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STABILITY AND MONOTONICITY PROPERTIES OF STIFF QUASILINEAR BOUNDARY PROBLEMS

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ABSTRACT

The paper is concerned with nonlinear two-point boundary value problems of the singular perturbation type. Besides giving existence and uniqueness statements we investigate two kinds of stability properties of the boundary problems, namely the stability with respect to time-evolution and the stability with respect to small perturbations. Both kinds of stability are essentially obtained by inverse-monotonicity properties of classes of linear boundary value problems.

1. INTRODUCTION

This paper is concerned with boundary value problems (P_ε) of the form

$$T_\varepsilon u \equiv -\varepsilon u'' + \alpha(x,u)u' + \beta(x,u) = 0, \quad x \in [0,1] ,$$

$$(P_\varepsilon) \qquad Ru = (u(0), u(1)) = (A,B),$$

where $\varepsilon > 0$ is small. Among others we shall give conditions implying the existence of a unique solution u of (P_ε) for $0<\varepsilon\leq\varepsilon_0$. If (P_ε) is treated numerically, let us say with a difference method on a grid $G = \{x_0,\ldots,x_n\}$, then the operator equation (P_ε) is replaced by a finite dimensional system of equations

$$T_{\varepsilon,G} u = \gamma_G, \ u \in \mathbb{R}^G ,$$

with a solution $u_{\varepsilon,G} \in \mathbb{R}^G$, let us say. The truncation error is

$$\tau_{\varepsilon,G} = T_{\varepsilon,G} u_\varepsilon|_G - T_{\varepsilon,G} u_{\varepsilon,G} ,$$

where $u_{\varepsilon|G}$ denotes the restriction of u_ε to the grid G. Thus it is the question of the stability of $T_{\varepsilon,G}$ to estimate the error $u_{\varepsilon|G} - u_{\varepsilon,G}$ by $\tau_{\varepsilon,G}$. The stability of $T_{\varepsilon,G}$ can be stated in the form of a stability inequality |5|.

$$||u-v|| \leq c_\varepsilon ||T_{\varepsilon,G}u-T_{\varepsilon,G}v||' \quad \text{for all } u,v \in \mathbb{R}^G ,$$

where $|| \; ||$, $|| \; ||'$ are suitable norms and c_ε is the stability constant. It cannot be expected that $T_{\varepsilon,G}$ has better stability properties than T_ε itself, and the stability properties of T_ε can be very bad for small ε. It follows from |1,19| that even in some linear cases of (P_ε) the stability constants for T_ε with respect to the maximum norm grow like $e^{\sigma/\varepsilon}$ with $\sigma > 0$ for $\varepsilon \to 0^+$. It is a purpose of this paper to separate cases with better stability behaviour: we shall give conditions under which the stability constants are $O(1)$ or $O(\varepsilon^{-1/2})$ or $O(\varepsilon^{-1})$ for $\varepsilon \to 0^+$. These cases should be treatable by suitable numerical methods for fairly small values of ε (cf.|2,6,11,13-15,20-24; 29-31|) though standard methods may fail.

Throughout the paper we shall work with monotonicity methods - with respect to the natural ordering, and there are some relations of our techniques to the method of differential inequalities as used in |9,16,17|. But the intention of the present paper is different. In |16,17| the essential assumption is that the reduced differential equation ($\varepsilon = 0$) has one or more solutions with certain properties, and then it is investigated whether there is a solution of the ε-dependent problem in a neighbourhood of these solutions. These valuable techniques might lead to a priori estimates for u_ε. In the present paper we shall investigate the inverse of (T_ε, R) without referring to a reduced problem, and we are motivated by the stability question as mentioned above. Actually, if a priori estimates for u_ε are known - for example obtained by the methods of |9,16,17|, then it is more likely that our conditions can be fulfilled by altering the coefficients $\alpha(x,u)$ and $\beta(x,u)$ outside the a priori domain. In this way the two techniques might be used together. Nevertheless, we always take global assumptions for $\alpha(x,u)$ and $\beta(x,u)$, and if these are met, then an a priori analysis is not necessary.

The solutions u_ε of (P_ε) describe the stationary states of the evolution equation

$$
\begin{aligned}
&u_t - \varepsilon u_{xx} + \alpha(x,u)u_x + \beta(x,u) = 0, \quad x \in [0,1], \ t>0,\\
(1)\qquad &u(0,t) = A, \quad u(1,t) = B, \quad t>0\\
&u(x,0) = u_o(x), \quad x \in [0,1],
\end{aligned}
$$

which arises among others in convective-diffusion type flow problems (cf. |10|). Under our conditions (to be described below) the solution u_ε of (P_ε) is always a stable state of the evolution equation. Concerning (1) it is of great interest whethet T_ε is monotone in the sense of Minty |25|. However, it turns out that T_ε is never Minty-monotone if α actually depends on u and ε is sufficiently small. This is one reason for us to investigate T_ε with monotonicity concepts as induced by the natural ordering.

Conceptually this paper owes to |6| where the same concepts of a stability inequality and of monotonicity are used, and where stability and inverse monotonicity for T_ε and for discrete analogues was proved under the special assumption $\alpha(x,u) = \alpha(x) \geq \alpha_o > 0$. I thank Prof.Dr.E.Bohl for his encouraging interest and for discussions about these problems.

2. PRELIMINARIES AND STATEMENT OF THE MAIN RESULTS

The problem (P_ε) is considered under the assumptions

$$A,B,\varepsilon \in \mathbb{R}, \ \varepsilon > 0, \ \alpha,\alpha_u,\beta,\beta_u \in C([0,1] \times \mathbb{R})$$

where $\alpha_u = \frac{\partial\alpha}{\partial u}$ etc. In cases II and IV below also $\alpha_x \in C([0,1] \times \mathbb{R})$ is required. R always denotes the boundary operator

$$Ru = (u(0), u(1))$$

and we use the pointwise ordering, i.e.

$$u \le v \iff u(x) \le v(x) \quad \text{for all} \quad x \in [0,1] \;,$$
$$u < v \iff u(x) < v(x) \quad \text{for all} \quad x \in [0,1] \;,$$
$$Ru \le Rv \iff u(0) \le v(0) \quad \text{and} \quad u(1) \le v(1)$$

where $u,v \in C[0,1]$. For any $u \in C^2[0,1]$ the linearization of T_ε at u is

(2) $$(DT_\varepsilon u)w = -\varepsilon w'' + \alpha(x,u)w' + \{\alpha_u(x,u)u' + \beta_u(x,u)\}w, w \in C^2[0,1] \;.$$

Concerning the evolution equation (1) the following definition is in common use.

DEFINITION 1. *A solution* u_ε *of* (P_ε) *is called stable, if the smallest eigenvalue of the eigenvalue problem*

$$(DT_\varepsilon u_\varepsilon)w = \lambda w, \quad Rw = (0,0)$$

is positive.

Because of the following Lemma the stability of u_ε in the sense of Definition 1 is equivalent to the inverse monotonicity of $(DT_\varepsilon u_\varepsilon, R)$.

