} \rho_n^2 \dfrac{h_i h_{i+1} (h_i + h_{i+1})}{h_{i+1}(e^{-\rho_n h_i} - 1) + h_i (e^{\rho_n h_{i+1}} - 1)} & i=m+1,\ldots,n-1 \end{cases}$$

$\rho_i = b(x_i)/\varepsilon$, $i=0,1,\ldots,n.$

The solution of (6) is $u_h = [u_o, u_1, \ldots, u_n]^T \in \mathbb{R}^{n+1}$ (here we omit the subscript ε). Let u_ε^h denote the restriction of u_ε to the mesh I_h : $u_\varepsilon^h = [u_\varepsilon(x_o), u_o(x_1), \ldots, u_\varepsilon(x_n)]^T \in \mathbb{R}^{n+1}$.

We use the maximum norm $\|v\| = \max_{0 \le i \le n} |v_i|$ for any vector

$v = [v_o, v_1, \ldots, v_n]^T \in \mathbb{R}^{n+1}$ and $\|A\| = \max_{0 \le i \le n} \sum_{j=o}^{n} |a_{ij}|$ for any matrix $A = [a_{ij}] \in \mathbb{R}^{n+1,n+1}$.

THEOREM 2. *It is valid that*

$$\|u_\varepsilon^h - u_h\| \le M h .$$

P r o o f. The consistency error of the operator L_ε^h is by definition:

$$r_i := L_\varepsilon^h u_\varepsilon(x_i) - (L_\varepsilon u_\varepsilon)_{x=x_i} = -\varepsilon^2(\sigma_i D_h u_\varepsilon(x_i) - u''_\varepsilon(x_i)) ,$$

$$i = 1,2,\ldots,n-1 .$$

Let $r_h = [0, r_1, r_2, \ldots, r_{n-1}, 0]^T \in \mathbb{R}^{n+1}$ and $f_h = [U_o, f(x_1), f(x_2), \ldots, f(x_{n-1}), U_1]^T \in \mathbb{R}^{n+1}$. Then (6) can be written in the form

$$A_\varepsilon^h u_h = f_h ,$$

where $A_\varepsilon^h \in \mathbb{R}^{n+1,n+1}$ is tridiagonal matrix. Since $\sigma_i > 0$, $i=1, 2,\ldots,n-1$, (see Lemma 1. below), A_ε^h is an L-matrix, inverse monotone and (cf. eg. |4|,|5|):

$$\| (A_\varepsilon^h)^{-1} \| \le \max(1, \beta^{-2}) .$$

Now we have

$$A_\varepsilon^h (u_\varepsilon^h - u_h) = f_h + r_h - r_h = r_h$$

and

$$\| u^h - u_h \| \le \| (A_\varepsilon^h)^{-1} \| \; \| r_h \| \le M \| r_h \| .$$

To complete the proof it is sufficiant to show that

$$\| r_h \| \le M h .$$

or

(7) $$|r_i| \le M h , \quad i=1,2,\ldots,n-1 .$$

First we consider $i=1,2,\ldots,m-1$. The fitting factor $\sigma_i = \sigma_i^o(\rho_o)$ is chosen so that

$$\sigma_i D_h v_\varepsilon(x_i) = v''_\varepsilon(x_i) ,$$

where $v_\varepsilon(x)$ is given Theorem 1. This fact and (3) imply that

(8) $$|r_i| = \varepsilon^2 |\sigma_i D_h z_{o,\varepsilon}(x_i) - z''_{o,\varepsilon}(x_i)| .$$

