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SEMIGROUPS IN WHICH SOME BI-IDEAL IS A GROUP

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Semigroups containing minimal ideals are considered by A.H.CLIFFORD, |2|. If semigroups S contains at least one minimal left and at least one minimal right ideal, then it has a completely simple kernel or equivalently it has a quasi-ideal which is a group (see Theorem 3.2.|2| and Theorem 5.14. |9|). The structural theorem of this class is given in |7| and |8| . Here we will characterize this class using the notions of bi-ideal and AB-ideal (Theorem 1.). Using Theorem 1 we give a characterization of a semigroup in which some quasi-ideal is a special power joined semigroup. At the end we give a characterization of a semigroup in which every proper subsemigroup is a special power joined semigroup (Theorem 3.).

For nondefinied notions we refer to |5| and |9|.

A nonempty subset G of a semigroup S is called a left-A-ideal (right-A-ideal) of S if $sG \cap G \neq \emptyset$ $(Gs \cap G \neq \emptyset)$ for any $s \in S$. This notion is introduced by O.GROŠEK and L.SATKO, |4|. In this note introduce the concept of almost bi-ideal (AB-ideal).

DEFINITION 1. *A nonempty subset* B *of a semigroup* S *is called an almost bi-ideal (AB-ideal) of* S *if* $BsB \cap B \neq \emptyset$ *for every* $s \in S$.

If a left (right) A-ideal is a semigroup, then it is an AB-ideal.

If B is an AB-ideal of a semigroup S and $B \subset C \subset S$, then C is an AB-ideal of S.

The union of two AB-ideals of a semigroup S is also an AB-ideal of S.

The proof of the following proposition is obvious.

PROPOSITION 1. *Every nonempty subset of a semigroup* S *is an AB-ideal of* S *if and only if* S *is a rectangular band.*

PROPOSITION 2. *A semigroup* S *has a proper AB-ideal if and only if there exists an element* $a \in S$ *such that* $(S \setminus a)s(S \setminus a) \cap (S \setminus a) \neq \emptyset$ *for every* $s \in S$.

P r o o f. If a semigroup S contains a proper AB-ideal B and $a \notin B$, then $B \subset S \setminus a$ and $S \setminus a$ is a proper AB-ideal of S, i.e. $(S \setminus a)s(S \setminus a) \cap (S \setminus a) \neq \emptyset$ for every $s \in S$.

The converse is obvious.

As the corollary of Proposition 2. we have

PROPOSITION 3. *A semigroup* S *has no proper AB-ideals if and only if for every* $a \in S$ *there exists* $s \in S$ *such that* $(S \setminus a)s(S \setminus a) = a$.

PROPOSITION 4. *Let* B *be an AB-ideal of a semigroup* S. *Then* xBy *is an AB-ideal of* S *for every* $x,y \in S$.

P r o o f. First we have $BysxB \cap B \neq \emptyset$ for any $x,y,s \in S$ and this implies $xBysxBy \cap xBy \neq \emptyset$, i.e. that xBy is an AB-ideal.

PROPOSITION 5. *If* B *is a subsemigroup of a semigroup* S *and a minimal AB-ideal of* S, *then* B *is a subgroup of* S.

P r o o f. Let B be a minimal AB-ideal of a semigroup S which a subsemigroup of S. Then by Proposition 4. b_1Bb_2 is an AB-ideal of S. Since B is a minimal AB-ideal we have $B = b_1Bb_2$ for every $b_1, b_2 \in B$. This implies that B is a subgroup of S.

The following lemma is known.

LEMMA 1. |7|. *Let* B *be a bi-ideal of a semigroup* S. *Then* B *is a minimal bi-ideal of* S *if and only if* B *is a group.*

PROPOSITION 6. *A group* G *is an AB-ideal of a semigroup* S *if and only if* G *is a minimal bi-ideal of* S.

P r o o f. Let G be an AB-ideal of S. Then for every $s \in S$ there exist $g_1, g_2 \in G$ such that $g_1 s g_2 \in G$ and $g_1^{-1} g_1 s g_2 g_2^{-1} \in G$, i.e. $ese \in G$, where e is the identity of G. If follows from this that $gsh \in G$ for every $g,h \in G$. Hence, G is a bi-ideal of S. By Lemma 1. we have that G is a minimal bi-ideal of S.

The converse follows immediately.

LEMMA 2. |7|. *The union of all minimal bi-ideals of a semigroup* S *is an ideal of* S.