LEMMA 1 *(|35, ch. 1|)* *Let* $p,q \in C[0,1]$, $\varepsilon > 0$,

(3) $$L_\varepsilon w = -\varepsilon w'' + pw' + qw, \quad w \in C^2[0,1].$$

The following conditions are equivalent:

(i) (L_ε, R) *is inverse monotone, i.e.*

$$L_\varepsilon w \ge 0, \; Rw \ge (0,0) \implies w \ge 0 \quad \textit{for all } w \in C^2[0,1] \;.$$

(ii) *The smallest eigenvalue of the eigenvalue problem*

$$L_\varepsilon w = \lambda w \;, \quad Rw = (0,0)$$

is positive.

(iii) *There exists* $e \in C^2[0,1]$ *with*

$$e \ge 0, \quad L_\varepsilon e \ge 0, \quad (L_\varepsilon Re) \ne (0,0,0) \;.$$

(iv) *Green's function* $G_\varepsilon(x,y)$ *corresponding to* (L_ε, R) *and the solutions* $g_\varepsilon, h_\varepsilon \in C^2[0,1]$ *of*

(4) $$L_\varepsilon g_\varepsilon = L_\varepsilon h_\varepsilon = 0, \quad Rg_\varepsilon = (1,0), \quad Rh_\varepsilon = (0,1)$$

exist and are nonnegative.

The following Theorem describes the sufficient conditions for the following properties of (P_ε), which can depend on ε .

PI: (P_ε) has a solution $u_\varepsilon \in C^2[0,1]$.

PII: (T_ε,R) is inverse monotone, i.e.

$$T_\varepsilon u \leq T_\varepsilon v, \quad Ru \leq Rv \Rightarrow u \leq v \quad \text{for all} \quad u,v \in C^2[0,1] .$$

Especially, (P_ε) has at most one solution.

PIII: The smallest eigenvalue of any eigenvalue problem

$$(5) \qquad (DT_\varepsilon u)w = \lambda w, \quad Rw = (0,0) ,$$

where $u \in C^2[0,1]$ is arbitrary, is positive. Especially, any solution of (P_ε) is stable. In some cases all eigenvalues of (5) have a common lower bound independent of u. This lower bound then is denoted by $\lambda_1(\varepsilon)$.

THEOREM 1 *Let the conditions of one of the following cases hold for all* $(x,u) \in [0,1] \times \mathbb{R}$ *where* $\alpha_0,\beta_0,\gamma_0 \in \mathbb{R}$:

Case I: $\alpha(x,u) = \alpha(x) \geq \alpha_0 > 0, \quad \beta_u(x,u) \geq \beta_0 ,$
$\alpha_0^2 + 4\varepsilon\beta_0 > 0, \quad \varepsilon > 0 .$

Case II: $\alpha(x,u) \geq \alpha_0 > 0, \quad (\beta_u - \alpha_x)(x,u) \geq \gamma_0 ,$
$\alpha_0^2 + 4\varepsilon\gamma_0 > 0, \quad \varepsilon > 0 .$

Case III: $\alpha(x,u) = \alpha(x), \quad \beta_u(x,u) \geq 0, \quad \varepsilon > 0.$

Case IV: $(\beta_u - \alpha_x)(x,u) \geq 0, \quad \varepsilon > 0 .$

Then P II *and* P III *hold. In cases* I,I,III *also* P I *holds.*

In the following Theorems we give the stability inequalities for cases I-IV separately, partly under further conditions. In some cases we also give lower bounds for $\lambda_1(\varepsilon)$ where $\lambda_1(\varepsilon)$ is defined in P III. The norms

$$\|u\|_\infty = \max\{|u(x)| : 0 \leq x \leq 1\}, \quad \|u\|_1 = \int_0^1 |u(x)|dx$$

are used.

THEOREM 2. *Assume case I with* $\beta_o \le 0$. *For all* $u,v \in C^2[0,1]$ *holds*

$$|u(x)-v(x)| \le e^{-2\beta_o/\alpha_o} \{H_\varepsilon \|T_\varepsilon u - T_\varepsilon v\|_1 + |u(0)-v(0)| + |u(1)-v(1)| e^{(x-1)\alpha_o/\varepsilon}\},\ x \in [0,1]$$

with $H_\varepsilon = (\alpha_o^2 + 4\varepsilon\beta_o)^{-1/2}$.

Furthermore $\lambda_1(\varepsilon) \ge \alpha_o^2/4\varepsilon + \beta_o$.

THEOREM 3. *Assume case II with* $\gamma_o \le 0$. *For all* $u,v \in C^2[0,1]$ *with* $u(0)=v(0)$ *holds*

$$|u(x)-v(x)| \le e^{-2\gamma_o/\alpha_o}\{\bar{H}_\varepsilon \|T_\varepsilon u - T_\varepsilon v\|_1 + |u(1)-v(1)| e^{(x-1)\alpha_o/\varepsilon}\},\ x \in [0,1] \tag{6}$$

with $\bar{H}_\varepsilon = (\alpha_o^2 + 4\varepsilon\gamma_o)^{-1/2}$.

Furthermore $\lambda_1(\varepsilon) \ge \alpha_o^2/4\varepsilon + \gamma_o$.

Assume Case IIa: the conditions of case II and $\alpha_1 \ge \alpha(x,u)$. Then for all $u,v \in C^2[0,1]$ holds

$$|u(x)-v(x)| \le S_\varepsilon + e^{-2\gamma_o/\alpha_o}|u(0)-v(0)|(1+\bar{H}_\varepsilon(\alpha_1-\alpha_o))$$

where S_ε is the right hand side of (6).

REMARK 1.

a) The stability inequality of Theorem 2 was proved in |6| for all u,v with $Ru = Rv$.

b) Since H_ε, $\bar{H}_\varepsilon \to 1/\alpha_o < \infty$ for $\varepsilon \to 0^+$ we have stability uniformly in ε. Notice that $e^{(x-1)\alpha_o/\varepsilon}$ is a boundary layer function tending to zero for $x \in [0,1)$ and $\varepsilon \to 0^+$.

c) Since $\lambda_1(\varepsilon) \to \infty$ for $\varepsilon \to 0^+$ the solutions u_ε as stationary states of (1) become 'more stable' if ε gets smaller.

d) Similar results as stated in Theorems 2 and 3 hold if $\alpha(x,u) = \alpha(x) \le -\alpha_o < 0$ or $\alpha(x,u) \le -\alpha_o < 0$. Just use the transformations $x \to 1-x$.

Theorems 2 and 3 are only applicable to problems without turning points. The following Theorems will allow for turning point problems. First consider case III, and let

$$\alpha_{max} = \max\{\alpha(x) : x \in [0,1]\},\quad \alpha_{min} = \min\{\alpha(x) : x \in [0,1]\} .$$

If $\alpha_{min} > 0$ or $\alpha_{max} < 0$ then Theorem 2 is applicable. Thus in the next Theorem assume $\alpha_{max} \geq 0 \geq \alpha_{min}$.

