Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 11 (1981)
Review of Research Faculty of Science-University of Novi Sad, Volume 11 (1981)

A THEOREM ON ALMOST CONTINUOUS SELECTION PROPERTY AND ITS APPLICATIONS

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1. INTRODUCTION

E.Michael and C.Pixley proved the following Theorem which unifies and generalizes some previously known results about the almost continuous selection property.

THEOREM 1. |8| *Let* X *be paracompact,* Y *be a Banach space,* $Z \subseteq X$ *with* $\dim_X Z \leq 0$ *and* $\phi : X \to F(Y)$ *a lower semicontinuous mapping with* $\phi(x)$ *convex for all* $x \in X \setminus Z$. *Then* ϕ *admits a selection.*

In this paper we shall prove that a similar result holds also if X is a normal topological space and Y is a paranormed space.

First, we shall give some notations and definitions. Let: $2^Y = \{S \mid S \subseteq Y, S \neq \emptyset\}$ and $F(Y) = \{S \mid S \in 2^Y$ and S is closed in Y$\}$. A mapping $\phi : X \to 2^Y$ is lower semicontinuous (l.c.s.) if and only if the set $\{x \mid x \in X, \phi(x) \cap V \neq \emptyset\}$ is open in X for every open V in Y. A selection for a mapping $\phi : X \to F(Y)$ is a continuous mapping $f : X \to Y$ such that $f(x) \in \phi(x)$ for all $x \in X$. Finally, if $Z \subseteq X$ then $\dim_X Z \leq 0$ means that $\dim E \leq 0$ for every set $E \subset Z$ which is closed in X.

Let E be a linear space over the real or complex number field. The function $\| \|^* : E \to [0,\infty)$ will be called a paranorm if and only if:

1. $\|x\|^* = 0 \iff x = 0$.
2. $\|-x\|^* = \|x\|^*$, for every $x \in E$.
3. $\|x+y\|^* \leq \|x\|^* + \|y\|^*$, for every $x, y \in E$.
4. If $\|x_n - x_o\|^* \to 0$, $\lambda_n \to \lambda_o$ then $\|\lambda_n x_n - \lambda_o x_o\|^* \to 0$.

Then we say that $(E, \| \|^*)$ is a paranormed space. The space E is also a topological vector space in which the fundamental system of neighbourhoods of zero in E is given by the family $\{U_\varepsilon\}_{\varepsilon>0}$ where $U_\varepsilon = \{x \mid x \in E, \|x\|^* < \varepsilon\}$.

In |9| the following fixed point theorem is proved.

THEOREM 2. *Let* K *be a bounded, closed and convex subset of* E *and* $T: K \to K$ *be a completely continuous operator on* K. *If there exists a number* $C(K) > 0$ *such that:*

(1) $\|\lambda x\|^* \le C(K)\lambda \|x\|^*$, *for every* $0 \le \lambda < 1$ *and* $x \in K-K$

then there exists an element $p \in K$ *such that* $Tp = p$.

In |9| Zima has given an example of the space E and of the set K such that the relation (1) is satisfied.

DEFINITION *Let* $(E, \| \|^*)$ *be a paranormed space and* K *be a nonempty subset of* E. *If there exists* $C(K) > 0$ *such that (1) holds we say that* K *satisfies the Zima condition.*

Some fixed point theorems in paranormed spaces are proved in |3|.

2. AN ALMOST CONTINUOUS SELECTION THEOREM

First, we shall prove the following Lemma.

LEMMA 1. *Let* $(Y, \| \|^*)$ *be a paranormed space,* K *be a compact and convex subset of* Y *which satisfies the Zima condition. Then for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that:*

$$\mathrm{co}((U_\delta + C) \cap K) \subseteq C + U_\varepsilon$$

for every closed and convex subset C *of* K.

P r o o f: Let $\delta > 0$ be such that $U_\delta + U_\delta \subseteq U_{\frac{\varepsilon}{C^2(K)}}$. Since the set C is closed and K is compact there exists a finite set $F = \{x_1, x_2, \dots, x_n\} \subseteq C$ so that

$$C \subseteq \bigcup_{i=1}^{n} \{x_i + U_\delta\}.$$

Then $(C+U_\delta)\cap K \subseteq \bigcup_{i=1}^{n} \{(x_i + U_{\frac{\varepsilon}{C^2(K)}})\cap K\}$.