THEOREM 1. *Let be a semigroup. Then the following conditions are equivalent:*

(i) S *has Ab-ideal which is a group;*
(ii) S *has a bi-ideal which is a group;*
(iii) S *has a quasi-ideal which is a group;*
(iv) S *contains a completely simple kernel.*

P r o o f. (i) ⇒ (ii). This implication follows by Lemma 1. and by Proposition 6. (ii) ⇒ (iii). Let S has a bi-ideal which is a group and let K be the union of all bi-ideals which are groups. Then by Lemmas 1. and 2. K is an ideal od S. As K is completely regular semigroup we have that every bi-ideal of K is a quasi-ideal (Corollary 3.3. |6|). By Theorem 5.3. |9| we have that S contains a quasi-ideal which is a group. (iii) ⇒ (iv). This implication follows by Theorem 5.14. |9|. (iv) ⇒ (i). If a semigroup S contains a completely simple kernel K, then the maximal subgroups of K are bi-ideals of S and the assertion follows by Proposition 6.

COROLLARY 1. *Let* S *be a semigroup. The following conditions are equivalent:*

(i) S *is the union of its minimal bi-ideal;*

(ii) S *is the union of its minimal quasi-ideal;*

(iii) S *is the union of its AB-ideals which are groups;*

(iv) S *is completely simple.*

COROLLARY 2. *Let* S *be a semigroup. Then some left ideal of a semigroup* S *is a group if and only if* S *contains a kernel which is a right group.*

DEFINITION 2. |1|. S *is a special power joined semigroup* (s.p.j.) *if for every* $a,b \in S$ *there exists a* $n \in N$ *such that* $a^n = b^n$.

LEMMA 3. |1| . S *is a* s.p.j. *semigroup if and only if* S *is a nil-extension of a periodic group.*

THEOREM 2. *Let* S *be a semigroup. Then some quasi-ideal of* S *is a* s.p.j. *semigroup if and only if* S *contains a completely simple periodic kernel.*

P r o o f. If some quasi-ideal Q of a semigroup S is a s.p.j. semigroup, then by Lemma 3. Q is a nil-extension of a periodic group G, and so G is a quasi-ideal od S. Hence, by Theorem 1. we have that S contains a completely simple kernel.

The converse is trivial.

COROLLARY 3. *Let* S *be a semigroup. Then some left ideal of* S *is a periodic group* (s.p.j. *semigroup) if and only if* S *contains a kernel which is a periodic right group.*

THEOREM 3. *Let* S *be a semigroup. Then the following conditions are equivalent:*

(1) Every proper subsemigroup of S *contains exactly one idempotent;*

(2) S *satisfies one of the following conditions:*

(i) $|S| = 2$;

(ii) S *is* s.p.j.;

(3) Every proper subsemigroup of S *is* s.p.j. .

P r o o f. (1) $\Rightarrow$ (2). Let S be a semigroup in which every proper subsemigroup has exactly one idempotent. It is clear that S is periodic. Let $|S| > 2$. If S has exactly one idempotent, then S is s.p.j. . If S has two or more than two idempotents, then $S = \langle e,f \rangle$, where e and f are distinct idempotents of S. If $ef = fe$, then $S = \{e,f,ef\}$ which is a contradiction. If $ef \neq fe$, then

$$S = \{e,f\} \cup \langle ef\rangle \cup \langle efe\rangle \cup \langle fe\rangle \cup \langle fef\rangle .$$

In this case $\langle ef\rangle \cup \langle efe\rangle$ is a proper subsemigroup of S with exactly one idempotent $(ef)^n$. If $\{e\} \cup \langle ef\rangle \cup \langle efe\rangle$ is a proper subsemigroup of S, then $e = (ef)^n$ and therefore $e = ef$. From this $S = \{e,f,fe\}$. As $\{f,fe\}$ is a subsemigroup of S we have that $f = fe$ So $|S| = 2$, which is a contradiction. If $S = \{e\} \cup \langle ef\rangle \cup \langle efe\rangle$, then $f = (ef)^n$ and so $ef = f$. From this $S = \{e,f,fe\}$. As $\{f,fe\}$ is a subsemigroup of S we have that $f = fe$, which is not possible.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1). These two implications follow immediately.

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REZIME

POLUGRUPE U KOJIMA NEKI BI-IDEAL JESTE GRUPA

U ovom radu daju se nove karakterizacije (pomoću bi-ideala i skoro bi-ideala (AB-ideala) za polugrupe koje sadrže potpuno prosto jezgro. Na osnovu ovog rezultata opisuju se polugrupe u kojima neki kvazi-ideal jeste specijalno stepeno povezana polugrupa. Na kraju opisuju se polugrupe u kojima svaka prava podpolugrupa sadrži tačno jedan idempotent.