THEOREM 4. *Assume case III and let*

$$\alpha_o = \min\{\alpha_{max}, -\alpha_{min}\} \geq 0 .$$

For all $u,v \in C^2[0,1]$ *holds*

$$||u-v||_\infty \leq c_\varepsilon \, ||T_\varepsilon u - T_\varepsilon v||_1 + \max\{|u(0)-v(0)|, |u(1)-v(1)|\}$$

with

(7) $\quad c_\varepsilon = 1/\varepsilon$ *for* $\alpha_o = 0$, $\quad c_\varepsilon = (e^{\alpha_o/\varepsilon} - 1)/\alpha_o$ *for* $\alpha_o > 0$.

Simple linear examples show that the stability constants c_ε can actually grow as $e^{\sigma/\varepsilon}$ $(\sigma > 0)$ for $\varepsilon \to 0^+$ under the general conditions of Theorem 4 (cf. |1|). The next two Theorems describe subcases of case III where better stability inequalities hold.

THEOREM 5. *Assume*

Case III a: $\alpha(x,u) = \alpha(x)$, $\quad \beta_u(x,u) \geq \beta_o > 0$.

For all $u,v \in C^2[0,1]$ *holds*

$$||u-v||_\infty \leq \max\left\{\frac{1}{\beta_o} ||T_\varepsilon u - T_\varepsilon v||_\infty,\ |u(0)-v(0)|,\ |u(1)-v(1)|\right\}.$$

Furthermore $\lambda_1(\varepsilon) \geq \beta_o > 0$.

THEOREM 6. *Assume*

Case III b: $\alpha(x,u) = \alpha(x)$, $\alpha(x) \geq 0$ *for* $0 \leq x \leq x_o$

$\alpha(x) \leq 0$ *for* $x_o \leq x \leq 1$, $\beta_u(x,u) \geq 0$.

For all $u,v \in C^2[0,1]$ *holds*

$$||u-v||_\infty \le \frac{1}{\varepsilon} \max\{x_o, 1-x_o\} \, ||T_\varepsilon u - T_\varepsilon v||_1 + \max\{|u(0)-v(0)|, \; |u(1)-v(1)|\} .$$

Furthermore $\lambda_1(\varepsilon) \ge \frac{\pi^2}{4} \varepsilon \min\{x_o^{-2}, \; (1-x_o)^{-2}\}$.

Now consider subcases of case IV where $\alpha(x,u)$ can actually depend on u. Also, the conditions allow for $\alpha(x,u) = 0$ for some $(x,u) \in [0,1] \times \mathbb{R}$, thus turning points can occur.

THEOREM 7. *Assume*

Case IV a: $(\beta_u - \alpha_x)(x,u) \ge 0$ *and* $\alpha(x,u) \ge -\alpha_o$ *or* $\alpha(x,u) \le \alpha_o$ *with* $\alpha_o \ge 0$.

For all $a,v \in C^2[0,1]$ *with* $Ru = Rv$ *holds*

$$||u-v||_\infty \le c_\varepsilon ||T_\varepsilon u - T_\varepsilon v||_1$$

with c_ε *as in* (7). *The problem* (P_ε) *has a unique solution* u_ε *for all* $\varepsilon > 0$.

THEOREM 8. *Assume*

Case IV b:

$$(\beta_u - \alpha_x)(x,u) \ge \gamma_o > 0 .$$

For all $u,v \in C^2[0,1]$ *with* $Ru = Rv$ *holds*

$$||u-v||_1 \le \frac{1}{\gamma_o} ||T_\varepsilon u - T_\varepsilon v||_1 .$$

Furthermore $\lambda_1(\varepsilon) \ge \gamma_o > 0$.

THEOREM 9. *Assume*

Case IV c: $(\beta_u - \alpha_x)(x,u) \ge \gamma_o > 0$ *and* $-\alpha_o \le \alpha(x,u) \le \alpha_o$.

For all $u,v \in C^2[0,1]$ *with* $Ru = Rv$ *holds*

$$||u-v||_\infty \le \frac{1}{4\varepsilon\gamma_o} (\alpha_o + (\alpha_o^2 + 4\varepsilon\gamma_o)^{1/2}) \, ||T_\varepsilon u - T_\varepsilon v||_1$$

Case IV c *is a subcase of* IV a *and* IV b, *thus* u_ε *exists and* $\lambda_1(\varepsilon) \ge \gamma_o > 0$.

REMARK 2. a) Theorems 2-9 state stability inequalities for the nonlinear operator T_ε. The results can be stated in terms of stability for all linearized problems

(8) $\quad (DT_\varepsilon u)w = f, \quad Rw = (a,b), \quad f \in C[0,1], \quad a,b \in \mathbb{R}$

where $u \in C^2[0,1]$. E.g., let the conditions of Theorem 2 hold. Then for all $u \in C^2[0,1]$ the solutions w of the linear problem (8) exists and

$$|w(x)| \le e^{-2\beta_o/\alpha_o}\{H_\varepsilon \|f\|_1 + |a| + |b| e^{(x-1)\alpha_o/\varepsilon}\}, \quad x \in [0,1] .$$

b) For simplicity we have always assumed global conditions for the functions α and β. But it is apparent from the proofs below that all stability inequalities hold, if the assumptions on α and β are valid only between the chosen functions u and v occurring in the inequality.

2. THE PROOFS OF THEOREMS 1-9

Let the general conditions as stated in section 2 hold. For $u,v \in C^2[0,1]$ define the linear operator

$$\Delta T_\varepsilon(u,v) = \int_0^1 DT_\varepsilon(v+s(u-v))ds,$$

i.e. $\Delta T_\varepsilon(u,v)w = -\varepsilon w'' + pw' + qw, \quad w \in C^2[0,1]$, where

(9) $\quad p(x) = \int_0^1 \alpha(x,z_s(x)ds, \quad x \in [0,1]$

(10) $\quad q(x) = \int_0^1 \{\alpha_u(x,z_s(x))z_s'(x) + \beta_u(x,z_s(x))\}ds, \quad x \in [0,1]$

with $z_s = v+s(u-v)$.

If $\alpha_x \in C[0,1] \times \mathbb{R})$ note that

$$\Delta T_\varepsilon(u,v)^*w = -\varepsilon w'' - pw' + rw, \quad w \in C^2[0,1],$$

where

(11) $\quad r(x) = (q - p')(x) = \int_0^1 (\beta_u - \alpha_x)(x,z_s(x))ds.$

By the mean value theorem it holds that

$$T_\varepsilon u - T_\varepsilon v = \Delta T_\varepsilon(u,v)(u-v)$$

and thus the following Lemma follows:

LEMMA 2. *Let* $L_\varepsilon = T_\varepsilon(u,v)$ *and assume*

(12) $$L_\varepsilon w = 0, \quad Rw = (0,0) \implies w = 0.$$

Let $G_\varepsilon, g_\varepsilon, h_\varepsilon$ *be defined as in Lemma 1. Then*

$$(u-v)(x) = \int_0^1 G_\varepsilon(x,y)(T_\varepsilon u - T_\varepsilon v)(y)dy$$

$$+ (u-v)(0)g_\varepsilon(x) + (u-v)(1)h_\varepsilon(x), \quad x \in [0,1].$$

Now the proofs of the stability inequalities proceed as follows: The assumptions on α and β lead to estimates for the coefficients p, q, r of L_ε or L_ε^* uniformly for all $u,v \in C^2[0,1]$. Thus we study linear operators L_ε and we estimate G_ε, g_ε and h_ε only using inequalities for the coefficients of L_ε. Apart from L_ε other differential operators like $\hat{L}_\varepsilon$ etc. will occur. Then $\hat{G}_\varepsilon$ denotes Green's function corresponding to $(\hat{L}_\varepsilon,R)$ and $\hat{g}_\varepsilon$, $\hat{h}_\varepsilon$ are defined similarly as g_ε, h_ε (see 4)) with L_ε replaced by $\hat{L}_\varepsilon$. The following three Lemmas are elementary and only stated for convenience of later use.