Let $\{\beta_k\}_{k=1}^{n}$ be a partition of the unity subordinated to the open covering $\{(x_i + U_{\frac{\varepsilon}{C^2(K)}})\}_{i=1}^{n}$. Suppose now that $z \in co\{(C+U_\delta)\cap K\}$.

Then there exist $\gamma_i \geq 0$ $(i=1,2,\dots,m)$ and $z_j \in \{(C+U_\delta)\cap K\}$ $(j=1,2,\dots,m)$ so that $z = \sum_{j=1}^{m} \gamma_j z_j$. Further, for every $j \in \{1,2,\dots,m\}$ let:

$$c_j = \sum_{k=1}^{n} \beta_k(z_j)x_k .$$

Then from the fact that C is convex it follows that $c_j \in C$, for every $j \in \{1,2,\dots,m\}$ and so $c = \sum_{j=1}^{m} \gamma_j c_j \in C$. Now, we shall prove

that $z - c \in U_\varepsilon$, which implies that $z \in U_\varepsilon + C$. Indeed, since the set K satisfies the Zima condition it follows that:

$$\|z-c\|^* = \|\sum_{j=1}^{m} \gamma_j z_j - \sum_{j=1}^{m} \gamma_j c_j\|^* \leq C(K)\sum_{j=1}^{m} \gamma_j \|z_j - c_j\|^* =$$

$$= C(K)\sum_{j=1}^{m} \gamma_j \|z_j - \sum_{k=1}^{n} \beta_k(z_j)x_k\|^* = C(K)\sum_{j=1}^{m} \gamma_j \|\sum_{k=1}^{n} \beta_k(z_j)(z_j-x_k)\|^* \leq$$

$$\leq C^2(K)\sum_{j=1}^{m} \gamma_j \Big(\sum_{\beta_k(z_j)\neq 0} \beta_k(z_j)\|z_j - x_k\|^*\leq$$

$$\leq C^2(K)\sum_{j=1}^{m} \gamma_j \Big(\sum_{\beta_k(z_j)\neq 0} \beta_k(z_j)\frac{\varepsilon}{C^2(K)}\Big) = \varepsilon$$

which means that $z - c \in U_\varepsilon$.

THEOREM 3. *Let* X *be a normal topological space,* $(Y, \|\ \|^*)$ *be a paranormed space,* $Z \subsetneq X$ *so that* $\dim_X Z \leq 0$ *and* $\phi : X \to F(Y)$ *be a lower semicontinuous mapping so that* $\phi(x)$ *is convex, for every* $x \in X \setminus Z$. *If* $\phi(X) \subseteq K$ *where* K *is a compact and convex subset of* Y *and* K *satisfies the Zima condition then* ϕ *has the almost continuous selection property which means that for every* $\varepsilon > 0$ *there exists a continuous mapping* $f : X \to K$ *such that* $f(x) \in U_\varepsilon + \phi(x)$, *for every* $x \in Y$.

P r o o f: For every $\varepsilon > 0$ we shall denote the set $U_\varepsilon + \phi(x)$ by $B_\varepsilon(\phi(x))$ for every $x \in X$. Let us prove that there exists $f : X \to Y$ such that $f(x) \in B_\varepsilon(\phi(x))$, for every $x \in X$. From Lemma it follows that for $\varepsilon > 0$ there exists $\delta > 0$ so that for every $x \in X \setminus Z$:

$$co((U_\delta + \phi(x)) \cap K) \subseteq U_\varepsilon + \phi(x) = B_\varepsilon(\phi(x)) .$$

Since $\{U_\delta + y\}_{y \in K}$ is an open covering of the set K and K is compact, there exists a finite subset $\{y_1, y_2, \ldots, y_n\} \subseteq K$ such that:

$$K \subseteq \bigcup_{i=1}^{n} \{y_i + U_\delta\} .$$

Further, for every $k=1,2,\ldots,n$ let $U_{y_k} \subseteq X$ be defined in the following way:

$$U_{y_k} = \{x | x \in X,\ y_k \in B_\delta(\phi(x))\} .$$

Since the mapping ϕ is lower semicontinuous it follows that every set U_{y_k} is an open subset of X. Since X is a normal topological space there exists an open covering $\{V_{y_k}\}_{k=1}^{n}$ such that $\bar{V}_{y_k} \subset U_{y_k}$ for every $k \in \{1,2,\ldots,n\}$. Further let:

$$F_x = \{y_k | y_k \in K,\ x \in \bar{V}_{y_k}\} , \text{ for every } x \in X .$$

