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EXIT CRITERIA FOR SOME ITERATIVE METHODS

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ABSTRACT

In this paper we consider the equation $f(x)=0$ on the interval $D=[a,b]$, for a real - valued function f. We use the iterative method $x_{n+1}=\phi(x_n)$, $n=0,1,\ldots,$ with a suitably chosen $\phi(x)$ and $x_o \in D$. We accept x_{n+1} as a sufficiently accurate approximation of the exact solution α of the given equation, if we have $|x_{n+1}-x_n|<\varepsilon$, where $\varepsilon>0$ is the pre-assinged tolerance, and if the stopping inequality $|x_{n+1}-x_n| \geq |x_{n+1}-\alpha|$ is valid. For the special functions ϕ we give sufficient conditions for the stopping inequality. As special cases we obtain both Newton's iterative method and the classical regula falsi method. Moreover, we prove the stopping inequality for $n=0,1,\ldots,$ for the class of iterative methods which are generated by inverse interpolation.

INTRODUCTION

In this paper we shall consider some iterative methods for determining the unique solution $\alpha \in D$ of the equation $f(x)=0$, where f is a real - valued function defined on an interval $D=[a,b]$. Most of the iterative methods can be writen in the form

(1) $$x_{n+1} = \phi(x_n) \quad ,$$

for some suitable function ϕ and an initial approximation x_o. Under certain conditions the iteration defined by (1) converges to the solution α of $f(x)=0$, i.e. $\alpha = \lim_{n\to\infty} x_n$. In many automated numerical algorithms, the calculations of interates x_n of (1) are stopped when the difference between two successive approximations is less than a pre-assigned tolerance. So, one evaluates $x_1, x_2, \ldots,$ and accepts x_{n+1} as a sufficiently accurate approximation of α when

(2) $$|x_n - x_{n+1}| < \varepsilon ,$$

where ε is the pre-assinged tolerance. Such a procedure may be justified in terms of a stopping inequality.

DEFINITION *The inequality*

$$|x_n - x_{n+1}| \geq |x_{n+1} - \alpha|$$

will be referred to as the stopping inequality.

The validity of the stopping inequality is sufficient to insure that the value x_{n+1}, accepted as the final result, by the above exit criterion (2), will be within the tolerance ε , i.e.

(3) $$|x_{n+1} - \alpha| < \varepsilon.$$

We shall give sufficient conditions for the stopping inequality for some iterative methods of (1), for the solution of the equation $f(x) = 0$.

As special cases are obtained Newton´s method and the secant iterative method (the classical regula falsi method). The stopping inequality for Newton´s method is proved in |1|.

1. SOME SUFFICIENT CONDITIONS FOR THE STOPPING INEQUALITY

In this section we shall consider the equation $f(x) = 0$ on the interval $D = [a,b]$ and the equation $x = \phi(x)$ which has roots which coincide with those of $f(x) = 0$ in the interval D and no others. First we shall give some notations and assumptions.

Let the function f satisfy the following conditions.

(F1) $f(a) < 0, \quad f(b) > 0,$ (F2) $f'(x) > 0, \quad x \in D.$

The condition (F1) implies that $f \in C(D)$ has a root $\alpha \in (a,b)$, and (F2) implies that $f \in C(D)$ has only one root in (a,b). Let

$$D^- = [a,b), \quad D^+ = (\alpha,b], \quad D_o^- = [a,\alpha], \quad D_o^+ = [\alpha,b] ,$$

and

$$u(x) = \frac{f(x)}{f'(x)} , \quad x \in D .$$

Then, under the assumptions (F1), (F2) for $f \in C(D)$ it holds

$$f(x) < 0, \quad u(x) < 0, \quad x \in D^-, \quad f(x) > 0, \quad u(x) > 0, \quad x \in D^+ .$$

If $f(x) = 0$ has a solution $\alpha \in D$ and if g is any function such that

$$0 < |g(x)| < \infty, \quad x \in D$$

then α is a solution of $f(x) = 0$ if and only if α is a solution $x = \phi(x)$, where