LEMMA 3. *Let*

(13) $$L_\varepsilon w = -\varepsilon w'' + pw' + qw, \quad w \in C^2[0,1]$$

with $p \in C^1[0,1]$, $q \in C[0,1]$ *and let*

(14) $$p_\varepsilon(x) = \exp\left(\frac{1}{\varepsilon}\int_0^x p(s)ds\right).$$

Then

(i) $L_\varepsilon(p_\varepsilon w) = p_\varepsilon L_\varepsilon^* w$ *for all* $w \in C^2[0,1]$.

(ii) $G_\varepsilon(x,y) = p_\varepsilon(x)G_\varepsilon^*(x,y)/p_\varepsilon(y) = G_\varepsilon^*(y,x)$.

(iii) $g_\varepsilon = p_\varepsilon g_\varepsilon^*$, $h_\varepsilon = p_\varepsilon h_\varepsilon^*/p_\varepsilon(1)$.

In (ii) *and* (iii) *we assumed* (12).

LEMMA 4. *Let* L_ε *be given as in* (13) *with* $p,q \in C[0,1]$.
For $\sigma \in \mathbb{R}$ *let*

$$L_{\varepsilon,\sigma}w = -\varepsilon w'' + (p - 2\varepsilon\sigma)w' + (-\varepsilon\sigma^2 + p\sigma + q)w .$$

Then

(i) $L_\varepsilon(e^{\sigma x}w) = e^{\sigma x}L_{\varepsilon,\sigma}w, \quad w \in C^2[0,1].$

(ii) $G_\varepsilon(x,y) = e^{\sigma(x-y)}G_{\varepsilon,\sigma}(x,y).$

(iii) $g_\varepsilon = e^{\sigma x}g_{\varepsilon,\sigma}, h_\varepsilon = e^{\sigma(x-1)}h_{\varepsilon,\sigma}$.

In (ii) *and* (iii) *we assumed* (12).

LEMMA 5. *Let* L_ε *be given as in* (13) *with* $p, q \in C[0,1]$, *and let* $\hat{L}_\varepsilon w = -\varepsilon w'' + pw' + \hat{q}w$ *with* $\hat{q} \in C[0,1]$, $\hat{q} \leq q$. *If* $(\hat{L}_\varepsilon, R)$ *is inverse monotone, then* (L_ε, R) *is inverse monotone as well and*

$$0 \leq G_\varepsilon \leq \hat{G}_\varepsilon,\ 0 \leq g_\varepsilon \leq \hat{g}_\varepsilon,\ 0 \leq h_\varepsilon \leq \hat{h}_\varepsilon .$$

After these general preliminaries we first prove P II under the conditions of Theorem 1. Let $u,v \in C^2[0,1]$ and set

$$L_\varepsilon w = \Delta T_\varepsilon(u,v)w = -\varepsilon w'' + pw' + qw, \quad w \in C^2[0,1].$$

It holds

in case I: $p = \alpha \geq \alpha_o > 0,\ q \geq \beta_o,\ \alpha_o^2 + 4\varepsilon\beta_o > 0$,

in case II: $p \geq \alpha_o > 0,\ r = q - p' \geq \gamma_o,\ \alpha_o^2 + 4\varepsilon\gamma_o > 0$,

in case III: $q \geq 0$,

in case IV: $r = q - p' \geq 0$.

If we show that (L_ε, R) is inverse monotone, then P II follows. In case I set

$$\sigma = \frac{1}{2\varepsilon}\,(\alpha_o + (\alpha_o^2 + 4\varepsilon\beta_o)^{1/2}) > 0$$

and aplly Lemma 4:

$$L_\varepsilon(e^{\sigma x}) = e^{\sigma x}(-\varepsilon\sigma^2 + p\sigma + q) \geq e^{\sigma x}(-\varepsilon\sigma^2 + \alpha_o\sigma + \beta_o) = 0.$$

In case II set

$$\sigma = \frac{1}{2\varepsilon}\,(-\alpha_o - (\alpha_o^2 + 4\varepsilon\gamma_o)^{1/2}) < 0$$

and notice

$$L_\varepsilon^*(e^{\sigma x}) = e^{\sigma x}(-\varepsilon\sigma^2 - p\sigma + r) \geq e^{\sigma x}(-\varepsilon\sigma^2 - \alpha_o\sigma + \gamma_o) = 0.$$

Let $\bar{e}(x) = 1$ for all $x \in [0,1]$. Then in case III holds $L_\varepsilon \bar{e} \geq 0$ and in case IV holds $L_\varepsilon^* \bar{e} \geq 0$. Thus in all cases (L_ε, R) is inverse monotone by Lemma 1, and P II follows.

To prove P III under the conditions of Theorem 1 we just notice that for any linearization (see (2))

$$(DT_\varepsilon u)w = -\varepsilon w'' + pw' + qw$$

the coefficients p, q and $r = q - p'$ satisfy the same inequalities as above. Thus with Lemma 1, (ii) the property P III follows.

P r o o f of Theorem 2. Let $L_\varepsilon = \Delta T_\varepsilon(u,v)$. Because of Lemma 2 we have to show that

$$G_\varepsilon(x,y) \leq e^{-2\beta_o/\alpha_o} H_\varepsilon,$$

(15) $$g_\varepsilon(x) \leq e^{-2\beta_o/\alpha_o}, h_\varepsilon(x) \leq e^{-2\beta_o/\alpha_o + (x-1)\alpha_o/\varepsilon}.$$

a) First assume $\beta_o = 0$, i.e. $q \geq 0$. If $\hat{L}_\varepsilon w = -\varepsilon w'' + pw'$, then elementary calculation shows (with p_ε as in (14)):

(16) $$\hat{G}_\varepsilon(x,y) = (\varepsilon p_\varepsilon(y) \int_o^1 p_\varepsilon ds)^{-1} \cdot \begin{cases} \int_o^x p_\varepsilon ds \int_y^1 p_\varepsilon ds, & x \leq y, \\ \int_o^y p_\varepsilon ds \int_x^1 p_\varepsilon ds, & y \leq x, \end{cases}$$

Since $p \geq \alpha_o > 0$ we have

$$\hat{G}_\varepsilon(x,y) \leq \hat{G}_\varepsilon(y,y) \leq \frac{1}{\varepsilon p_\varepsilon(y)} \int_o^y p_\varepsilon ds$$

$$= \frac{1}{\varepsilon} \int_o^y \exp\left(-\frac{1}{\varepsilon} \int_s^y p d\sigma\right) ds \leq \frac{1}{\varepsilon} \int_o^y e^{-(y-s)\alpha_o/\varepsilon} ds$$

$$\leq (1 - e^{-\alpha_o/\varepsilon})/\alpha_o < 1/\alpha_o .$$

Because of Lemma 5 we thus have $G_\varepsilon(x,y) \leq 1/\alpha_o$.

b) Let $\beta_o < 0$, $\sigma = \frac{1}{2\varepsilon}(\alpha_o - (\alpha_o^2 + 4\varepsilon\beta_o)^{1/2}) > 0$.