Since there exists a point $\theta_i \in [x_{i-1}, x_{i+1}]$ such that

$$D_h z_{o,\varepsilon}(x_i) = z''_{o,\varepsilon}(\theta_i) ,$$

from (8) it follows that

$$|r_i| \le \varepsilon^2 |1-\sigma_i| |z''_{o,\varepsilon}(\theta_i)| + \varepsilon^2 |z''_{o,\varepsilon}(\theta_i) - z''_{o,\varepsilon}(x_i)| \le$$
$$\le \varepsilon^2 |1-\sigma_i| K_2 + \varepsilon^2 h_{i+1} K_3 ,$$

where

$$K_j = \max_{0\le x\le 1/2} |z_{o,\varepsilon}^{(j)}| \ , \quad j=2,3 \ .$$

From Theorem 1. we have

$$K_j \le M\ \varepsilon^{1-j} \ , \quad j=2,3$$

and

$$|r_i| \le M(\varepsilon|1-\sigma_i| + h) \ .$$

In Lemma 2. we shall prove that

$$\varepsilon|1-\sigma_i| \le M\, h_{i+1} \ .$$

Because of that, (7) is valid for i=1,2,...,m-1.

In the case when i=m+1,...,n-1 the proof of (7) is analogous. We use (4) and

$$\sigma_i^1(\rho_n) D_h w_\varepsilon(x_i) = w_\varepsilon''(x_i) \ .$$

When i=m, we have

$$|r_m| \le \varepsilon^2 h \max_{x_{m-1}\le x\le x_{m+1}} |u_\varepsilon^{(3)}(x)| \ ,$$

and because of Theorem 1. :

$$|r_m| \le \varepsilon^2\, hM(1+\varepsilon^{-2}+\varepsilon^{-3}(\exp(-\frac{b(0)}{\varepsilon}\, x_{m-1}) + \\ + \exp(-\frac{b(1)}{\varepsilon}\,(1-x_{m+1})))) \ .$$

Since $x_{m-1} = 1/2 - h_m \ge 1/4$ and $x_{m+1} = 1/2 + h_{m+1} \le 3/4$, we get

$$|r_m| \le M\, h \ .$$

Here we used $\varepsilon^{-1}\exp(-\frac{\beta}{4\varepsilon}) \le M$. This completes the proof of Theorem 2.

LEMMA 1. *For $\rho > 0$ it is valid that $\sigma_i^o(\rho) > 0$, i=1,2, ...,m-1.*

P r o o f. Let i=1,2,...,m-1 and

$$g(\rho) := h_{i+1}(e^{\rho h_i} - 1) + h_i(e^{-\rho h_{i+1}} - 1) \ .$$

We have $g(0) = 0$ and $g'(\rho) > 0$ for $\rho > 0$. Now it follows that $g(\rho) > 0$ and $\sigma_i^o(\rho) > 0$, $\rho > 0$.

LEMMA 2. *For* $\rho > 0$ *it is valid that*

(9) $$|1-\sigma_i^o(\rho)| \le h_{i+1}\rho, \quad i=1,2,\dots,m-1 .$$

P r o o f. Let $i=1,2,\dots,m-1$ and let us denote $k=h_i$, $K_i=h_{i+1}$. First we shall prove

(10) $$\sigma_i^o(\rho) \le 1+K\rho .$$

This inequality is equivalent with

(11) $$\sum_{j=3}^{\infty} \frac{\rho^j}{j!} K(k+K) R_j \ge 0 ,$$

where

$$R_j = j(k+K)^{j-2}K + (k+K)^{j-1} - (j+1)K^{j-1} - \frac{1}{2} j(j-1)kK^{j-2} .$$

Now we have that $R_3 = k^2 + kK \ge 0$ and from $R_j \ge 0$, $j \ge 3$, it follows that

$$\begin{aligned} R_{j+1} &\ge (k+K)((j+1)K^{j-1} + \frac{1}{2} j(j-1)kK^{j-2}) + (k+K)^{j-1}K - \\ &- (j+2)K^j - \frac{1}{2}(j+1)jkK^{j-1} \ge \\ &\ge \frac{1}{2} jkK^{j-2}((j-1)(k+K)-(j+1)K) + (j+1)kK^{j-1} = \\ &= \frac{1}{2} j(j-1)k^2K^{j-2} + kK^{j-1} \ge 0 . \end{aligned}$$

Hence, (11) is valid and so is (10). Using a similar technique we can get

(12) $$\sigma_i^o(\rho) \ge 1-\rho K .$$

The inequalities (11) and (12) imply (9) .

