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A NOTE ON QUOTIENT SPACES AND PARACOMPACTNESS

Ilija Kovačević
Fakultet tehničkih nauka. Institut za primenjene osnovne discipline, 21000 Novi Sad, ul. Veljka Vlahovića 3, Jugoslavija

ABSTRACT

The aim of the present paper is to study some properties of space $\mathcal{D}$, where $\mathcal{D}$ is an almost-upper semicontinuous decomposition of a topological space X with quotient topology.

The notation is standard except that $\alpha(A)$ will be used to denote the interior of the closure of A.

1. DEFINITIONS AND SOME KNOWN RESULTS

DEFINITION 1.1. *A subset of a space is said to be regularly open iff it is the interior of some closed set or equivalently iff it is the interior of its own closure. A set is said to be regularly closed iff it is the closure of some open set or equivalently iff it is the closure of its own interior,* |1|.

A subset is regularly open iff its complement is regularly closed.

DEFINITION 1.2. *A space* X *is said to be almost regular iff for any regularly closed set* F *and any point* $x \notin F$, *there exist disjoint open sets containing* F *and* x *respectively,* |10|.

DEFINITION 1.3. *A space* X *is said to be almost normal iff for every pair of disjoint sets* A *and* B *, one of which is closed and the other regularly closed, there exist disjoint open sets* U *and* V *such that* $A \subset U$, $B \subset V$, |12|.

DEFINITION 1.4 *A subset* A *of a space* X *is* α*-nearly compact (N-closed) iff every* X*-regularly open cover of* A *has a finite subcovering,* |11|.

DEFINITION 1.5. *A space* X *is locally nearly compact iff each point of* X *has an open neighbourhood* U *such that* $\bar{U}$ *is* α*-nearly compact,* |2|.

DEFINITION 1.6 *A space* X *is nearly paracompact iff every regularly open cover of* X *has a locally finite open refinement,* |12|.

DEFINITION 1.7. *Let* X *be a topological space and* A *a subset of* X. *The set* A *is* α*-nearly paracompact iff every* X*-regularly open cover of* A *has an* X*-open* X*-locally finite refinement which covers* A, |6|.

DEFINITION 1.8. *A function* $f : X \to Y$ *is said to be almost continuous iff for each point* $x \in X$ *and each open neighbourhood* V *of* f(x) *in* Y, *there exists an open neighbourhood* U *of* x *in* X *such that* $f(U) \subset \alpha(V)$, |9|.

A function is almost continuous iff the inverse image of every regularly open set is open, |9|.

DEFINITION 1.9. *A function* $f : X \to Y$ *is said to be almost closed (almost-open) iff for every regularly closed (regularly open) set* F *of* X f(F) *is closed (open) in* Y, |9|.

DEFINITION 1.10. *A decomposition* $\mathcal{D}$ *of a topological space* X *is almost-upper semicontinuous iff for each* D *in* $\mathcal{D}$ *and each regularly open set* U *containing* D *there exists an open set* V *such that* $D \subset V \subset U$ *and* V *is the union of members of* $\mathcal{D}$, |7|.

THEOREM 1.1 *A space* X *is almost regular iff for each point* $x \in X$ *and each regularly open set* V *containing* x, *there exists a regularly open set* U *such that* $x \in U \subset \bar{U} \subset V$, |10|.

THEOREM 1.2. *Let* $\mathcal{D}$ *be a decomposition of a topological space* X *and let* $\mathcal{D}$ *have a quotient topology. A decomposition* $\mathcal{D}$ *is almost-upper semicontinuous iff the projection* P *of* X *onto* $\mathcal{D}$ *is almost closed*, |7|.

THEOREM 1.3. *Let* X *be a topological space. Let* $\mathcal{D}$ *be an almost-upper semicontinuous decomposition of* X *whose members are* α*-nearly compact subsets of* X *and let* $\mathcal{D}$ *have a quotient topology. Then* $\mathcal{D}$ *is, respectively, Hausdorff or almost regular, provided* X *has a corresponding property*, |7|.

THEOREM 1.4. *If* A *is an* α*-nearly paracompact subset of a Hausdorff space* X *and* x *a point* $X \setminus A$, *then there are disjoint regularly open neighbourhoods of* x *and* A, |5|.

THEOREM 1.5. *If* f *is an almost closed and continuous mapping of a normal (almost normal) space* X *onto a space* Y, *then* Y *is normal (almost normal)*, |4|.

THEOREM 1.6. *Let* X *be a nearly paracompact almost regular space. If* $f : X \to Y$ *is an almost continuous, almost closed surjection, such that* $f^{-1}(y)$ *is* α*-nearly compact for each point* $y \in Y$, *then* Y *is nearly paracompact almost regular*, |3|.