Use the transformation described in Lemma 4 and notice that the proved part a) is applicable to $L_{\varepsilon,\sigma}$ with

$$\alpha_{o,\sigma} = \alpha_o - 2\varepsilon\sigma = (\alpha_o^2 + 4\varepsilon\beta_o)^{1/2} .$$

Lemma 4 yields

$$G_\varepsilon(x,y) \le e^{\sigma}(\alpha_o^2 + 4\varepsilon\beta_o)^{-1/2} \le e^{-2\beta_o/\alpha_o} H_\varepsilon$$

where the last estimate follows from

$$\sigma = \frac{1}{2\varepsilon}(\alpha_o - \alpha_o(1 + 4\varepsilon\beta_o/\alpha_o^2)^{1/2})$$

$$\le \frac{1}{2\varepsilon}(\alpha_o - \alpha_o(1 + 4\varepsilon\beta_o/\alpha_o^2)) = -2\beta_o/\alpha_o .$$

c) Since $\beta_o \le 0$, both roots

$$\sigma_{1,2} = \frac{1}{2\varepsilon}(\alpha_o \pm (\alpha_o^2 + 4\varepsilon\beta_o)^{1/2})$$

of $-\varepsilon\sigma^2 + \alpha_o\sigma + \beta_o = 0$ are nonnegative, and thus $p \ge \alpha_o$, $q \ge \beta_o$ implies $L_\varepsilon(e^{\sigma_i x}) \ge 0$. Since $R(e^{\sigma_2 x}) \ge (1,0)$, $R(e^{\sigma_1(x-1)}) \ge (0,1)$ it follows that $g_\varepsilon(x) \le e^{\sigma_2 x}$, $h_\varepsilon(x) \le e^{\sigma_1(x-1)}$, which leads to to (15) since $\sigma_2 \le -2\beta_o/\alpha_o$, $\sigma_1 = \alpha_o/\varepsilon - \sigma_2 \ge \alpha_o/\varepsilon + 2\beta_o/\alpha_o$.

d) To show $\lambda_1(\varepsilon) \ge \alpha_o^2/4\varepsilon + \beta_o$, just consider $e(x) = \exp(\alpha_o x/2\varepsilon)$ and notice

$$(DT_\varepsilon u)e \ge (\alpha_o^2/4\varepsilon + \beta_o)e .$$

Thus $(DT_\varepsilon u - (\alpha_o^2/4\varepsilon + \beta_o), R)$ in inverse monotone, and by Lemma 1 the assertion follows.

P r o o f of Theorem 3. Let $L_\varepsilon = \Delta T_\varepsilon(u,v)$, $L_\varepsilon^* w = -\varepsilon w'' - pw' + rw$ with $r \ge \gamma_o$. We have to show that

$$G_\varepsilon(x,y) \le e^{-2\gamma_o/\alpha_o} \bar{H}_\varepsilon , \tag{17}$$

$$h_\varepsilon(x) \le e^{-2\gamma_o/\alpha_o + (x-1)\alpha_o/\varepsilon} , \tag{18}$$

and in case II a

$$g_\varepsilon(x) \le e^{-2\gamma_o/\alpha_o}(1 + \bar{H}_\varepsilon(\alpha_1 - \alpha_o)) . \tag{19}$$

a) By the proof of Theorem 2 it follows that
$G_\varepsilon^*(x,y) \le e^{-2\gamma_o/\alpha_o} \bar{H}_\varepsilon$, which proves (17).

b) To show (18) and (19) first assume $\gamma_o = 0$. With p_ε defined by (14) consider

$$\bar{g}_\varepsilon(x) = p_\varepsilon(x) \int_x^1 \frac{ds}{p_\varepsilon(s)} \Big/ \int_o^1 \frac{ds}{p_\varepsilon(s)} ,$$

$$\bar{h}_\varepsilon(x) = \frac{p_\varepsilon(x)}{p_\varepsilon(1)} \int_o^x \frac{ds}{p_\varepsilon(s)} \Big/ \int_o^1 \frac{ds}{p_\varepsilon(s)} .$$

With Lemma 3, (i) it follows that $L_\varepsilon \bar{g}_\varepsilon \ge 0$ and $L_\varepsilon \bar{h}_\varepsilon \ge 0$. Since $R\bar{g}_\varepsilon = (1,0)$, $R\bar{h}_\varepsilon = (0,1)$ we thus have $g_\varepsilon \le \bar{g}_\varepsilon$, $h_\varepsilon \le \bar{h}_\varepsilon$. Now $p \ge \alpha_o > 0$ implies

$$\bar{h}_\varepsilon(x) \le p_\varepsilon(x)/p_\varepsilon(1) = \exp(-\frac{1}{\varepsilon}\int_x^1 p(s)ds) \le e^{(x-1)\alpha_o/\varepsilon} .$$

If $0 \le \alpha_o \le p \le \alpha_1$, then

$$p_\varepsilon(x) \int_x^1 \frac{ds}{p_\varepsilon(s)} = \int_x^1 \exp(-\frac{1}{\varepsilon}\int_x^s p(\sigma)d\sigma)ds \le \varepsilon(1 - e^{-\alpha_o/\varepsilon})/\alpha_o ,$$

and

$$1/p_\varepsilon(s) = \exp(-\frac{1}{\varepsilon}\int_o^s p(\sigma)d\sigma) \ge e^{-s\alpha_1/\varepsilon} ,$$

$$\int_o^1 \frac{ds}{p_\varepsilon(s)} \ge \varepsilon(1 - e^{-\alpha_1/\varepsilon})/\alpha_1 .$$

If follows that $\bar{g}_\varepsilon \le \alpha_1/\alpha_o$, and (18), (19) are proved for $\gamma_o = 0$.

c) Let $\gamma_o < 0$, $\sigma = \frac{1}{2\varepsilon}(\alpha_o - (\alpha_o^2 + 4\varepsilon\gamma_o)^{1/2}) > 0$.