(4) $$\phi(x) = x - u(x)g(x) .$$

THEOREM 1. *Let* $f \in C^2(D)$ *satisfy condition* (F2) *and let* $\alpha \in D$ *be a solution of* $f(x) = 0$. *Let* ϕ *be of the form* (4), *where*

(G) $$g \in C^1(D), \quad g(\alpha) = 1.$$

Then there exists the interval $D_\rho(\alpha) = \{x \mid |x-\alpha| \le \rho\} \subset D, \rho > 0$, *such that for any* $x_o \in D_\rho(\alpha)$ *the iterates* $x_n = \phi(x_{n-1})$, $n=1,2,\dots$, *converge to* α *and the stopping inequality is valid for all* $n = 0,1,\dots$

P r o o f. From (4) we obtain

$$\phi'(x) = 1 - g(x) + u(x) \left(\frac{f''(x)}{f'(x)} g(x) - g'(x) \right) ,$$

and $\phi \in C^1(D)$, $\phi'(\alpha) = 0$. Thus, there exists $D_\rho(\alpha)$ such that

$$|\phi'(x)| \le L \le 0.5, \quad x \in D_\rho(\alpha) ,$$

and

$$0 < g(x), \quad x \in D_\rho(\alpha) .$$

Now, from

$$|\phi(x)-\alpha| = |\phi(x)-\phi(\alpha)| \le L|x-\alpha| < \rho, \quad x \in D_\rho(\alpha)$$

we conclude that ϕ is a contraction type mapping of $D_\rho(\alpha)$ into itself. Thus, ϕ has a fixed point in $D_\rho(\alpha)$, and this point is α. It is, also, well known that iterates x_n converge to α and that it,

$$|x_{n+1}-\alpha| \le \frac{L}{1-L}|x_{n+1}-x_n|, \quad n=0,1,2,\dots,$$

holds.

Since $L \in [0, 0.5]$, from this inequality follows the stopping inequality for all $n=0,1,\dots$.

REMARK 1. The conditions (G) for $g(x)$ are not too strong. For example, the condition $g(\alpha)=1$ is satisfied for all eight one-point iteration functions ϕ from |2|, and for all iteration functions generated by the inverse interpolation, paragraph 3. As a special case for $g(x)=1$, $x \in D$ we have Newton´s method, for $g(x) = 1+u(x) \cdot \frac{f''(x)}{2f'(x)}$, $x \in D$, Chebyshev´s method of degree 3 and for

$$g(x) = 1/(1-u(x)\frac{f''(x)}{2f'(x)}), \quad x \in D, \text{ Halley´s method.}$$

THEOREM 2. *Let* $f \in C^2(D)$ *satisfy* (F1), (F2) *and*

(F3) $$f''(x) > 0, \quad x \in D.$$

Let

(F4) $$f'(b) \le 2f'(a),$$

(5) $$g \in C^1(D_o^+), \quad g(x) \ge 1, \quad x \in D_o^+$$

(6) $$\phi'(x) \ge 0, \quad x \in D_o^+.$$

Then for any $x_o \in D^+$ *the iterates* $x_n = \phi(x_{n-1})$, $n=1,2,\dots$, *converge to the unique solution* α *of* $f(x)=0$ *in* D, *and the stopping inequality is valid for all* $n=0,1,\dots$.

P r o o f. The conditions (F1) and (F2) assure us of the existence and uniqueness of a solution α of the equation $f(x)=0$. From (6) it follows that for $x_0 \in D^+$ we have

$$\alpha < x_n < x_{n-1} \le b$$

either for all $n=1,2,\ldots$, or for $n=1,2,\ldots,k$ with some fixed $k \in \mathbb{N}$ and $\alpha = x_{k+i}$, $i=1,2,\ldots$. Thus $x_n \in D_0^+$, $n=1,2,\ldots$, and we can conclude that the limit of the sequence defined by $x_n = \phi(x_{n-1})$, $n=1,2,\ldots$, exists and that this limit is a solution of the equation $f(x)=0$. Thus, $\alpha = \lim_{n\to\infty} x_n$. From $x_{n+1} = \phi(x_n)$, $n=0,1,\ldots$, we obtain for all $n=0,1,\ldots$,