THEOREM 1.7. *If* f *is an almost closed and continuous mapping of a Hausdorff paracomapct space* X *onto a* T_1 *space* Y, *then* Y *is paracompact regular*, |4|.

THEOREM 1.8. *If* X *is an almost regular topological space,* A *an* α*-nearly paracompact subset,* U *a regularly open neighbourhood of* A, *then there exists a regularly open neighbourhood* V *of* A *such that* $A \subset V \subset \bar{V} \subset U$, |5|.

THEOREM 1.9. *If a mapping* $f : X \to Y$ *is almost continuous and almost closed, then for each regularly closed (regularly* *open) set* F *of* Y*,* $f^{-1}(F)$ *is regularly closed (regularly open) in* X*,* |3|.

THEOREM 1.10. *A surjection mapping* $f : X \to Y$ *is almost closed iff for any subset* B *of* Y *and any regularly open set* $U \subset X$ *containing* $f^{-1}(B)$*, there exists an open set* V *in* Y *such that* $B \subset V$ *and* $f^{-1}(V) \subset U$*,* |8|.

2. SOME CHARACTERIZATIONS OF QUOTIENT SPACES AND PARACOMPACTNESS

THEOREM 2.1. *Let* X *be a Hausdorff space. Then for any disjoint subsets* A *and* B*, where* A *is* α*-nearly paracompact* B *is* α*-nearly compact, there exist disjoint regularly open sets* U *and* V *such that* $A \subset U$ *and* $B \subset V$.

P r o o f. For each point $x \in B$, by Theorem 1.4, there exist disjoint regularly open sets U_x and V_x such that $A \subset U_x$, $x \in V_x$. The family

$$\mathcal{V} = \{V_x : x \in B\}$$

is a cover of B by regularly open sets of X. Sinse B is α-nearly compact, there exist a finite number of points $x_1, x_2, \dots$ $\dots, x_n$ in B such that

$$B \subset V_{x_1} \cup V_{x_2} \cup \dots \cup V_{x_n} \quad .$$

Let

$$U = U_{x_1} \cap U_{x_2} \cap \dots \cap U_{x_n} \quad \text{and} \quad V_1 = V_{x_1} \cup V_{x_2} \cup \dots \cup V_{x_n} \quad .$$

Now, we have $A \subsetneq U$, $B \subset V_1$ and $U \cap V_1 \neq \emptyset$. Let $V = \alpha(V_1)$. Now, U and V are regularly open subsets of X such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

THEOREM 2.2. *If* f *is an almost closed continuous mapping of a regular space* X *onto a space* Y *such that* $f^{-1}(y)$ *is* α*-nearly paracompact for each point* $y \in Y$*, then* Y *is regular.*

P r o o f. Since, in regular space, every α-nearly paracompact subset is α-paracompact, then $f^{-1}(y)$ is α-paracompact for each point $y \in Y$. Let $y \in Y$ and V be an open set containing y. Then $f^{-1}(V)$ is an open set in X containing $f^{-1}(y)$. Since X is regular and $f^{-1}(y)$ is α-paracompact there exists a regularly open set U in X such that

$$f^{-1}(y) \subset U \subset \bar{U} \subset f^{-1}(V).$$

Since f is almost closed, then, by Theorem 1.10, there exists an open set W in Y such that $y \in W$ and $f^{-1}(W) \subset U$. Therefore,we have

$$y \in W \subset f(U) \subset f(\bar{U}) \subset V.$$

Since f is almost closed and $\bar{U}$ is regularly closed, $f(\bar{U})$ is closed. Hence we have

$$y \in W \subset \alpha(W) \subset \bar{W} \subset V.$$

We have, that for every point $y \in Y$ and every open neighbourhood V of y there exists an open set W such that $y \in W \subset \bar{W} \subset V$, hence Y is regular.

COROLLARY 2.1. *Regularity is preserved under perfect mappings.*

THEOREM 2.3. *If f is an almost closed almost continuous mapping of an almost regular space X onto a space Y such that $f^{-1}(y)$ is α-nearly paracompact for each point $y \in Y$, then Y is almost regular.*

P r o o f. Let $y \in Y$ and V be a regularly open set of Y containing y. Then, since f is almost closed and almost continuous, by Theorem 1.9, $f^{-1}(V)$ is a regularly open set in X containing $f^{-1}(y)$. Since X is almost regular and $f^{-1}(y)$ is α-nearly paracompact, then, by Theorem 1.8, there exists a regularly open set U in X such that

$$f^{-1}(y) \subset U \subset \bar{U} \subset f^{-1}(V).$$

Since f is almost closed, then there exists an open set W in Y such that $y \in W$ and $f^{-1}(W) \subset U$. Therefore, we have

$$y \in W \subset f(U) \subset f(\bar{U}) \subset V.$$

Since f is almost closed, $f(\bar{U})$ is closed. Hence we have

$$y \in W \subset \alpha(W) \subset \bar{W} \subset V.$$

Since $\overline{\alpha(W)} = \bar{W}$, then $\alpha(W)$ is a regularly open set containing y such that $y \in \alpha(W) \subset \overline{\alpha(W)} \subset V$, hence, by Theorem 1.1, Y is almost regular.

