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ON THE IMPOSSIBILITY OF EXTENDING THE PRODUCT OF CONTINUOUS FUNCTIONS TO THE FIELD OF MIKUSIŃSKI OPERATORS

Bogoljub Stanković
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića 4, Jugoslavija

ABSTRACT

Our aim is to show that the usual product of continuous functions can not be extended on the whole field M (the field of Mikusiński operators) or on some special subsets of M, as an inner operation, if we require the extended operation to preserve some natural properties of the product of continuous functions.

INTRODUCTION

The field M of Mikusiński operators |1| was constructed by the extension of the ring C of continuous functions $f=\{f(t)\}$ defined on the interval $[0,\infty)$ and provided with two operations: sum and convolution ($fg = \int_0^t f(t-u)\cdot g(u)du$). In C we have the countable and saturated family of semi-norms: $\|f\|_n = \max_{0\le t\le n}|f(t)|$ and in this way a topology. In M we define a convergence class too. Usually we work with the following one (called type I) : the sequence $\{a_n\}\subset M$ converges in M to a if and only if there exists $q\in M$ such that $qa_n \in C$ and qa_n converges to qa in C. This class is not topological.

The field M contains the ring L of local integrable functions defined on the interval $[0,\infty)$.

The default of an inner operation in M (let us denote it by ..), with the restriction on C as the usual product, diminishes the possibility of applying operators, especially in the theory of differential equations. This is the reason to ask the following question: Is it possible to define in M such an operation $\cdot\cdot$ which preserves some natural properties of the product of continuous functions? The answer to this question is the subject of discussion of this paper.

1. EXTENSION OF THE CLASSICAL DERIVATIVE TO THE ELEMENTS OF C

In the field M, the element ℓ (the integral operator which is from C) has, in M, the inverse element s (which is not from C). Using the operator s, we can enlarge the classical derivative to every continuous function $f \in C$: $Df = sf - f(0)I$; I is the unit element in the field M. This derivative D has the two following properties:

- If the function $f \in C$ has the usual derivative $f' \in L$, then $Df = f'$. This follows from the fact that in this case $sf = \{f'(t)\} + f(0)I$.
- If the functions f and g from C have as their derivatives f' and g' belonging to L, then in M there is:

$$D\{f(t)\cdot g(t)\} = \{g(t)\cdot Df(t)\} + \{f(t)\cdot Dg(t)\} . \tag{1}$$

2. THE SET P

By P we denote every subset from M which contains at least the ring C and elements of the form Df, $f \in C$. Let us suppose that in P is defined an inner operation $\cdot\cdot$ which is associative, and has the unit element $\{1\}$ and the restriction of which on C is the usual product. For this new operation we suppose something more: If f and g belong to C and their derivatives belong to L, then:

$$D(f \cdot\cdot g) = Df \cdot\cdot g + f \cdot\cdot Dg . \tag{2}$$

In P we induce the convergence class from M.

PROPOSITION. *The mapping* P *into* P : $a \to a_o \cdot\cdot a$, *where* a_o *is any element from* P, *is not sequentially continuous.*

The proof of this proposition is based on two lemmas.

LEMMA 1. *The inverse element of* $\{qt^p\} \in C$, $q \neq 0$, $0 < p < 1$, *exists in* P *for the operation* $\cdot\cdot$ *and it is just* $D\{rt^{1-p}\}$, $q\cdot r = \frac{1}{1-p}$.

P r o o f of lemma 1. - Let h be an element of P such that:

$$D\{r\ t^{1-p}\} \cdot\cdot \{q\ t^{p}\} = h\ .$$

Because of the associativity of the operation .. :

$$D\{r\ t^{1-p}\} \cdot\cdot \{q\ t^{p+1}\} = h \cdot\cdot \{t\}\ .$$

Using relation (2) and the properties of D we have:

$$D[\{rt^{1-p}\} \cdot \{qt^{p+1}\}] - [\{rt^{1-p}\} \cdot\cdot D\{qt^{p+1}\}] = h \cdot\cdot \{t\}\ ,$$

respectively:

$\{t\} = h \cdot\cdot \{t\}$, which gives $h = \{1\}$.

In a like manner we can prove that:

$$\{qt^{p}\} \cdot\cdot D\{rt^{1-p}\} = \{1\}\ .$$

LEMMA 2. $\lim\limits_{x \to 0^+} \Gamma(x)\ell^{x}$, $x > 0$, *does not exist in* M .

P r o o f of lemma 2. - Let us suppose that such a limit exists; then there exists a $q \in M$ such that $\Gamma(x)\ell^{x}q$ belongs to C and converges in C. From the continuity of the second operation in M it follows that there exists an element $w \in C$ such that $\pm\Gamma(x)\ell^{2+x}\Gamma(2+x)w$ belongs to C converges in C too.

