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NOTE ON THE SPANNING TREES OF A CONNECTED DIGRAPH

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ABSTRACT

Our aim in this paper is to give some relations between the spanning trees and some determinants obtained from the incidence matrix of a connected digraph. The spanning trees that differ by one edge are also investigated.

Let $D = (V,E)$ be a connected digraph (directed graph) with $V = \{v_1, v_2, \dots, v_p\}$ the set of vertices, $E = \{e_1, e_2, \dots, e_q\}$ the set of edges, and $S = (s_{ij})$, $i=1,2,\dots,p$; $j=1,2,\dots,q$, the incidence martix, where

$$s_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is the initial vertex of } e_j, \\ -1, & \text{if } v_i \text{ is the terminal vertex of } e_j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{S}$ be the matrix obtained from S by deleting the line corresponding to the vertex v_p. If $T = \{e_{j_1}, e_{j_2}, \dots, e_{j_m}\}$ $(m = p-1)$ is a spanning tree of D, we shall denote by $\bar{S}(T)$ the square submatrix of $\bar{S}$ obtained with the lines of $\bar{S}$ and the columns $j_1, j_2, \dots, j_m$. Because T is a spanning tree, there exists a unique chain connecting any two vertices in the graph (V,T). Let $c_i(v_i, v_p)$ such a chain connecting v_i with v_p, $i=1,2,\dots,m$, and $E(c_i)$ the edge-set of c_i. Let $e_{\alpha(i)}$, $\alpha(i) \in \{j_1, j_2, \dots, j_m\}$ the unique edge incident with the vertex v_i, $i=1,2,\dots,m$, for which $e_{\alpha(i)} \in E(c_i) \cap T$, and $\varepsilon(i) \in \{1,2,\dots,m\}$ such that $\alpha(i) = j_{\varepsilon(i)}$.

We consider the matrix $\bar{\bar{S}}(T) = [\bar{\bar{S}}(T)]_{i\beta}$, $i,\beta=1,2,\dots,m$, where

$$[\bar{\bar{S}}(T)]_{i\beta} = \begin{cases} 0 \ , \text{ if } \beta \neq \varepsilon(i), \\ [\bar{S}(T)]_{i,\varepsilon(i)}, \text{ if } \beta = \varepsilon(i). \end{cases}$$

THEOREM 1. $\det[\bar{S}(T)] = \det[\bar{\bar{S}}(T)]$.

P r o o f. We denote by $S(T)$ the submatrix of S obtained with the lines od S and the columns $j_1, j_2, \dots, j_m$.

Let v_{t_1} $(v_{t_1} \neq v_p)$ a terminal vertex of $T^{(1)} = T$. Adding the line t_1 of the matrix $S(T)$ to the line corresponding to the other vertex of $e_{\alpha(t_1)}$ we obtain the matrix $S_1(T)$. We consider now the tree $T^{(2)}$ obtained from $T^{(1)}$ by deleting the vertex v_{t_1} and the edge $e_{\alpha(t_1)}$. Let v_{t_2} $(v_{t_2} \neq v_p)$ a terminal vertex of $T^{(2)}$. Adding the line t_2 of the matrix $S_1(T)$ to the line corresponding to the other vertex of $e_{\alpha(t_2)}$ we obtain the matrix $S_2(T)$. Repeating the above thus give rise to the matrix $S_m(T)$. For this matrix the p-th line is null.

Denoting by $\bar{S}_k(T)$, $k=1,2,\dots,m$, the matrix obtained from $S_k(T)$ by deleting the p-th line, then $\bar{\bar{S}}(T) = \bar{S}_{p-1}(T)$.

On the other hand, according to the properties of determints we have

$$\det[\bar{\bar{S}}(T)] = \det[\bar{S}_{p-1}(T)] = \det[\bar{S}_{p-2}(T)] = \dots = \det[\bar{S}_1(T)] =$$

$\det[\bar{S}(T)]$, and the theorem is proved.