COROLLARY 2.2. *If f is an almost closed, almost continuous mapping of an almost regular spaces X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each point $y \in Y$, then Y is almost regular,* |5|.

P r o o f. Every α-nearly compact is α-nearly paracompact.

THEOREM 2.4. *Let X be a topological space. Let $\mathcal{D}$ be an almost-upper semicontinuous decomposition of X whose members are α-nearly paracompact subsets of X and let $\mathcal{D}$ have a quotient topology. If X is regular, then the projection P of a space X onto a space $\mathcal{D}$ is closed. If X is Hausdorff (regular, almost regular), then $\mathcal{D}$ is T_1 (regular, almost regular).*

P r o o f. Let X be a regular. Now, we have that for every point $A \in \mathcal{D}$, A is an α-nearly paracompact subset in X. Since every α-nearly paracompact subset in regular space is α-paracompact, then A is an α-paracompact subset in X. Let U be an open neighbourhood of A in X. Since A is α-paracompact, there exists an open neighbourhood W of A such that

$$A \subset W \subset \bar{W} \subset U.$$

Now, $\alpha(W)$ is a regularly open neighbourhood of A. Since $\mathcal{D}$ is almost-upper semicontinuous, then there exists an open set V such that $A \subset V \subset \alpha(W)$ and V is the union of members of $\mathcal{D}$. Now, we have that for each $A \in \mathcal{D}$ and each open set U containing A, there exists an open set V such that $A \subset V \subset U$ and V is the union of members of $\mathcal{D}$, hence $\mathcal{D}$ is an upper semicontinuous decomposition of X. Since $\mathcal{D}$ is an upper semicontinuous decomposition of

X, the projection P of a space X onto a space $\mathcal{D}$ is closed. If X is regular (almost regular) then $\mathcal{D}$ is, by Theorem 2.2 (Theorem 2.3) regular (almost regular).

Now, let X be a Hausdorff space. Let A be any point of $\mathcal{D}$. Then the set A is an α-nearly paracompact subset in X. By Theorem 1.4 A is closed in X. Since a subset F of $\mathcal{D}$ is closed iff $P^{-1}(F)$ is closed in X, then the point $A \in \mathcal{D}$ is closed in $\mathcal{D}$, hence $\mathcal{D}$ is T_1.

THEOREM 2.5. *Let* X *be a topological spaces. Let* $\mathcal{D}$ *be an almost-upper semicontinuous decomposition of* X *whose members are α-nearly paracompact subset of* X *and let* $\mathcal{D}$ *have a quotient topology. If* X *is a Hausdorff paracompact space, then* $\mathcal{D}$ *is paracompact regular.*

P r o o f. Since X is a Hausdorff paracompact space, then X is regular. By Theorem 2.4, $\mathcal{D}$ is regular. Since X is a Hausdorff space, then $\mathcal{D}$ is T_1. Since every Hausdorff paracompact space is normal, X is normal. Since the projection of a space X onto a space $\mathcal{D}$ is almost closed and continuous, it follows easily from Theorem 1.7, that $\mathcal{D}$ is a regular and paracompact space.

THEOREM 2.6. *Let* X *be a topological spaces. Let* $\mathcal{D}$ *be an almost-upper semicontinuous decomposition of* X *whose members are α-nearly compact subsets of* X *and let* $\mathcal{D}$ *have a quotient topology. If* X *is paracompact regular, then* $\mathcal{D}$ *is paracompact regular.*

P r o o f. Since the members of $\mathcal{D}$ are α-nearly compact subsets of X and the projection P of X onto $\mathcal{D}$ is almost closed and continuous, then $\mathcal{D}$ is paracompact regular.

THEOREM 2.7. *Let* X *be a topological space. Let* $\mathcal{D}$ *be an almost-upper semicontinuous decomposition of* X *and let* $\mathcal{D}$ *have a quotient topology. If* X *is normal (almost normal), then* $\mathcal{D}$ *is normal (almost normal).*

P r o o f. Since the projection P of a space X onto a space $\mathcal{D}$ is almost closed and continuous, then, by Theorem 1.5 $\mathcal{D}$ is normal (almost normal).