$$x_{n+1} - \alpha = x_n - \alpha - \frac{f(x_n)-f(\alpha)}{f'(x_n)} g(x_n),$$

$$= (x_n-\alpha)\left(1 - \frac{f'(\alpha_n)}{f'(x_n)} g(x_n)\right), \quad \alpha_n \in (\alpha, x_n),$$

and

$$x_{n+1} - x_n = (x_n-\alpha)\frac{f'(\alpha_n)}{f'(x_n)} g(x_n) .$$

Now we have

$$x_{n+1}-\alpha = \left(1 - \frac{f'(x_n)}{f'(\alpha_n)g(x_n)}\right)(x_{n+1}-x_n), \quad n=0,1,\ldots,$$

and the stopping inequality is valid if

$$\left|1 - \frac{f'(x_n)}{f'(\alpha_n)g(x_n)}\right| \le 1, \quad n=0,1,\ldots,$$

i.e., if

(7) $$0 \le \frac{f'(x_n)}{f'(\alpha_n)g(x_n)} \le 2, \quad n=0,1,\ldots .$$

By the assumption x_0 is in D^+ and so $x_1, x_2, \ldots$, are also. So we have $g(x_n) \ge 1$, $n=0,1,\ldots$. Now, from (F3) and (7) we conclude that

$$0 < \frac{f'(x_n)}{f'(\alpha_n)g(x_n)} \le \frac{f'(x_n)}{f'(\alpha_n)} < \frac{f'(b)}{f'(a)} \le 2, \quad n=0,1,\ldots,$$

which completes the proof.

REMARK 2. For Newton's method we have $g(x)=1$, $x \in D$ and $\phi'(x) = u(x)f''(x)/f'(x)$. Under the assumptions $f \in C^2(D)$ and (F1)-(F4) we have $\phi'(x) \geq 0$, $x \in D_o^+$ and Theorem 2 shows that for Newton's method the stopping inequality is valid. This is the result of |1|.

THEOREM 3. *Let* $f \in C^2(D)$ *satisfy the conditions* (F1)-(F3) *and let*

$$g \in C^1(D_o^-), \quad g(x) \geq 0.5, \quad x \in D_o^-$$

$$\phi'(x) \geq 0, \quad x \in D_o^-$$

Then for any $x_o \in D^-$ *the iterates* $x_n = \phi(x_{n-1})$, $n=1,2,\ldots$, *converge to the unique solution* $\alpha \in D$ *of* $f(x)=0$ *and the stopping inequality is valid for all* $n=0,1,\ldots$.

P r o o f. One can easily show that for $x_o \in D^-$

$$a \leq x_{n-1} < x_n < \alpha$$

holds either for all $n=1,2,\ldots$, or for $n=1,2,\ldots,k$ with some fixed $k \in \mathbb{N}$ and $x_{k+i} = \alpha$, $i=1,2,\ldots$. Thus $x_n \in D_o^-$, $n=1,2,\ldots$ and $\alpha = \lim_{n\to\infty} x_n$.

As in the proof of Theorem 2, the stopping inequality is valid if (7) holds, where $\alpha_n \in (x_n, \alpha)$, $n=0,1,\ldots$. From (F3) we have $0 < f'(x_n) < f'(\alpha_n)$, $n=0,1,\ldots$. Then, since $g(x_n) \geq 0.5$, $n=0,1,\ldots$, we obtain

$$0 < \frac{f'(x_n)}{f'(\alpha_n)g(x_n)} < \frac{1}{g(x_n)} \leq 2, \quad n=0,1,\ldots ,$$

which implies the stopping inequality.

REMARK 3. Let $f \in C^2(D)$. Under the assumptions (F1)-(F4) the classical regula falsi method can be written as

$$x_o \in D^-,$$

$$x_{n+1} = x_n - \frac{x_n - b}{f(x_n) - f(b)} f(x_n), \quad n=0,1,\ldots ,$$

with

$$g(x) = \frac{f'(x)}{f(x)-f(b)}(x-b), \quad x \in D.$$

This method is of the from $x_{n+1} = \phi(x_n)$, $n=0,1,\ldots$, where $\phi(x) = x-u(x)g(x)$. For some $\beta \in (x,b)$, $x \in D^-$, by the mean value theorem we have $g(x) = f'(x)/f'(\beta)$. Now from (F3), (F4) and $x < \beta$ we conclude

$$g(x) = \frac{f'(x)}{f'(\beta)} > \frac{f'(a)}{f'(b)} \geq 0.5, \quad x \in D_0^-.$$

In this case we have

$$\phi'(x) = 1 + \frac{1}{f(b)-f(x)}\left(f(x)-f(b)\frac{f'(x)}{f'(\beta)}\right) > 0, \; x \in D_0^- .$$

Now from Theorem 3 it follows that for the classical <u>regula</u> falsi method the stopping inequality is valid.