From the definition of F_x it follows that for every $x \in X$ $F_x \subset B_\delta(\phi(x))$. Let $S = X \setminus Z$ and for every $s \in S$:

$$G_s = \{x | x \in X,\ co\ F_s \subset B_\varepsilon(\phi(x))\} \setminus \{\bigcup_{y_k \notin F_s} \bar{V}_{y_k}\}.$$

Every set G_s is nonempty, since $s \in G_s$. Namely:

$$co\ F_s \subset co(B_\delta(\phi(s)) \cap K) \subset B_\varepsilon(\phi(s)) .$$

Using Lemma 1 we conclude that for every $s \in S$ the set G_s is open. Further, for every $x \in G_s$ we have that $F_x \subset F_s$ since $x \notin \bar{V}_{y_k}$ if $y_k \notin F_s$. Let $G = \bigcup_{s \in S} G_s$ and $E = X \setminus G$. Since G is open the set E is closed and $\dim E \leq 0$. For the relatively open covering $\{V_{y_k} \cap E\}_{k=1}^{n}$ of E there exists a relatively open, disjoint covering $\{D_{y_k}\}_{k=1}^{n}$ such that $D_{y_k} \subseteq V_{y_k} \cap E$ and let $W_{y_k} = V_{y_k} \cap (D_{y_k} \cup G)$ for every $k=1,2,\ldots,n$. Now, we shall define the mapping $f : X \to K$ in the following way:

$$f(x) = \sum_{k=1}^{n} p_{y_k}(x) y_k , \quad \text{for every } x \in X$$

and $\{p_{y_k}\}_{k=1}^{n}$ is the partition of the unity subordinated to $\{W_{y_k}\}_{k=1}^{n}$. It is obvious that the mapping f is continuous. Let $x \in E$. Then there exists one y_k such that $x \in D_{y_k}$ and so:

$$f(x) = y_k \in B_\delta(\phi(x)) \cap K \subseteq B_\varepsilon(\phi(x)) .$$

Suppose that $x \in G = \bigcup_{s \in S} G_s$ and let $x \in G_s$, for $s \in S$. Then from the definition of the mapping f it follows that $f(x) \in \text{co } F_x$. Further, $x \in G_s$ implies $F_x \subset F_s$ and so:

$$f(x) \in \text{co } F_x \subseteq \text{co } F_s \subset B_\varepsilon(\phi(x)) .$$

Using Theorem 3 we can formulate the following fixed point theorem for multivalued mapping in paranormed space. Some fixed point theorems for multivalued mappings in paranormed spaces are proved in |3|.

THEOREM 4. *Let* Y *be a complete paranormed space,* K *be a compact and convex subset which satisfies the Zima condition* $(K \subseteq Y)$, $Z \subseteq K$ *with* $\dim Z \leq 0$ *and* $\phi : K \to F(K)$ *be a lower semicontinuous mapping such that* $\phi(x)$ *is convex, for every* $x \in K \setminus Z$. *Then there exists at least one fixed point of the mapping* ϕ.

P r o o f: Since every compact topological space is normal, from Theorem 3 it follows that there exists a continuous mapping $f : K \to K$ such that all the conditions of Theorem 2 are satisfied and so there exists $x \in K$ so that $x = f(x)$. Then x is a fixed point of the mapping ϕ.

COROLLARY 1. *Let* Y *be a complete paranormed space,* K *be a closed and convex subset of* Y, $\phi : K \to F(K)$ *be a lower semicontinuous mapping such that* $\overline{\phi(K)}$ *is compact,* $\overline{\text{co}}\ \phi(K)$ *satisfies the Zima condition and there exists* $M \subseteq \overline{\text{co}}\ \phi(K)$ *such that* $\dim_Y M \leq 0$ *If* $\phi(x)$ *is convex, for every* $x \in \overline{\text{co}}\ \phi(K) \setminus M$ *then there exists* $x \in K$ *such that* $x \in \phi(x)$.

P r o o f: We shall apply Theorem 4 taking for the set K the set $\overline{\text{co}}\ \phi(K)$. Since Y is a complete paranormed space and $\overline{\phi(K)}$ is complete, it follows |6| that the set $\overline{\text{co}}\ \phi(K)$ is also compact. Further, from $\phi(K) \subset K$ it follows that $\overline{\text{co}}\ \phi(K) \subset K$, since K is closed and convex. This implies that $\phi(\overline{\text{co}}\ \phi(K)) \subseteq \phi(K) \subseteq \overline{\text{co}}\ \phi(K)$. So we have that:

$$\Phi : \overline{co}\ \Phi(K) \to F(\overline{co}\ \Phi(K))$$

and all the conditions of Theorem 4 are satisfied, which implies that there exists $x \in \overline{co}\ \Phi(K)$ such that $x \in \Phi(x)$.