The proved part b) is applicable to $L_{\varepsilon,\sigma}$ as defined in Lemma 4, and we get

$$g_{\varepsilon,\sigma}(x) \le (\alpha_1 - 2\varepsilon\sigma)/(\alpha_o - 2\varepsilon\sigma) = 1 + \bar{H}_\varepsilon(\alpha_1 - \alpha_o) ,$$

$$h_{\varepsilon,\sigma}(x) \le e^{(x-1)(\alpha_o - 2\varepsilon\sigma)/\varepsilon} ,$$

$$g_\varepsilon(x) \le e^\sigma(1 + \bar{H}_\varepsilon(\alpha_1 - \alpha_o)) ,$$

$$h_\varepsilon(x) \le e^{\sigma + (x-1)\alpha_o/\varepsilon} ,$$

and (18), (19) follow $\sigma \leq -2\gamma_o/\alpha_o$.

d) The assertion $\lambda_1(\varepsilon) \geq \alpha_o^2/4\varepsilon + \gamma_o$ follows as in the proof of Theorem 2 since DT_ε and $(DT_\varepsilon u)^*$ have the same eigenvalues.

P r o o f of Theorem 4. Let $L_\varepsilon = \Delta T_\varepsilon(u,v)$, $\hat{L}_\varepsilon w = -\varepsilon w'' + pw'$. By Lemma 5 holds $G_\varepsilon(x,y) \leq \hat{G}_\varepsilon(x,y)$, and as in the proof Theorem 2 it follows that $\hat{G}_\varepsilon(x,y) \leq c_\varepsilon$. Since $L_\varepsilon \bar{e} \geq 0$, $R\bar{e} = (1,1)$ we have $g_\varepsilon + h_\varepsilon \leq \bar{e}$. With Lemma 1 the Theorem is proved.

P r o o f of Theorem 5. Let $L_\varepsilon w = \Delta T_\varepsilon(u,v)w = -\varepsilon w'' + pw' + qw$ with $p = \alpha$, $q \geq \beta_o > 0$. Set

$$c = \max\{\frac{1}{\beta_o}\,\|T_\varepsilon u - T_\varepsilon v\|_\infty,\ |u(0)-v(0)|,\ |u(1)-v(1)|\} ,$$

then $L_\varepsilon(c\bar{e} \pm (u-v)) \geq c\beta_o\bar{e} \pm (T_\varepsilon u - T_\varepsilon v) \geq 0$,

$$R(c\bar{e} \pm (u-v)) \geq (0,0) ,$$

thus $\pm(u-v) \leq c\bar{e}$. The inequality $\lambda_1(\varepsilon) \geq \beta_o$ is obvious since $((DT_\varepsilon u) - \beta_o, R)$ is inverse-monotone.

P r o o f of Theorem 6.

a) We have to show that $\hat{G}_\varepsilon(x,y) \leq \frac{1}{\varepsilon}\max\{x_o, 1-x_o\}$ where $\hat{G}_\varepsilon$ is given in (16) and

$$\hat{G}_\varepsilon(x,y) \leq \hat{G}_\varepsilon(y,y) = \int_o^y p_\varepsilon ds \int_y^1 p_\varepsilon ds \,/\, (\varepsilon p_\varepsilon(y) \int_o^1 p_\varepsilon ds) .$$

Since $p_\varepsilon(s)$ is monotonically increasing for $0 \leq s \leq x_o$ and decreasing for $x_o \leq s \leq 1$ we have for $0 \leq y \leq x_o$

$$\hat{G}_\varepsilon(y,y) \leq \frac{1}{\varepsilon}\int_o^y p_\varepsilon(s)ds \,/\, p_\varepsilon(y) \leq y/\varepsilon \leq x_o/\varepsilon ,$$

and similarly for $x_o \leq y \leq 1$

$$\hat{G}_\varepsilon(y,y) \leq \frac{1}{\varepsilon}\int_y^1 p_\varepsilon(s)ds \,/\, p_\varepsilon(y) \leq (1-y)/\varepsilon \leq (1-x_o)/\varepsilon .$$

b) To prove the statement about $\lambda_1(\varepsilon)$ first assume $\frac{1}{2} \leq x_o \leq 1$ and let

$$e(x) = \sin\left(\frac{\pi x}{2x_o} \right) . \tag{20}$$

Since $e \geq 0$, $e'(x) \geq 0$ in $[0,x_o]$, $e'(x) \leq 0$ in $[x_o,1]$ it holds that

$$(DT_\varepsilon u)e \geq -\varepsilon e'' = \frac{\pi^2}{4} \varepsilon x_o^{-2} e = \frac{\pi^2}{4} \varepsilon \min\{x_o^{-2}, (1-x_o)^{-2}\} e .$$

In case that $0 \leq x \leq x_o$, consider instead of (20) the function $e(x) = \sin\left(\frac{\pi(1-x)}{2(1-x_o)} \right)$.

P r o o f of Theorem 7. Let $L_\varepsilon = \Delta T_\varepsilon(u,v)$, $L_\varepsilon^* w = -\varepsilon w'' - - pw' + rw$ with $r \geq 0$. As in the proof of Theorem 2 it follows that $G_\varepsilon^*(x,y) \leq c_\varepsilon$, which proves the stability inequality. The existence result follows from Theorem 10 below.

P r o o f of Theorem 8. Let L_ε^* be defined as above. Since $r \geq \gamma_o > 0$ we have with $\bar{e}(x) \equiv 1$:

$$L_\varepsilon^* \bar{e} \geq \gamma_o \bar{e} \ , \text{ thus } G_\varepsilon^* \bar{e} \leq \frac{1}{\gamma_o} \bar{e}, \text{ i.e. } \int_0^1 G_\varepsilon(x,y)dx \leq \frac{1}{\gamma_o} \ ,$$

and the stability inequality follows. The smallest eigenvalue of the eigenvalue problem

$$(DT_\varepsilon u)^* w = \lambda w, \quad Rw = (0,0)$$

is $> \gamma_o$, thus $\lambda_1(\varepsilon) \geq \gamma_o$.

P r o o f of Theorem 9. Let $L_\varepsilon = \Delta T_\varepsilon(u,v)$, $L_\varepsilon^* w = -\varepsilon w'' - - pw' + rw$. It holds that $r \geq \gamma_o > 0$ and $-\alpha_o \leq p \leq \alpha_o$, and we have to show the estimate

$$G_\varepsilon^*(x,y) \leq \frac{1}{4\varepsilon\gamma_o} \left(\alpha_o + (\alpha_o^2 + 4\varepsilon\gamma_o)^{1/2}\right) \tag{21}$$