REMARK 4. If we replace (F3) and (F4) by

(F3´) $f''(x) < 0, \quad x \in D,$

(F4´) $f'(a) < 2f'(b)$,

and D^+, D_0^+, by D^-, D_0^- Theorems 2 is also valid.

If we replace, in Theorem 3, the condition (F3) by (F3´), and D^-, D_0^- by D^+, D_0^+ Theorem 3 is true.

3. THE ORDER OF ITERATION FUNCTIONS GENERATED BY INVERSE INTERPOLATION

In this section we shall study the iteration functions generated by inverse interpolacion which are given in |2|.

Let $\alpha \in D$ be a solution of $f(x) = 0$. Let f' be non-zero in a neighbourhood D of α and let $f^{(s)}$ be continuous in this neighbourhood. Then f has an inverse F and $F^{(s)}$ is continuous in a neighbourhood of zero. Let Q_s be the polinomial whose first s-1 derivative agree with F at the point $y = f(x)$. Then

$$F(t) = Q_s(t) + \frac{F^{(s)}(z(t))}{s!}(t-y)^s$$

and

$$Q_s(t) = \sum_{j=o}^{s-1} \frac{F^{(j)}}{j!} (t-y)^j ,$$

where $z(t)$ lies in the interval determined by y and t.
Define

$$E_s = Q_s(0) .$$

Hence

$$E_s = \sum_{j=o}^{s-1} \frac{(-1)^j}{j!} F^{(j)} f^j ,$$

or

$$E_s = x - \sum_{j=1}^{s-1} \frac{(-1)^{j-1} F^{(j)}}{j!\,(F')^j} u^j .$$

With the definition

$$Y_j(x) = \left. \frac{(-1)^{j-1} F^{(j)}(y)}{j!\,(F'(y))^j} \right|_{y=f(x)}$$

we can write

$$E_s = x - \sum_{j=1}^{s-1} Y_j u^j ,$$

and

$$\alpha = E_s + \frac{(-1)^s F^{(s)}(z(o))}{s!\,(F')^s} u^s .$$

Now we can write $E_s(x)$ in the form (4):

$$E_s(x) = x - u(x)g(x) ,$$

where

(8) $$g(x) = \sum_{j=1}^{s-1} Y_j u^{j-1} .$$

If $x_o \in D$ is some initial estimate of α, we form the sequence

(9) $$x_{n+1} = E_s(x_n), \quad n=0,1,\ldots .$$

Under certain conditions these iterates converge to α. We shall give a sufficient condition for the stopping inequality for the iterative method (9). First we shall give some assumptions.

In this section let α be the unique solution of $f(x)=0$ in $D=[a,b]$ and let D^-, D_o^-, D^+, D_o^+ be defined as before. Let $s \in N$, $s \geq 2$.

THEOREM 4. *Let* $f \in C^s(D)$ *satisfy* (F1), (F2), (F4) *and*

(F5) $\quad \operatorname{sgn}(f^{(j)}(x)) = (-1)^j, \quad x \in D, \ j=2,3,\dots,s \ .$

Then for any $x_o \in D^+$ *the iterates* x_n, *defined by* (9), *converge to the unique solution* $\alpha \in D$ *of* $f(x)=0$ *and the stopping inequality is valid for all* $n=0,1,\dots$.

