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A NOTE ON THE SPACE $\mathcal{D}'_A$

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1. In |4| a subspace $\mathcal{D}'_A \subset \mathcal{D}'$ is investigated. It is shown that the elements of $\mathcal{D}'_A$ may be uniquely expanded into a Hermitte series. First the space A is defined as the space of functions defined on R such that $f \in A$ iff for some $a > 4$, $F(x)e^{-x^2/a} \in L_2(R)$ Furthermore, the set $\mathcal{D}'_A$ is defined such that $f \in \mathcal{D}'_A$ iff there exist $k \in N_o$, $F \in A$ such that

$$(1) \qquad f = D^k(F)$$

where D is the tempered derivative given in |1| with

$$(2) \qquad D^k(F) = \exp(-x^2/4)(\exp/x^2/4)F)^{(k)}, \quad D^o(F) = F$$

and k-th derivative is in the distributional sense. It is shown that $\mathcal{D}'_A$ is a subspace of $\mathcal{D}'$, and an estimate of Hermitte coefficients of elements from $\mathcal{D}'_A$ is given.

In this paper, using the elementary approach, we shall characterize the elements from $\mathcal{D}'_A$. Also, we shall introduce the convergence in $\mathcal{D}'_A$ and characterize it. In this way we shall show that $\mathcal{D}'_A$ is identical with the space of the $K'(M_p)$- type, introduced in |2|, for the special sequence (M_p).

2. Let us put

$$(3) \qquad M_p(x) = e^{x^2/(4-1/p)}, \quad p=1,2,\dots .$$

THEOREM 1. $f \in \mathcal{D}'_A$ *iff there exist* $p \in N, m \in N_o$ *and bounded measurable functions* f_j, $0 \le j \le m$, *such that*

$$f = \sum_{j=o}^{m} (M_p\, f_j)^{(j)} \,. \tag{4}$$

P r o o f. Let $f \in \mathcal{D}'_A$. This means that there exist $k \in N_o$ and $F \in A$ such that (1) holds.

The tempered integral given in |1| is defined on the space of locally integrable functions by

$$S^o(G) = G; \quad S(G) = e^{-x^2/4} \int_o^x e^{t^2/4} G(t)dt; \quad S^k(G) = S(S^{k-1}G) \tag{5}$$

As $A \subset L_1^{loc}$, if we put $F_1(x) = S(F(x))$, then

$$|F_1(x)| \le e^{-x^2/4} \left| \int_o^x F(t)\; e^{-t^2/a}\; e^{+t^2/a}\; e^{+t^2/4}\; dt \right| \le$$

$$\le \; e^{-x^2/4} \left(\int_o^x (|F(t)|e^{-t^2/a})^2 dt \right)^{1/2} \left(\int_o^x e^{2(t^2/a+t^2/4)} dt \right)^{1/2} \le$$

$$\le \; K\sqrt{x}\; e^{x^2/a} \quad \text{where} \quad K = \left(\int_R (|F(t)|e^{-t^2/a})^2 dt \right)^{1/2} .$$

There exists $p \in N$ such that $a > 4 + 1/p$. This implies that for a suitable constant K_o

$$|F_1(x)| \le K_o M_p(x) \;\; \text{i.e.} \;\; M_{p+1}^{-1}(x) F_1(x) \in L_2$$

Since $D(f) = f' + x \cdot f/2$ and $D(S(F)) = F$, the members of the expression $D^{k+1}(F_1) = f$ are of the form $c_{j,\ell}\, x^j (F_1(x))^{(\ell)}$, $j \le k+1$, $\ell \le k+1$ and $c_{j,\ell}$ are real numbers.

After using Leibnitz's formula we obtain that in $D^{k+1}(F_1)$ the members are of the form $\tilde{c}_{j,\ell}(x^j F_1(x))^{(\ell)}$ with some other coefficients $\tilde{c}_{j,\ell}$. In the same way as for $F_1(x)$, we can prove that $|x^j F_1(x)| \le K_j \cdot M_p(x)$. By putting

$$f_\ell = \sum_{j \le k+1} \tilde{c}_{j,\ell}\, x^j F_1(x) \quad \text{we get}$$

$$f(x) = \sum_{\ell \le k+1} (M_{p+1}(x) f_\ell(x))^{(\ell)}$$

which means that f has the representation of the form (4).

Let us now assume that f has the representation of the form (4). We denote $M_p f_j = F_j$. Since $M_{p+1}^{-1} F_j \in L_2$, similarly as in the first part of that proof, it follows that $S(x^r F_j(x) M_{p+3}^{-1}(x)) \in \in L_2$ for any $r \in N_o$ From the formula

(6) $\quad f' = D(f) - x\cdot f/2$ and $x'D(f) = D(xf) - f$,

taking operator S enough times, we obtain that

$$\sum_{j \le m} (F_j(x))^{(j)} = D^k(H(x))$$

for some $k \in N$, and some function H(x) such that

$$M_{p+p_o}^{-1}(x) H(x) \in L_2 .$$

The integer p_o depends on the number of applications of operator S. So we have proved that $f \in \mathcal{D}'_A$.