We know that $\int_0^t u^{1+x} \cdot w(t-u)du$ converges uniformly to $\int_0^t u \cdot w(t-u)du$ in every interval $[0,T]$, $T < \infty$, as $x \to 0_+$. Because of the Titchmarsh theorem |2| there exists $t_0 \in (0,T_0)$ such that $\int_0^{t_0} u \cdot w(t_0-u)du = r \neq 0$. We can suppose that $r > 0$ (without any restriction). Then we also have an interval $[t_0-\eta,\ t_0+\eta] \subset [0,T_0]$ such that the continuous function $\int_0^t u \cdot w(t-u)du > r' > 0$. Let us chosse an ε in such a way that $0 < \varepsilon < r'$. For this ε there exists $\delta(\varepsilon)$ such that:

$$\int_0^t u^{1+x} w(t-u)du \geq \int_0^t u \cdot w(t-u)du - \varepsilon,\quad 0 < x < \delta(\varepsilon)\ ,$$

for every $t \in [0, T_o]$ and also for $t \in [t_o - \eta,\ t_o + \eta]$.

We know that $\Gamma(x) > 0$, $x > 0$ and $\Gamma(x) \to \infty$ for $x \to 0$, $x > 0$. Consequently:

$$\Gamma(x) \int_0^t u^{1+x} \cdot w(t-u)du \geq (r' - \varepsilon)\Gamma(x)$$

for $t \in [t_o - \eta,\ t_o + \eta]$ and $\Gamma(x)\Gamma(2+x)\ell^{2+x}w$ cannot converge in C.

P r o o f of the proposition. - Let us start from the sequence $\{a_n\} \subset P$:

$$a_n = \{t^v\} \cdot\cdot\ D\{nt^{1/n}\}, \qquad 0 < v < 1 .$$

First we shall find an other analytical expression for it. For this aim let us "multiply" a_n by

$$\{\frac{1}{n-1}\, t^{1-v-1/n}\} \in C, \qquad n \geq n_o > 1, \qquad 1-v-1/n_o > 0 :$$

$$\{\frac{1}{n-1}\, t^{1-v-1/n}\} \cdot\cdot\ a_n = \{\frac{t^{1-1/n}}{n-1}\} \cdot\cdot\ D\{nt^{1/n}\} =$$

$$= D\{\frac{n}{n-1}\, t\} \cdot\cdot\ D\{\frac{t^{1-1/n}}{n-1}\} \cdot\cdot\ \{nt^{1/n}\}.$$

By lemma 1 we have:

$$= \{\frac{n}{n-1}\} - \{1\} = \{\frac{1}{n-1}\} .$$

Now we can conclude that:

$$\{t^{1-v-1/n}\} \cdot\cdot\ a_n = \{1\}$$

and a_n, $n \geq n_o > 1$, $1-v-1/n_o > 0$ is the inverse element to $\{t^{1-v-1/n}\}$. Lemma 1 says that:

$$a_n = D\{\frac{t^{v+1/n}}{v+1/n}\} = \Gamma(v+1/n)\ell^{v+1/n} .$$

Now we shall prove that a_n converges in P to $D\{\frac{t^v}{v}\}$. a_n belongs to C and converges in C to $\Gamma(v)\ell^{1+v} = \ell\,\Gamma(v)\ell^v = \ell D\{\frac{t^v}{v}\}$.

The element $D\{\frac{t^{1-v}}{1-v}\} \in P$ can be our a_o from the proposition. "Multiplying" a_n by the so defined a_o and taking care of lemma 1 we have:

$$D\{\frac{t^{1-v}}{1-v}\} \cdot\cdot \{t^v\} \cdot\cdot D\{nt^{1/n}\} = D\{nt^{1/n}\} = \Gamma(1/n)\ell^{1/n}$$

and this sequence does not converge in P (by lemma 2).

CONSEQUENCE OF THE PROPOSITION. The consequence of our proposition is that the usual product of continuous functions cannot be extended to a set of the form P (as well as M) as an inner operation, if we require of the extended operation: to be associative; to have the unit element |1| ; for $f,g \in C$ with derivatives $f',g' \in L$, the relation (2) to be satisfied and the application $a \to a_o \cdot\cdot a$, P into P, to be sequentially continuous for every $a_o \in P$.

REMARK. Our proposition remains true if we change the convergence class in M and consequently in P under only one condition: that $\Gamma(x)\ell^x$ does not converge when $x \to 0_+$, in this new convergence class. This is the case if we replace the defined convergence class by the so-called convergence class type II.

REFERENCES

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REZIME

O NEMOGUĆNOSTI PROŠIRENJA NEPREKIDNIH FUNKCIJA NA POLJE OPERATORA MIKUSIŃSKOG

Obeležićemo sa C prsten neprekidnih funkcija, a sa L prsten lokalno integrabilnih funkcija definisanih nad [0,∞). M je polje operatora Mikusińskog |1|, a D operacija koja presli kava C u M i koja uopštava klasični izvod: Df = sf - f(0) I gde su s i I elementi iz M (s operator diferenciranja, a I jedinični elemenat). P će biti svaki podskup od M koji sadrži bar C i elemente oblika Df, f ∈ C.

Pokazano je da se u P (pa time i u M) ne može definisati unutrašnja operacija ·· ako se od nje zahteva da je njena restrikcija nad C obično množenje, da je asocijativna, da ima jedinični elemenat {1}, da za f i g iz C koje imaju izvode f′ i g′ iz L bude zadovoljena relacija (2) i da je preslikavanje $a \to a_o \cdot\cdot a$ sekvencijalno neprekidno za svako $a_o \in P$. Sekvencijalna neprekidnost se odnosi na konvergentnu klasu tipa I i II kao i svaku drugu u kojoj ne postoji $\lim_{x \to o^+} \Gamma(x) \ell^x$.