P r o o f. From (F2) we have $f'(x)>0$, $x \in D$, and f has an inverse F. In a neighbourhood of zero $F^{(s)}$ is continuous. In |2| the formula for the derivative of the inverse function is given. We have

$$F^{(j)} = (f')^{-j} \sum (-1)^r (j+r-1)! \prod_{i=2}^{j} \frac{(A_i)^{B_i}}{B_i!} \ , \quad j=1,2,\dots,s \ ,$$

with the sum taken over all non-negative integers B_i such that

$$\sum_{i=2}^{j} (i-1)B_i = j-1 \ ,$$

and where

$$r = \sum_{i=2}^{j} B_i, \quad A_j(x) = \frac{f^{(j)}(x)}{j!\,f'(x)} \ , \quad j=1,2,\dots,s \ .$$

For $j=1$, $B_i=0$ for all i.

Now (F2) and (F5) imply

$$\operatorname{sgn}(A_j(x)) = \operatorname{sgn}(f^{(j)}(x)) = (-1)^j, \quad j=2,\dots,s \ ,$$

and

$$\operatorname{sgn}\Big((-1)^r (j+r-1)! \prod_{i=2}^{j} \frac{(A_i)^{B_i}}{B_i!}\Big) = (-1)^r \prod_{i=2}^{j} \operatorname{sgn}(A_i^{B_i}) =$$

$$= (-1)^r \prod_{i=2}^{j} (-1)^{iB_i} \ .$$

Observe that

$$(-1)^r \prod_{i=2}^{j} (-1)^{iB_i} = (-1)^r (-1)^{\sum_{i=2}^{j} iB_i} =$$

$$= (-1)^r (-1)^{j-1} (-1)^r = (-1)^{j-1} .$$

Thus

$$\text{(10)} \qquad \operatorname{sgn}(F^{(j)}(f(x))) = (-1)^{j-1}, \quad j = 1,2,\dots,s .$$

This implies

$$\operatorname{sgn}(Y_j(x)) = (-1)^{j-1} \operatorname{sgn}(F^{(j)}(f(x))) = 1, \quad j=1,2,\dots,s .$$

From (8) and $F'(f(x)) = (f'(x))^{-1}$ we have $Y_1 = 1$, $g \in C^1(D)$ and

$$\text{(11)} \qquad g(x) = 1 + \sum_{j=2}^{s-1} Y_j u^{j-1} \geq 1, \quad x \in D_o^+ .$$

In |2| it is proved that

$$E_{s+1} = E_s - \frac{u}{s} E_s' .$$

Thus

$$E_s' = \frac{(-1)^{s-1} F^{(s)}}{(s-1)!\,(F')^s} u^{s-1} .$$

Now for $x \in D_o^+$ $u^{s-1}(x) \geq 0$ holds and

$$\operatorname{sgn}(E_s'(x)) = (-1)^{s-1} \operatorname{sgn}(F^{(s)}(f(x))) = 1, \quad x \in D ,$$

i.e. $E_s'(x) \geq 0$, $x \in D_o^+$.

From (F5) we have $f''(x) > 0$, $x \in D$ and from (11) follows (5). Now we can apply Theorem 2.

REMARK 5. For $s = 2$ the iterates x_n, defined by (9), are Newton's iterates, and for $s = 3$ Chebyshev's.

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REZIME

IZLAZNI KRITERIJUM ZA NEKE ITERATIVNE METODE

U radu se posmatra rešavanje jednačine $f(x)=0$ u intervalu $D=[a,b]$, pri čemu je f realna funkcija realne promenljive, itarativnim postupkom $x_{n+1}=\phi(x_n)$, $n=0,1,\ldots$, sa pogodno izabranim $\phi(x)$ i $x_o \in D$. Kao dobra aproksimacija tačnog rešenja α date jednačine uzima se x_{n+1} ako važi $|x_{n+1}-x_n|<\varepsilon$ sa unapred izabranim $\varepsilon>0$ i ako važi nejednačina zaustavljanja

$$|x_{n+1}-x_n| \geq |x_{n+1}-\alpha| \ . \tag{1}$$

Za posebno izabrane funkcije ϕ dati su dovoljni uslovi za nejednačinu (1), a kao specijalne slučajeve dobijamo Njutnov iterativni postupak i klasičan postupak regula falsi. Takodje je za klasu iterativnih postupaka generisanih inverznom interpolacijom dokazana nejednačina (1) za $n=0,1,\ldots$.