4. SOME REMARKS

The difference scheme used in (6) has quasi-constant fitting factors in the sense of |5|. Namely, when the mesh is uniform ($h_i = h$, $i=1,2,\dots,n$), we get the constant fitting factors from |1|:

$$\sigma_i^o(\rho_o) = (\rho_o h/2)^2 \mathrm{sh}^{-2}(\rho_o h/2), \quad \sigma_i^1(\rho_n) = \sigma_i^o(\rho_n) .$$

Because of the non-uniform mesh we have proved linear convergence uniform in ε. In |1| it is shown that on the uniform mesh the quadratic convergence uniform in ε can be obtained when $u_\varepsilon \in C^4(I)$ and $b(0) = b(1)$.

A similar scheme with quasi-constant fitting factor was obtained in |5|, but our scheme seems to be simpler.

It is possible to construct a scheme with variable fitting factors. We can take (6) with

$$\sigma_i = \begin{cases} \sigma_i^0(\rho_i) , & i=1,2,\dots,m-1 \\ 1 , & i=m \\ \sigma_i^1(\rho_i) , & i=m+1,\dots,n-1 \end{cases}$$

Such a scheme was also considered in |1| on the uniform mesh.

5. NUMERICAL EXAMPLE

We shall condider a simple test problem from |7|,|4|:

$$-\varepsilon^2 u'' + u = 1, \quad x \in I, \quad u(0) = u(1) \ .$$

For the solution of this problem we have $u_\varepsilon(1/2+x) = u_\varepsilon(1/2-x)$, $x \in [0,1/2]$ and we shall give numerical results on $[0,1/2]$.

Let m_1 be the number of mesh points x_i in $[0,\varepsilon]$ and let

$$E = \max\{|u_\varepsilon(x_i) - u_i| \, | \, x_i \in (\varepsilon,1/2]\} \ ,$$

$$E_1 = \max\{|u_\varepsilon(x_i) - u_i| \, | \, x_i \in [0,\varepsilon]\} \ ,$$

$$P = \max\{|u_\varepsilon(x_i) - u_i| / |u_\varepsilon(x_i)| \, | \, x_i \in (\varepsilon,1/2]\} \cdot 100 \ ,$$

$$P_1 = \max\{|u_\varepsilon(x_i) - u_i| / |u_\varepsilon(x_i)| \, | \, x_i \in [0,\varepsilon]\} \cdot 100 \ .$$

It is convenient to use such a mesh which changes automatically when ε does. This enables us to achieve the same number of mesh points in the boundary layer whatever value ε takes. A special mesh of a such type was given in |3|, where the mesh points are determined as values of function $\lambda_\varepsilon(s)$ at equidistant points (see |4| as well). This mesh implies a convergence uniform in ε for the classical scheme

$$-\varepsilon^2 D_h u_i + b^2(x_i)u_i = f(x_i).$$

Here we shall use a similar but simpler function $\lambda_\varepsilon(s)$:

$$\lambda_\varepsilon(s) = \begin{cases} \psi_\varepsilon(s) , & s \in [0,\alpha] \\ \psi_\varepsilon(\alpha) + \psi_\varepsilon'(\alpha)(s-\alpha) , & s \in [\alpha,1/2] \end{cases}$$

where $\psi_\varepsilon(s) = a\varepsilon s/(q-s)$, $s \in [0,q)$. q and a are fixed constants: $q \in (0,1/2)$ and $a>0$, and $(\alpha,\psi_\varepsilon(\alpha))$ is the point of contact of a tangent line taking the value 1/2 at $s=1/2$, to the curve $\psi_\varepsilon(s)$. We take an $\varepsilon \le q$. Then there exists unique $\alpha \in [0,q)$ and it be easily determined from

$$\psi_\varepsilon(\alpha) = 1/2 + \psi_\varepsilon'(\alpha)(\alpha-1/2),$$

which reduces to a quadratic equation.