Now G_ε^* has a representation

$$G_\varepsilon^*(x,y) = \frac{1}{\varepsilon W_\varepsilon(y)} \cdot \begin{cases} \phi_\varepsilon(x)\psi_\varepsilon(y), & x \leq y \\ \phi_\varepsilon(y)\psi_\varepsilon(x), & y \leq x \end{cases}$$

where $L_\varepsilon^*\phi_\varepsilon = L_\varepsilon^*\psi_\varepsilon = 0$, $\phi_\varepsilon(0) = \psi_\varepsilon(1) = 0$, $\phi_\varepsilon'(0) = -\psi_\varepsilon(1) = 1$, $W_\varepsilon = \phi_\varepsilon'\psi_\varepsilon - \psi_\varepsilon'\phi_\varepsilon$. Since $\phi_\varepsilon' > 0 > \psi_\varepsilon'$, $W_\varepsilon > 0$ we have

$$G_\varepsilon^*(x,y) \leq G_\varepsilon^*(y,y) = \frac{1}{\varepsilon}\frac{\phi_\varepsilon\psi_\varepsilon}{\phi_\varepsilon'\psi_\varepsilon - \psi_\varepsilon'\phi_\varepsilon}(y) ,$$

and thus for $0 < y < 1$:

$$(G_\varepsilon^*(y,y))^{-1} = \varepsilon\left(\frac{\phi_\varepsilon'}{\phi_\varepsilon} - \frac{\psi_\varepsilon'}{\psi_\varepsilon}\right)(y) .$$

The function $p_\varepsilon = \phi_\varepsilon/\phi_\varepsilon' \in C^1[0,1]$ satisfies the Riccati equation

(22) $$\varepsilon\rho' - \varepsilon\rho - p\rho + r\rho^2 = 0, \quad \rho(0) = 0.$$

Since the constant $e(x) = c$ is an upper solution of (22) for

$$c = \frac{1}{2\gamma_o}(\alpha_o + (\alpha_o^2 + 4\varepsilon\gamma_o)^{1/2}) > 0$$

we conclude |33, ch. 2| that $\rho_\varepsilon \leq c$. Similarly, $-\psi_\varepsilon/\psi_\varepsilon' \leq c$, thus

$$(G_\varepsilon^*(y,y))^{-1} \geq 2\varepsilon / c$$

and (21) follows.

P r o o f of the existence statements

In many cases the existence of a solution of (P_ε) is most easily established by the Nagumo-Lemma |18,26| which implies

LEMMA 6. *Let* $\alpha,\beta \in C([0,1] \times \mathbb{R})$, $\varepsilon > 0$, *and assume for two functions* $u,v \in C^2[0,1]$ *holds*

$$u \leq v, \quad T_\varepsilon u \leq 0 \leq T_\varepsilon v, \quad Ru \leq (A,B) \leq Rv .$$

Then (P_ε) *has a solution* $u_\varepsilon \in C^2[0,1]$ *with* $u \leq u_\varepsilon \leq v$.

Also notice that in cases I and II, where α is independent of u, the solvability of (P_ε) follows from |3, Th. 7.6|. (The value of $\alpha(L_2,K_2)$ in |3, Th. 7.6)| is ∞ under our conditions). The existence statements for all our cases I, II, III, IVa follow from the next Theorem where we just require a stability inequality in the maximum norm.

THEOREM 10*. *Let* $\alpha,\beta \in C([0,1] \times \mathbb{R})$,

$$Tu = -u'' + \alpha(x,u)u' + \beta(x,u), \quad u \in U = \{v \in C^2[0,1],\ Rv = 0\} .$$

Suppose for some $k > 0$

(23) $$\|u-v\|_\infty \le k\|Tu - Tv\|_\infty \quad \forall\ u,v \in U .$$

Then T *is a homemorphism from* $(U,\|\ \|_{2,\infty})$ *onto* $(C[0,1],\|\ \|_\infty)$ *with* $\|u\|_{2,\infty} = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty$.

P r o o f. a. Let $Fu = \alpha(\cdot,u)u' + \beta(\cdot,u)$ for $u \in U$, and let G denote Green's function to $(-u'',R)$. Thus for all $r \in C[0,1]$ holds $Tu = r,\ u \in U \iff u + GFu = Gr,\ u \in U$. Since F is completely continuous from U into $C[0,1]$ and since G is bounded from $C[0,1]$ into U the operator

$$K = GF : U \to U$$

is completely continuous.

Also, $I + K$ is injective on U, since

$$u + GFu = v + GFv,\ u,v \in U \implies Tu = Tv \implies u = v \quad \text{by (23)}.$$

Now the Brouwer-Schauder Theorem of the invariance of domain (cf. |34, Satz 15.3|) yields: $(I+K)(U)$ is open in U and

$$(I+K)^{-1} : (I+K)(U) \subset U \to U$$

is continuous.

b. It remains to show that $(I+K)(U)$ is closed in U. To prove this let

$$u_n \in U,\quad v_n = u_n + Ku_n \to \bar{v} \in U \quad \text{for} \quad n \to \infty .$$

Then $v_n - v_m = u_n + Ku_n - u_m - Ku_m$ implies $-v_n'' + v_m'' = Tu_n - Tu_m$, thus

$$\|Tu_n - Tu_m\|_\infty \to 0 \quad \text{for} \quad n,m \to \infty .$$

*) I would like to thank Dr. W.-J. Beyn for helpful remarks and discussions concerning this result.

(23) yields $u_n \to \bar{u} \in C[0,1]$ for $n \to \infty$ with respect to $\| \ \|_\infty$. Define the linear operator

$$Lw = -w'' + \alpha(\cdot, \bar{u}(\cdot))w', \quad w \in U,$$

and solve $L\bar{w} = -\bar{v}''(\cdot) - \beta(\cdot, \bar{u}(\cdot))$, $\bar{w} \in U$. The linear operator L satisfies a stability inequality on U:

$$\|w\|_{2,\infty} \le \bar{k}\|Lw\|_\infty \quad \text{for all} \quad w \in U,$$

and taking $w = u_n - \bar{w}$ we find

$$\begin{aligned}\|u_n - \bar{w}\|_{2,\infty} &\le \bar{k}\|Lu_n - L\bar{w}\|_\infty \\ &= \bar{k}\|-u_n'' + \alpha(\cdot,\bar{u})u_n' - L\bar{w}\|_\infty \\ &= \bar{k}\|-v_n'' - \alpha(\cdot,u_n)u_n' - \beta(\cdot,u_n) + \alpha(\cdot,\bar{u})u_n' \\ &\quad + \bar{v}'' + \beta(\cdot,\bar{u})\|_\infty \le \eta_n + \|(\alpha(\cdot,\bar{u}) - \alpha(\cdot,u_n))(u_n' - \bar{w}')\|_\infty \\ &\quad + \|(\alpha(\cdot,\bar{u}) - \alpha(\cdot,u_n))\bar{w}'\|_\infty \le \bar{\eta}_n\|u_n' - \bar{w}'\|_\infty + \bar{\bar{\eta}}_n\end{aligned}$$

where η_n, $\bar{\eta}_n$, $\bar{\bar{\eta}}_n \to 0$ for $n \to \infty$. Thus $u_n \to \bar{w}$ in U, and thus

$$\bar{v} = \bar{w} + K\bar{w} \in (I+K)(U).$$

c. $I + K = I + GF$ is a homeomorphism from U onto itself, and therefore

$$T = -D^2(I + GF) : U \to C[0,1]$$

is also a homeomorphism.

4. APPLICATIONS AND FURTHER REMARKS

In this section we consider a class of examples and characterize the Minty-monotonicity of T_ε.