COROLLARY 2. *Let* Y *be a complete paranormed space,* K *be a closed and convex subset of* Y, $\Phi : K \to F(K)$ *be a lower semicontinuous mapping such that* $\overline{co}\ \Phi(K)$ *satisfies the Zima condition and that the following two conditions are satisfied:*

(i) There exists a nonempty set $C \subset K$ *such that* $C \subset \Phi(C)$ *and* $M \subseteq C$ *with* $\dim_Y M \leq 0$ *so that* $\Phi(x)$ *is convex for every* $x \in K \setminus M$.

(ii) If $Q = \overline{co}\, Q \subset K$ *and* $Q = \overline{co}\ \Phi(Q)$ *then* Q *is compact.*

Then there exists $x \in K$ *so that* $x \in \Phi(x)$.

P r o o f: The proof is similar to the proof of the Theorem from |1|. Let $A = \{Q \mid Q \subseteq K, Q = \overline{co}\ Q, C \subseteq Q, \Phi(Q) \subseteq Q\}$. Then $A \neq \emptyset$ and $Q \in A$ implies $\overline{co}\ \Phi(Q) \in A$. Let $C_o = \bigcap_{Q \in A} Q$. Since $C \subseteq C_o$, C_o is a nonempty, closed and convex subset of K. Further, $C_o = \overline{co}\ \Phi(C_o)$ and from (ii) it follows that C_o is compact. Since $M \subseteq C$ and $C \subseteq C_o$ we conclude that $M \subseteq C_o$ and $K \setminus M \supseteq C_o \setminus M$. This implies that $\Phi | C_o$ satisfies all the conditions of Theorem 4 and there exists $x \in K$ such that $x \in \Phi(x)$.

R e m a r k: From the proof of Theorem 3 it is easy to see that we can suppose that Y is a topological vector space, K is such that for every open neighbourhood V of zero in Y there exists an open neighbourhood U of zero in Y such that for every closed and convex subset C of K:

$$co((U + C) \cap K) \subseteq V + C$$

and Φ is a lower semicontinuous mapping from X into K such that, for every open neighbourhood U of zero in Y the set:

$$T = \{x \mid x \in X, C' \subseteq U + \Phi(x)\}$$

is open, where, C' is a compact subset of Y.

Now, we shall give an example of topological vector space Y such that for every neighbourhood U of Y there exists a neighbourhood V of Y such that $co((V+C)\cap K)\subseteq C+U$, where K is a compact and convex subset of Y and C be an arbitrary closed and convex subset of K.

First, we shall give some notations and notions from |7| and |2|. A linear mapping Φ of a topological semifield E into another F is said to be positive if $\Phi(x)\geq 0$ in F, for every $x\in E$ with $x\geq 0$. Let $\|\ \|$ be a mapping of a linear space X over R into a topological semifield E and Φ be a continuous positive linear mapping of E into itself. The triplet $(X,\|\ \|,\Phi)$ is called a paranormed space over E and $\|\ \|$ a Φ-paranorm on X over E if the following conditions are satisfied:

(P1) $\|x\|\geq 0$, for every $x\in X$.

(P2) $\|\lambda x\| = |\lambda|\ \|x\|$, for every real λ and every $x\in X$.

(P3) $\|x+y\|\leq\Phi(\|x\|+\|y\|)$, for every $x,y\in X$.

DEFINITION *A set* $K, K\subseteq X$, *where* $(X,\|\ \|,\Phi)$ *is a* Φ *paranormed space, is said to be of type* Φ *if and only if for every* $n\in N$, *every* $x_1,x_2,\ldots,x_n\in K-K$ *and every* $\lambda_i, 0\leq\lambda_i<1$ $(i=1,2,\ldots,n)$ *such that* $\sum_{i=1}^{n}\lambda_i=1$, *we have:*

$$\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|\leq\sum_{i=1}^{n}\lambda_i\Phi(\|x_i\|)$$

In |7| is proved that every topological vector space is a Φ-paranormed space over a topological semifield R_Δ. We shall use in the further text the following notation:

$$U_{\mu,\varepsilon}=\{x\mid x\in X,\ \|x\|(t)<\varepsilon,\ \text{for every } t\in\mu\}$$

where μ is a finite subset of Δ and $\varepsilon>0$. Then X is a topological vector space in which $\{U_{\mu,\varepsilon}\}_{\varepsilon>0,\mu\in\Delta}$ is the base of the fundamental system of zero in E. Similarly as in Lemma 1 we shall prove the following Lemma.