We shall take mesh I_h with

$$x_i = \lambda_\varepsilon(i/n), \quad i=0,1,\dots,m,$$

and test our scheme (6).

TABLE 1. $a=1$, $q=0.4$, $m=20$

ε	m_1	E	E_1	P	P_1
10^{-2}	9	$1.19\cdot10^{-7}$	$2.09\cdot10^{-7}$	$1.19\cdot10^{-5}$	$3.23\cdot10^{-4}$
10^{-4}	9	$5.96\cdot10^{-8}$	$1.64\cdot10^{-7}$	$8.24\cdot10^{-6}$	$2.54\cdot10^{-4}$
10^{-6}	9	$1.79\cdot10^{-7}$	$2.98\cdot10^{-7}$	$2,47\cdot10^{-5}$	$3.12\cdot10^{-4}$
10^{-10}	9	$1.19\cdot10^{-7}$	$2.09\cdot10^{-7}$	$1.65\cdot10^{-5}$	$3.12\cdot10^{-4}$

TABLE 2. $a=0.3$, $q=0.499$, $m=20$

ε	m_1	E	E_1	P	P_1
10^{-2}	16	$7.75\cdot10^{-7}$	$2.04\cdot10^{-6}$	$1.10\cdot10^{-4}$	$1.89\cdot10^{-3}$
10^{-4}	16	$1.07\cdot10^{-6}$	$2.29\cdot10^{-6}$	$1.53\cdot10^{-4}$	$1.99\cdot10^{-3}$
10^{-6}	16	$1.07\cdot10^{-6}$	$2.40\cdot10^{-6}$	$1.53\cdot10^{-4}$	$2.04\cdot10^{-3}$
10^{-10}	16	$1.19\cdot10^{-6}$	$2.35\cdot10^{-6}$	$1.70\cdot10^{-4}$	$2.10\cdot10^{-3}$

With a small number of mesh points we achieve extremely good results. The errors change slightly for different values of ε. A high percentage of mesh points in the boundary layer is obtained.

REFERENCES

|1| Doolan,E.P.,J.J.H.Miller, W.H.A.Schilders, *Uniform numerical methods for problems with initial and boundary layers, Boole Press, Dublin, 1980.*

|2| Kellog R.B., A.Tsan, *Analysis of some difference approximations for a singular perturbation problem without turning points, Math. Comp. 32(1978), 1025-1039.*

|3| Bahvalov A.S., *K optimizacii metodov rešenija kraevyh zadač pri naličii pograničnogo sloja, Ž.vyčisl. mat. i mat.fiz., T9, No 4, 841-859, 1969.*

|4| Herceg,D., R.Vulanović, *Some finite difference schemes for a singular perturbation problem on a non-uniform mesh, Zbornik radova PMF u Novom Sadu, Ser.Mat. 11(1981),117-134.*

|5| Herceg,D., *A uniformly convergent scheme with quasi-constant fitting factors, Zbornik radova PMF u Novom Sadu, Ser.Mat. 11 (1981), 105-115.*

|6| Pearson,C.E., *On a differential equation of boundary layer type, J. Math. Phys. 47 (1968), 134-154.*

|7| Herceg,D., *O korišćenju neekvidistantne mreže kod diferencnih postupaka, Zbornik radova PMF u Novom Sadu, knjiga 10(1980), 102-112.*

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REZIME

JEDNA EKSPONENCIJALNO FITOVANA ŠEMA NA NERAVNOMERNOJ MREŽI

U radu se posmatra diskretizacija problema (1) na neravnomernoj mreži. Koristi se eksponencijalno fitovana šema (6) i dokazuje linearna konvergencija uniformna po malom parametru ε . U slučaju kada je mreža ekvidistantna posmatrana šema svodi se na poznatu šemu iz |1| sa konstantnim faktorom fitovanja.

Numerički rezultati dobijeni su na specijalno konstruisanoj mreži, koja je formirana slično kao u |3|.