EXAMPLE.

$$(24) \qquad T_\varepsilon u \equiv -\varepsilon u'' + uu' + \mu u = \gamma(x), u(0) = A, \quad u(1) = B$$

with $\varepsilon > 0$, μ, $A, B \in \mathbb{R}$, $\gamma \in C[0,1]$. These examples include the stationary Burgers equation $-\varepsilon u'' + uu' = 0$ (cf. |4,7,28|) and Cochran's equation $-\varepsilon u'' + uu' + u = 0$ (cf. |8,12,27|). With Lemma 6 it follows that (24) has a solution u_ε. (Upper and lower solutions-even independent of ε - of the form

$$v(x) = cx + d, \quad c > -\mu, \quad d \in \mathbb{R}$$

are easily computed.) If $\mu \geq 0$, then Theorem 1, case IV shows the uniqueness and monotone dependence of u_ε upon γ, A and B. For $\mu < 0$ a solution of (24) will not be unique in general. Take e.g. $\gamma = 0$, $A = B = 0$. Then $u_\varepsilon = 0$ is a solution for all ε, and bifurcation occurs at $\varepsilon = -\mu/(n^2\pi^2) > 0$, $n \in \mathbb{N}$. Assume $\mu \geq 0$ and $\max\{\|\gamma\|_\infty, |A|, |B|\} \leq M$. Then $\|u_\varepsilon\|_\infty \leq M+1$ since $u(x) = x - M - 1$ is a lower solution and $v(x) = x + M$ is an upper solution for (24). Thus it is sufficient to obtain stability inequalities for all $u, v \in C^2[0,1]$ with $\|u\|_\infty \leq \alpha_o$, $\|v\|_\infty \leq \alpha_o$. Because of Remark 2b, Theorem 7 and Theorem 9 the following holds:

Let $u, v \in C^2[0,1]$, $\|u\|_\infty \leq \alpha_o$, $\|v\|_\infty \leq \alpha_o$, $Ru = Rv$, and let T_ε be given as in (24). Then

$$\|u - v\|_\infty \leq s_{\varepsilon\mu} \|T_\varepsilon u - T_\varepsilon v\|_1$$

where
$$s_{\varepsilon\mu} = \begin{cases} \frac{1}{4\varepsilon\mu} (\alpha_o + (\alpha_o^2 + 4\varepsilon\mu)^{1/2}) & \text{for} \quad \mu > 0, \\ \frac{1}{\alpha_o} (e^{\alpha_o/\varepsilon} - 1) & \text{for} \quad \mu = 0. \end{cases}$$

Actually, for $\mu = 0$ the growth of the stability constants as $e^{\sigma/\varepsilon}$ $(\sigma > 0)$ for $\varepsilon \to 0^+$ cannot be improved. To see this, consider Burgers equation

$$(25) \qquad -\varepsilon u'' + uu' = 0, \quad u(0) = 1, \quad u(1) = -1 + \eta \quad .$$

(25) can be solved analytically (cf. |28, pp. 9-12|), and it follows that the solution for $\eta = e^{-\sigma/\varepsilon}$ with suitable $\sigma > 0$ differs from the solution for $\eta = 0$ by at least one. A transformation of (25) to homogeneous boundary conditions then proves our statement.

The following Theorem concerns the monotonicity of T_ε in the sense of Minty |25|. Here T is considered as an operator mapping $U = \overset{o}{W}{}^2[0,1]$ into $L_2[0,1]$, and

$$(u,v) = \int_0^1 uv dx \quad \text{for} \quad u,v \in L_2[0,1] \; .$$

THEOREM 11. *Assume*

$$\alpha, \alpha_x, \alpha_u, \beta, \beta_u \in C([0,1] \times \mathbb{R}) \quad \textit{and} \quad \varepsilon > 0.$$

(i) *If*

(26) $\alpha(x,u) = \alpha(x), \beta_u(x,u) - \frac{1}{2}\alpha_x(x) \geq 0$

then

$$(T_\varepsilon u - T_\varepsilon v,\ u-v) \geq \varepsilon\pi^2 (u-v,\ u-v) \ \forall u, v \in U,$$

i.e. T_ε *is uniformly Minty-monotone for each* $\varepsilon > 0$.

(ii) *If (26) is violated then* T_ε *is not Minty-monotone for* $0 < \varepsilon \leq \varepsilon_0$.

P r o o f. a) Let $L_\varepsilon w = -\varepsilon w'' + pw' + qw$, $w \in U$ with $p \in C^1[0,1]$, $q \in C[0,1]$. Since

$$(pw', w) = (w, -p'w - pw'), \text{ i.e. } (pw', w) = -\frac{1}{2}(p'w, w),$$

we have

(27) $$(L_\varepsilon w, w) = \varepsilon(w', w') + ((q - \frac{1}{2}p')w, w) .$$

If $q - \frac{1}{2}p' \geq 0$ then

$$(L_\varepsilon w, w) \geq \varepsilon\pi^2 (w,w) \quad \text{for all} \quad w \in U,$$

and taking $L_\varepsilon = \Delta T_\varepsilon(u,v)$ we have proved (i).

b) First note that T_ε is Minty-monotone iff

$$((DT_\varepsilon u)w, w) \geq 0 \quad \text{for all} \quad u, w \in U.$$

Now $(DT_\varepsilon u)w = -\varepsilon w'' + pw' + qw$ with

(28) $p(x) = \alpha(x, u(x)),\ q(x) = \beta_u(x, u(x)) + \alpha_u(x, u(x))u'(x)$,

and if $(q - \frac{1}{2}p')(x) < 0$ for some $x \in [0,1]$, then $L_\varepsilon = DT_\varepsilon u$ is not Minty-monotone for $0 < \varepsilon \leq \varepsilon_0$ because of (27). Here it holds that

$$(q - \frac{1}{2}p')(x) = \beta_u(x, u(x)) - \frac{1}{2}\alpha_x(x, u(x)) + \frac{1}{2}\alpha_u(x, u(x))u'(x).$$

If $\alpha_u(x_0, u_0) \neq 0$ for some $(x_0, u_0) \in (0,1) \times \mathbb{R}$, then solve the initial value problem

$$\beta_u(x, \bar{u}) - \frac{1}{2}\alpha_x(x, \bar{u}) + \frac{1}{2}\alpha_u(x, \bar{u})\bar{u}' = -1, \quad \bar{u}(x_0) = u_0$$

for $x \in (x_o - \sigma, x_o + \sigma)$. A simple process of approximation and extension thus leads to a function $u \in U$, for which the functions p and q defined in (28) fulfill

$$(q - \frac{1}{2} p')(x_o) < 0.$$

The case $\alpha_u \equiv 0$, $\beta_u(x_o, u_o) - \frac{1}{2} \alpha_x(x_o) < 0$ for some (x_o, u_o) is treated similarly.

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REZIME

OSOBINE STABILNOSTI I MONOTONIJE ZA ČVRST KVAZILINEARAN GRANIČNI PROBLEM

U ovom radu ispituju se osobine stabilnosti i monotonije jedne klase kvazilinearnog graničnog problema.