Let T_1 and T_2 two spanning trees of D. By |1| and theorem 1 it follows that

$$(1)\qquad \det[\bar{\bar{S}}(T_1)] = \pm 1 , \quad \det[\bar{\bar{S}}(T_2)] = \pm 1 .$$

Obviously, in $\bar{\bar{S}}(T_1)$ and $\bar{\bar{S}}(T_2)$ each line and each column contains a single nonull element (equal to ± 1). For an arbitrary column, if we want to have on the same place the nonull element of $\bar{\bar{S}}(T_2)$ as in $\bar{\bar{S}}(T_1)$, we must permute two columns in $\bar{\bar{S}}(T_2)$. Let π the total number of permutations necessary for all nonull elements of $\bar{\bar{S}}(T_2)$.

Let σ the total number of exchanges of sing such that each nonull element of $\bar{\bar{S}}(T_2)$ in the same place as in $\bar{\bar{S}}(T_1)$, to have the same sign

But, every permutation and every exchange of sing multiplies the value of $\det[\bar{\bar{S}}(T_2)]$ by -1. Hence, by (1) we have

$$(2)\qquad \det[\bar{\bar{S}}(T_2)] = (-1)^{\pi+\sigma} \det[\bar{\bar{S}}(T_1)] .$$

By (2) and theorem 1 it follows that

$$(3)\qquad \det[\bar{S}(T_1)] \det[\bar{S}(T_2)] = (-1)^{\pi+\sigma} .$$

Let T_1 and T_2 two spanning trees of D such that $|T_1 - T_2| = k$. Deleting the k distinct edges, every spanning tree becomes a graph containing k+1 connected components. Moreover, the k+1 connected components in T_1 and T_2 are identical, and only one of them contains the vertex v_p. The k components that do not contain v_p are called principal.

Obviously, every vertex of a principal component is connected by a unique chain with v_p in (V,T_i), $i=1,2$, and every chain (one of T_1 and other of T_2) contains an unique edge (one of T_1-T_2 and other of T_2-T_1) incident with the principal component. We call these edges principal.

If the principal edges have the same orientation related to the principal component, then this component is positive and negative otherwise.

We consider the graph having as vertices the k+1 components and as edges the principal edges (from T_1 and T_2) incident to the above components. We denote by $\sigma(T_1,T_2)$ the number of positive components from which we subtract the number of cycles in the graph above considered. By |2| we have

$$(4) \qquad \det[\bar{S}(T_1)]\ \det[\bar{S}(T_2)] = (-1)^{\sigma(T_1,T_2)} .$$

Let T_1,T_2 two spanning trees of D for which $T_1-T_2=\{a\}$ and $T_2-T_1=$ $=\{b\}$, $a\neq b$.

We denote by $\omega(T_1,b)$ the unique cycle contained in $(V,T_1\cup$ $\cup\{b\})$.

THEOREM 2.

$$\det[\bar{S}(T_1)]\det[\bar{S}(T_2)] = \begin{cases} -1, & \textit{if a and b have the same orientation in } \omega(T_1,b), \\ 1, & \textit{otherwise.} \end{cases}$$

P r o o f. Deleting the edge a from T_1 we obtain a graph that contains two connected components; one of them contains the vertex v_p and the other is principal.

Obviously, a and b are principal edges. If a and b have the same orientation in $\omega(T_1,b)$, then the principal component is negative, i.e., $\sigma(T_1,T_2)=-1$. This, by (4), it follows that

$$\det[\bar{S}(T_1)]\det[\bar{S}(T_2)] = -1 .$$

If a and b have not the same orientation in $\omega(T_1,b)$, then the principal component is negative, i.e., $\sigma(T_1,T_2)=0$. Then by (4) it follows that $\det[\bar{S}(T_1)]\det[\bar{S}(T_2)]=1$, and the theorem is proved. Let $T=\{e_{j_1},e_{j_2},\ldots,e_{j_m}\}$, (m=p-1) a spanning tree of D. Deleting from T the edge e_{j_h} $(1\leq h\leq m)$ we obtain two connected components V_h and $\bar{V}_h$.

To the bipartition $(V_h,\bar{V}_h)$ we can associate a cocycle $C(e_{j_h},T)$ that contains the edge e_{j_h}.

