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FUZZY GENERALIZED EQUIVALENCE RELATIONS AND PARTITIONS*)

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1. In |1| , Pickett defined a generalized equivalence relation on a set X containing at least n elements, as the (n+1)-ary relation E_n on X, satisfying the following three conditions, where S_{n+1} denotes a set of permutations on $\{1,\dots,n+1\}$:

E_{1n} : for all $x_1,\dots,x_n, (x_1,x_2,\dots,x_n,x_1) \in E_n$;

E_{2n} : for all $x_1,\dots,x_{n+1}$ and for each permutation $s \in S_{n+1}$, if $(x_1,x_2,\dots,x_{n+1}) \in E_n$, then $(x_{s(1)},x_{s(2)},\dots,x_{s(n+1)}) \in E_n$;

E_{3n} : for all $x_o,\dots,x_{n+1}$, $x_i \neq x_j$, for $i \neq j$, $i,j \in \{1,\dots,n\}$, if $(x_o,\dots,x_n) \in E_n$ and $(x_1,\dots,x_{n+1}) \in E_n$ then $(x_o,\dots,x_{n-1},x_{n+1}) \in E_n$

2. Fuzzy relations are discussed in |2| in the following way:

Let X be a set and $J([0,1],\wedge,\vee,\bar{},0,1)$ the distributive lattice on the unit interval $[0,1]$, where for $a,b \in [0,1]$,

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$$a \wedge b = \min(a,b), \quad a \vee b = \max(a,b), \quad \bar{a} = 1-a.$$

3. The fuzzy n-ary relation R on X is defined by

$$\underline{R} \stackrel{def}{=} \{((x_1,\ldots,x_n),\ m_{\underline{R}}(x_1,\ldots,x_n)) \mid x_i \in X,\ i=1,\ldots,n\ ,$$
$$m_{\underline{R}} : X^n \to [0,1]\}.$$

4. Hartmanis |3| defined a partition of type n on a set X with at least n elements, as a family P_n of subsets of X satisfying:

i) each $P \in P_n$ contains at least n elements; and

ii) each n different elements from X belong to exactly one $P \in P_n$.

As it is shown in |1|, each partition of type n determines one generalized equivalence relation on the same set, and vice versa, each (n+1)-ary equivalence determines one partition of type n.

In this article a fuzzy generalized equivalence relation will be defined and its properties will be described. In the second part, the notion of a fuzzy partition of type n will be given and it will be shown that there is a natural connection between fuzzy generalized equivalences and these partitions.

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DEFINITION 1. *Let X be a set and J the distributive lattice given in |2|. A fuzzy generalized equivalence relation on X is (n+1)-ary, fuzzy relation $\underline{R}_n$ on X, satisfying the following conditions:*

$\underline{E}_{1n}$: *for all* $x_1,\ldots,x_n$, $m_{\underline{R}_n}(x_1,\ldots,x_n,x_1) = 1$

$\underline{E}_{2n}$: *for all* $x_1,\ldots,x_{n+1}$, *and for each permutation* $s \in S_{n+1}$
$m_{\underline{R}_n}(x_1,\ldots,x_{n+1}) = m_{\underline{R}_m}(x_{s(1)},\ldots,x_{s(n+1)})$

$\underline{E}_{3n}$: *for all* $x_o,\ldots,x_{n+1}$, *if* $m_{\underline{R}_n}(x_o,\ldots,x_n) = a$ *and*
$m_{\underline{R}_n}(x_1,\ldots,x_{n+1}) = b$ *then* $m_{\underline{R}_n}(x_o,\ldots,x_{n-1},x_{n+1}) \geq a \wedge b$,

(a *and* b *are from* $[0,1]$, *and* $x_i \neq x_j$ *for* $i \neq j$, $i,j \in \{1,\ldots,n\}$)

The following two propositions are the generalizations of the Theorems of decomposition and synthesis for fuzzy relations (|2|), connecting the fuzzy generalized equivalences and (n+1)-ary equivalence relations given in |1|.

PROPOSITION 1. *Let* $\underset{\sim}{R}_n$ *be a fuzzy generalized equivalence relation on* X. *Then*

$$\underset{\sim}{R}_n = \bigcup_{a\in[0,1]} a\cdot R_a,$$

where

$$a_1 \le a_2 \quad \textit{implies} \quad R_{a_2} \le R_{a_1}$$

R_a $(a\in[0,1])$ *is the generalized equivalence relation in the sense of* |1|.