THEOREM 2.8. *Let* X *be a topological space. Let* $\mathcal{D}$ *be an almost-upper semicontinuous decomposition of* X *whose members are* α*-nearly compact subsets of* X *and let* $\mathcal{D}$ *have a quotient topology. If* X *is nearly paracompact almost regular, then* $\mathcal{D}$ *is nearly paracompact almost regular.*

P r o o f. Since the members of $\mathcal{D}$ are α-nearly compact subsets of X and the projection P of X onto $\mathcal{D}$ is almost closed and continuous, then, by Theorem 1.6, $\mathcal{D}$ is nearly paracompact almost regular.

LEMMA 2.1. *If* $f : X \to Y$ *is an almost closed almost continuous mapping of a space* X *onto a space* Y, *then the image of every* α*-nearly compact subset in* X *is an* α*-nearly compact subset in* Y.

P r o o f. Let K be any α-nearly compact subset in X. Let

$$\mathcal{U} = \{U_\alpha : \alpha \in I\}$$

be any Y-regularly open cover of f(K). Then

$$\{f^{-1}(U_\alpha) : \alpha \in I\}$$

is an X-regularly open cover of K. Since K is α-nearly compact, there exists a finite subset I_o of I such that

$$K \subset \bigcup\{f^{-1}(U_\alpha) : \alpha \in I_o\}.$$

Now, we have

$$f(K) \subset \bigcup\{U_\alpha : \alpha \in I_o\},$$

hence f(K) is α-nearly compact.

THEOREM 2.9. *Let* $f : X \to Y$ *be an almost closed almost continuous mapping of a space* X *onto a space* Y *such that* $f^{-1}(y)$ *is* α*-nearly compact for each point* $y \in Y$. *If* X *is a Hausdorff* *locally nearly compact space, then* Y *is locally nearly compact Hausdorff.*

P r o o f. Y is a Hausdorff space. We shall show that Y is locally nearly compact. Let y be any point of Y. Since X is locally nearly compact Hausdorff, for each point $x \in f^{-1}(y)$, there exists a regularly open neighbourhood K_x such that $\bar{K}_x$ is α-nearly compact. Now, the family

$$\mathcal{K} = \{K_x : x \in f^{-1}(y)\}$$

is an X-open cover of $f^{-1}(y)$. Since $f^{-1}(y)$ is α-nearly compact, then there exist a finite number of points $x_1, x_2, \dots, x_n$ in $f^{-1}(y)$ such that

$$f^{-1}(y) \subset \cup \{ K_{x_i} : i=1,2,\dots,n\} .$$

Let

$$K = \cup \{ \bar{K}_{x_i} : i=1,2,\dots,n\}.$$

$f^{-1}(y) \subset K^o$. Since f is almost closed, then there exists an open set V_y containing y such that $f^{-1}(V_y) \subset K^o$. Hence, we have

$$y \in V_y \subset f(K^o) \subset f(K) .$$

Since f is almost continuous and almost closed and K is α-nearly compact, then f(K) is α-nearly compact. Since Y is Hausdorff f(K) is closed. Therefore we have

$$\bar{V}_y \subset f(K).$$

Since, every Y-regularly closed subset of an α-nearly compact subset is α-nearly compact, then $\bar{V}_y$ is α-nearly compact.

Now, V_y is a Y-open neighbourhood of y such that $\bar{V}_y$ is α-nearly compact, hence Y is locally nearly compact.

COROLLARY 2.3. *Let* $f:X \to Y$ *be an almost closed almost continuous surjection with* N-*closed point inverses. If* X *is locally compact Hausdorff, then* Y *is locally nearly compact Hausdorff* , |8|.

THEOREM 2.10. *Let X be a topological space. Let $\mathcal{D}$ be an almost-upper semicontinuous decomposition of X whose members are α-nearly compact subsets of X and let $\mathcal{D}$ have a quotient topology. If X is locally nearly compact Hausdorff, then $\mathcal{D}$ is locally nearly compact Hausdorff.*

P r o o f. Since the members of $\mathcal{D}$ are α-nearly compact subsets of X and the projection P of X onto $\mathcal{D}$ is almost closed and continuous, then, by Theorem 2.9, $\mathcal{D}$ is locally nearly compact Hausdorff.

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REZIME

KOLIČNIK PROSTORI I PARAKOMPAKTNOST

Cilj rada je ispitivanje osobina prostora $\mathcal{D}$, gde je $\mathcal{D}$ skoro polu-neprekidno razlaganje prostora X sa količnik topologijom.