Obviously, if T_1 and T_2 are two spanning trees for which $T_2-T_1=\{b\}$ and $T_1-T_2=\{a\}$, then $b\in C(a,T_1)$. Moreover, if $c\in C(a,T_1)$, then $(T_1-\{a\})\cup\{c\}$ is a spanning tree.

Let T_o a spanning tree and $T_1,T_2,\dots,T_r$ all spanning trees for which $T_o-T_k=\{a_o\}$ and $T_k-T_o=\{a^{(k)}\}$, $k=1,2,\dots,r$. Thus $C(a_o,T_o)=\{a_o,a^{(1)},\dots,a^{(r)}\}$.

Moreover, if $b\in C(a_o,T_o)$, then $(T_o-\{a_o\})\cup\{b\}$ is one of $T_1,T_2,\dots,T_r$.

Let $A(T_o,a_o)=\{T_1,T_2,\dots,T_r\}$. Obviously, we have

$$A(T_o,a_o)=\bigcup_{\substack{b\in C(a_o,T_o)\\ b\neq a_o}}\{(T_o-\{a_o\})\cup\{b\}\}\ .$$

If $T_o=\{e_{j_1},e_{j_2},\dots,e_{j_m}\}$, then every spanning tree T with $|T_o-T|=1$ belongs to one of $A(T_o,e_{j_h})$, $h=1,2,\dots,m$. Also, all spanning trees of $A(T_o,e_{j_h})$ are distinct. Indeed, each $T\in A(T_o,e_{j_t})$ does not contain the edge e_{j_t}.

On the other hand, for every $T\in A(T_o,e_{j_t})$ holds $|T_o-T|=1$, i.e., all edges of T_o (except for e_{j_t}) belong to T. Hence, T does not belong to $A(T_o,e_{j_s})$ with $t\neq s$.

Let $v_o\in V$ and $e_o\in E$ arbitrary choosen, such that e_o is incident with the vertex v_o. We denote by $C(v_o)$ the cocycle associated to the bipartition $(\{v_o\},V-\{v_o\})$.

Let $A(e_o)$ the set of spanning trees that contain the edge e_o and $\bar{A}(e_o)$ the set of spanning trees that do not contain e_o.

THEOREM 3.

$$(5)\qquad \bar{A}(e_o)=\bigcup_{\substack{T\in A(e_o)\\ b\in C(e_o,T)\cap C(v_o)}}\{(T-\{e_o\})\cup\{b\}\}\ .$$

P r o o f. Obviously, every spanning tree obtained by (5) belongs to $\bar{A}(e_o)$. Suppose now there exists $T \in \bar{A}(e_o)$ such that it cannot be obtained by (5).

Let $\omega(T,e_o)$ the unique cycle contained in the graph $(V, T \cup \{e_o\})$. If $b (b \neq e_o)$ is an edge of $\omega(T,e_o)$ incident with the vertex v_o, then the spanning tree $T' = (T-\{b\}) \cup \{e_o\}$ belongs to $A(e_o)$. But $b \in C(e_o,T') \cap C(v_o)$, i.e., T can be obtained from T' by (5); contradiction. Hence each spanning tree of $\bar{A}(e_o)$ can be obtained by (5) and the theorem is proved.

THEOREM 4. *Every element of* $\bar{A}(e_o)$ *is obtained only once by (5).*

P r o o f. Suppose that T is a spanning tree of $\bar{A}(e_o)$ often generated by (5). In this case there exist at least two distinct edges c and d in T incident with the vertex v_o such that $(T-\{c\}) \cup \{e_o\}$ and $(T-\{d\}) \cup \{e_o\}$ are distinct elements of $A(e_o)$, i.e., c and d belong to the unique cycle $\omega(T,e_o)$. This is impossible. Hence the theorem is true.

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REZIME

NOTA O POKRIVAJUĆIM STABLIMA
ORIJENTISANOG DIGRAFA

U ovom radu se ispituju odnosi izmedju pokrivajućih stabala orijentisanog digrafa i determinanata nekih podmatrica matrice incidencije toga digrafa. Takodje su ispitivani parovi pokrivajućih stabala digrafa koja se razlikuju u orijentaciji samo jedne grane.