Let us define the convergence structure in the space $\mathcal{D}'_A$.

We say that the sequence (f_n) from $\mathcal{D}'_A$ converges to $f \in \mathcal{D}'_A$ iff there exist a sequence of functions (F_n), a function F, $k \in N_o$ and $a > 0$ such that

(7) $\quad D^k(F_n) = f_n \ , \quad D^k(F) = f$

and $(e^{-x^2/a} F_n(x))$ is the sequence from L_2 which converges to $e^{-x^2/a} F(x)$ in the L_2 norm.

THEOREM 2. *The sequence (f_n) from $\mathcal{D}'_A$ converges to $f \in \mathcal{D}'_A$ iff there exist $p \in N, m \in N_o$, sequences of bounded measurable functions $(F_{j,n})$, $0 \le j \le m$, and bounded measurable functions F_j, $0 \le j \le m$, such that*

$$F_{j,n} \to F_j \qquad \textit{almost everywhere}$$

(8) $$f_n = \sum_{j \le m} (M_p F_{j,n})^{(j)} \qquad f = \sum_{j \le m} (M_p F_j)^{(j)} .$$

P r o o f. If condition (7) is satisfied we put $F_{1,n} = S(F_n)$ and $F_1 = S(F)$. From the estimate

$$|F_{1,n}(x) - F_1(x)| \le e^{-x^2/4} \left(\int_0^x |F_{1,n}(t) - F_1(t)|^2 e^{-2t^2/a} dt \right)^{1/2} \cdot \left(\int_0^x e^{2(t^2/a + t^2/4)} dt \right)^{1/2}$$

if follows that $(F_{1,n}(x))$ converges almost everywhere to F_1. Similarly, as in Theorem 1, observing the expressions $D^{k+1}F_{1,n}$ and $D^{k+1}F_1$ we can prove that (8) holds.

Let us suppose that (8) holds. If we denote

$$F_j = M_p(x) \cdot f_j \quad \text{and} \quad F_{j,n} = M_p(x) \cdot f_{j,n}$$

similarly as in the proof of Theorem 1, we can show that

$$x^r F_{j,n} M_{p+3}^{-1}(x) \in L_2 \quad \text{and} \quad x^r F_j M_{p+3}^{-1}(x) \in L_2 ,$$

$0 \le j \le m$; $r \in N_o$. From the Lebesque theorem it follows that

$$x^r F_{j,n} M_{p+3}^{-1}(x) \xrightarrow{L_2} x^r F_j M_{p+3}^{-1}(x) ,$$

In the same way as in the proof of Theorem 1, using the formulae of (6) and applying S sufficiently many times we get the assertion.

As the tempered derivative D characterizes the space S' which is proved in |1|, from Theorems 1. and 2. we get

COROLLARY. *(i)* $f \in \mathcal{D}'_A$ *iff for some* $a > 4$, $m \in N_o$ *and a bounded continuous function* $F(x)$

$$f = (e^{x^2/a} F(x))^{(m)}$$

(ii) The sequence $(f_n) \in \mathcal{D}'_A$ *converges to* $f \in \mathcal{D}'_A$ *iff there exist bounded continuous functions* F_n, $n \in N$, *a bounded continuous function* F, $a > 4$, *and* $m \in N_o$, *such that*

$$f_n = (e^{x^2/a} F_n(x))^{(m)}, \quad f(x) = (e^{x^2/a} F(x))^{(m)}$$

and $(F_n(x))$ *converges to* $F(x)$ *almost uniformly.*

P r o o f. We only have to apply the operator S sufficiently many times in (4) and (8).

3. In |2| the spaces of the type $K'(M_p)$ are introduced and investigated.

Using the theory from |2| and |3|, from (4) and (8) directly it follows that spaces $\mathcal{D}_A'$ and $K'(\exp(-x^2/(4+1/p))$ are indentical both in a set theoretical and a topological sense.

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REZIME

PRILOZI TEORIJI PROSTORA $\mathcal{D}_A'$

U ovom radu, koristeći elementarni pristup, karakterišemo elemente iz $\mathcal{D}_A'$ uvedenog u |4|. Takodje uvodimo konvergenciju u prostor $\mathcal{D}_A'$ i karakterišemo je. Tako pokazujemo da je $\mathcal{D}_A'$ identičan sa prostorom $K'(M)$- tipa uvedenog u |2|, za specijalan niz $(M_p(X))$.