LEMMA 2. *Let* $(Y,\|\ \|,\Phi)$ *be a paranormed space,* K *be a compact and convex subset of* Y *of type* Φ. *Then for every* $U\in\{U_{\mu,\varepsilon}\}$ *there exists* $V\in\{U_{\mu,\varepsilon}\}$ *such that:*

$$\mathrm{co}((V+C)\cap K)\subseteq C+U$$

for every closed and convex subset C *of* K.

P r o o f: Let $U=U_{\mu,\varepsilon}$, $\mu\subset\Delta$ and $\varepsilon>0$. Since the mapping Φ is linear and continuous it follows that $N_1=(\Phi^2)^{-1}(U_{\mu,\varepsilon})$ is a neighbourhood of zero in R_Δ and let $\mu'\subset\Delta$ and $\varepsilon'>0$ be such that:

$$U_{\mu',\varepsilon'}\subseteq\{x \mid \|x\|\in N_1\}.$$

Let $\mu''\subset\Delta$ and $\varepsilon''>0$ be such that:

$$U_{\mu'',\varepsilon''}+U_{\mu'',\varepsilon''}\subseteq U_{\mu',\varepsilon'}\quad .$$

We shall prove that $\mathrm{co}((C+U_{\mu'',\varepsilon''})\cap K))\subseteq C+U_{\mu,\varepsilon}$. Let z and c be as in Lemma 2, where we take $U_{\mu'',\varepsilon''}$ instead of U_δ and $U_{\mu',\varepsilon'}$ instead of $U_{\frac{\varepsilon}{C^2(K)}}$. Then we have that $t\in\mu$ implies:

$$\|z-c\|(t)=\Big\|\sum_{j=1}^{m}\gamma_j z_j-\sum_{j=1}^{m}\gamma_j c_j\Big\|(t)\le$$

$$\le\sum_{j=1}^{m}\gamma_j\Phi(\|z_j-c_j\|)(t)=$$

$$=\sum_{j=1}^{m}\gamma_j\;\Phi\Big(\sum_{k=1}^{n}\beta_k(z_j)\;\Phi(\|z_j-x_k\|)\Big)(t)\le$$

$$\le\sum_{j=1}^{m}\gamma_j\;\Big(\sum_{k=1}^{n}\beta_k(z_j)\;\Phi^2(\|z_j-x_k\|)(t)\Big)=$$

$$=\sum_{j=1}^{m}\gamma_j\Big(\sum_{\substack{k=1\\ \beta_k(z_j)\neq 0}}^{n}\beta_k(z_j)\;\Phi^2(\|z_j-x_k\|)(t)\Big)$$

Since $\beta_k(z_j)\neq 0$ implies that $z_j-x_k\in U_{\mu',\varepsilon'}$ and so $\|z_j-x_k\|\in N_1$ we have that

$$\|z-c\|(t)\le\sum_{j=1}^{m}\gamma_j\sum_{\substack{k=1\\ \beta_k(z_j)\neq 0}}^{n}\beta_k(z_j)\varepsilon=\varepsilon \text{ and so } z-c\in U_{\mu,\varepsilon}\;.$$

Using this Lemma we can formulate the following fixed point theorem.

THEOREM 5. *Let* Y *be a complete* Φ *paranormed space,* K *be a compact, convex subset of type* Φ *of* Y *and* $Z \subset K$ *with* $\dim_Y Z \le 0$. *Further, let* $\phi: K \to F(K)$ *be a lower semicontinuous mapping such that* $\phi(x)$ *is convex, for every* $x \in K \setminus Z$ *and the set* $\{x | x \in K, C \subseteq U + \phi(x)\}$ *is open, for every* $C \subseteq Y$ *be compact and* U *be an arbitrary open neighbourhood of zero in* Y. *Then there exists at least one fixed point of the mapping* ϕ.

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REZIME

TEOREMA O OSOBINI GOTOVO NEPREKIDNE SELEKCIJE
I PRIMENA

U ovom radu dokazano je uopštenje teoreme Michaela i Pixleya o gotovo neprekidnoj selekciji u paranormiranim prostorima.