P r o o f. Let

$$R_a \overset{\text{def}}{=} \{(x_1,\dots,x_{n+1}) \mid x_i\in E,\quad i=1,\dots,n\ ,$$
$$m_{\underset{\sim}{R}_n}(x_1,\dots,x_{n+1}) \ge a\ ,\quad a\in[0,1]\ \}.$$

It is obvious that

$$a_i \le a_j \quad \text{implies} \quad R_{a_j} \le R_{a_i}\ .$$

R_a, thus, is not a fuzzy relation.

Let now for a and b from $[0,1]$

$$m_b\cdot R_a(x_1,\dots,x_{n+1}) \overset{\text{def}}{=} b\wedge m_{R_a}(x_1,\dots,x_{n+1}).$$

Then,

$$m_{\bigcup_{a\in[0,1]} a\cdot R_a}(x_1,\dots,x_{n+1}) = \bigvee_a a\cdot m_{R_a}(x_1,\dots,x_{n+1}) =$$

$$\bigvee_{a\le m_{\underset{\sim}{R}_n}} (x_1,\dots,x_{n+1}) = m_{\underset{\sim}{R}_n}(x_1,\dots,x_{n+1}),$$

since

$$m_{R_a}(x_1,\dots,x_n) = \begin{cases} 1, & \text{for } a\le m_{\underset{\sim}{R}_n}(x_1,\dots,x_{n+1}) \\ 0, & \text{otherwise.}\end{cases}$$

The decomposition of $\underline{R}_n$ into (n+1)-ary relations R_a is thus proved. Now we have to show that these relations are generalized equivalences.

Since $\underline{R}_n$ satisfies $\underline{E}_{n1}$, for all $x_1, \dots, x_n$ from X

$$m_{\underline{R}_n}(x_1, \dots, x_n, x_1) = 1,$$

then $(x_1, \dots, x_n, x_1)$ belongs to R_a for each a from $[0,1]$, i.e. R_a is reflexive (in the sense of E_{n1}) .

If $(x_1, \dots, x_{n+1})$ is in R_a, this means that $m_{\underline{R}_n}(x_1, \dots$ $\dots, x_{n+1}) \geq a$, and by symmetry (E_{3n}) for each $s \in S_{n+1}$

$$m_{\underline{R}_n}(x_{s(1)}, \dots, x_{s(n+1)}) \geq a \text{ , and thus}$$

$(x_{s(1)}, \dots, x_{s(n+1)}) \in R_a$, i.e. the condition E_{n2} is satisfied.

Finally if $(x_o, \dots, x_n)$ and $(x_1, \dots, x_{n+1})$ both belong to R_a, then

$$m_{\underline{R}_n}(x_o, \dots, x_n) \geq a \quad \text{and } m_{\underline{R}_n}(x_1, \dots, x_{n+1}) \geq a \text{ ,}$$

and by E_{3n}

$$m_{\underline{R}_n}(x_o, \dots, x_{n-1}, x_{n+1}) \geq a \text{ ,}$$

i.e. $(x_o, \dots, x_{n-1}, x_{n+1}) \in R_a$,

completing the proof.

It is obvious that the method used in the proof of the previous proposition can be applied in the opposite way also, i.e. that the converse is also true:

PROPOSITION 2. *If* $R_a (a \in [0,1])$ *are generalized equivalence relations such that*

$$a_i \leq a_j \qquad \textit{implies} \qquad R_{a_j} \leq R_{a_i} \text{ ,}$$

then $\underline{R}_n$, *defined by*

$$\underline{R}_n = \bigcup_{a \in [0,1]} a \cdot R_a$$

is a fuzzy generalized equivalence relation (in the sense of Definition 1).

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DEFINITION 2. *Let* $\underline{R}$ *be the fuzzy generalized equivalence relation on* X. *The equivalence class of the level* p ($p \in [0,1]$) *denote by* $|\{x_1^n\}|_p$ *) , *for* $x_1,\dots,x_n \in X$ *is defined by*

$$|\{x_1^n\}|_p = \{y \mid m_{\underline{R}}(x_1,\dots,x_n,y) \geq p\}.$$

DEFINITION 3. *Fof* $p \in [0,1]$, *the quotient-set of the level* p *for* $\underline{R}$ *on* X, *denoted by* $(X/\underline{R})_p$, *is*

$$(X/\underline{R})_p = \{|\{x_1^n\}|_p \mid x_1,\dots,x_n \in X\}.$$

DEFINITION 4. *The fuzzy quotient-set* $X/\underline{R}$ *is a fuzzy set on* $\bigcup_{p\in[0,1]} (X/\underline{R})_p$, *such that*

$$m_{(X/\underline{R})}(|\{x_1^n\}|_p) = \sup\, (\rho \mid |\{x_1^n\}|_\rho = |\{x_1^n\}|_p).$$

PROPOSITION 3. *If* $\underline{R}$ *is a fuzzy generalized equivalence relation on* X , *then* $(X/\underline{R})_p$, $p \in [0,1]$, *is a partition of type* n , *in the sense of 4.*

P r o o f. By Proposition 1., R_p is a generalized equivalence relation. From that, and the fact that

$$|\{x_1^n\}|_p = \{y \mid m_{\underline{R}}(x_1,\dots,x_n,y) \geq p\} = \{y \mid (x_1,\dots,x_n,y) \in R_p\}$$

it follows that conditions (i) and (ii), 4., are satisfied.

DEFINITION 5. *Let* X *be a set containing at least* n *elements. A fuzzy partition of type* n *on* X *is a fuzzy set* $\underline{\Pi}(X)$ *on* P(X), *satisfying:*

a) $\{P \mid P \in P(x) \quad m_{\underline{\Pi}(x)}(P) = 1\}$

is a partition of type n *on* X.

b) Let $P,Q \in P(x)$, $m_{\underline{\Pi}(x)}(P) \neq 0$, $m_{\underline{\Pi}(x)}(Q) \neq 0$. *Now, if*

*) $|\{x_1^n\}|_p$ stands for $|\{x_1,\dots,x_n\}|_p$

card $(P \cap Q) \geq n$ *then*

$$m_{\underline{\Pi}(x)}(P) \leq m_{\underline{\Pi}(x)}(Q) \qquad \textit{iff} \qquad Q \subset P$$

(where the equality holds on the left side iff it holds on the right).

The following proposition is the direct consequence of Definitions 4. and 5.

PROPOSITION 4. *Let* $\underline{R}$ *be the fuzzy generalized equivalence relation on* X. *Then* $X/\underline{R}$ *is a fuzzy partition of type* n *on* X.

PROPOSITION 5. *Let* $\underline{\Pi(X)}$ *be the fuzzy partition of type* **n** *on* X. *Then the fuzzy* (n+1)*-ary relation* $\underline{R}_\Pi$ *on* X *defined by*

$$(*) \qquad m_{\underline{R}_\Pi}(x_1,\dots,x_{n+1}) = \begin{cases} p, \text{ if there is } P \in \mathcal{P}(x) \text{ such that} \\ x_1,\dots,x_{n+1} \in P \\ \quad \text{and} \\ m_{\underline{\Pi}(x)}(P) = p, \\ 0 \qquad \text{otherwise}, \end{cases}$$

is a fuzzy quivalence relation on X.

P r o o f. $\underline{R}_\Pi$ is reflexive: (a) implies that

$$m_{\underline{R}_\Pi}(x_1,\dots,x_n,x_1) = 1.$$

$\underline{R}_\Pi$ is symmetric: If $m_{\underline{R}_\Pi}(x_1,\dots,x_{n+1}) = p$, by (*) then there is $P \in \mathcal{P}(x)$ such that $x_1,\dots,x_{n+1} \in P$ and $m_{\underline{\Pi}(x)}(P) = p$, which holds for each permutation of $x_1,\dots,x_{n+1}$.

$\underline{R}_\Pi$ is tranzitive: Let $m_{\underline{R}_\Pi}(x_o,\dots,x_n) = p$

$m_{\underline{R}_\Pi}(x_1,\dots,x_{n+1}) = q$, $x_i \neq x_j$, for $i \neq j$, $i,j \in \{1,\dots,n\}$.
Then there are P and Q in $\mathcal{P}(X)$ such that

$$x_o,\dots,x_n \in P \text{ and } x_1,\dots,x_{n+1} \in Q.$$

First let $p < q$. Then by (b) $Q \subset P$ and $x_o,\dots,x_{n+1} \in P$, i.e.

$$m_{\underline{R}_\Pi}(x_o,\dots,x_{n-1},x_{n+1}) = p \geq p \wedge q \ .$$

If $p = q$, then $P = Q$, by (b) (the part concerning the equality). Then again

$$x_o,x_1,\dots,x_n,x_{n+1} \in P = Q \ , \quad \text{and thus}$$

$$m_{\underline{R}_\Pi}(x_o,\dots,x_{n-1},x_{n+1}) = p = q.$$

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REZIME

RASPLINUTE UOPŠTENE RELACIJE EKVIVALENCIJE
I PARTICIJE

U radu se definišu rasplinute uopštene relacije ekvivalencije i dokazuju stavovi o dekompoziciji i sintezi tih relacija, pomoću ekvivalencija definisanih u |1|. Daje se i pojam rasplinute particije tipa n (uopštenje pojma iz |3|), i pokazuje da postoji veza izmedju tih particija i rasplinutih generalisanih ekvivalencija, preko odgovarajućih rasplinutih faktor - skupova.