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A NOTE ON GENERALIZED PSEUDO-BOOLEAN FUNCTIONAL EQUATIONS WITH CONSTANT COEFFICIENTS AND n VARIABLES

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Let $L \neq \emptyset$ be a finite set; L^n its direct product, and let $(P, \oplus, \cdot, I)$ be a commutative ring, and I its identity. A function $f : L^n \to P$ is called a generalized pseudo-Boolean function.

DEFINITION 1. *The partial derivative of a generalized pseudo-Boolean function* $f : L^n \to P$ *with respect to variable* x_i $(1 \leq i \leq n)$ *is the generalized pseudo-Boolen function*

$$\frac{\partial f_a}{\partial x_i} : L^n \to P, \quad a \in L, \quad (1 \leq i \leq n)$$

where

$$\frac{\partial f_a}{\partial x_i}(x) = f(x_1,\ldots,x_{i-1},a,x_{i+1},\ldots,x_n) - f(x)$$

$a \in L$, $x = (x_1,\ldots,x_n)$, $(1 \leq i \leq n)$.

The following notations will be used:

$m(\alpha) = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, $\alpha_i \in \{0,1\}$ (+ ordinary addition in the set of real numbers)

$M_n = \{\alpha \mid \alpha = \alpha_1\alpha_2\ldots\alpha_n,\ \alpha_i \in \{0,1\},\ m(\alpha) \geq 1\}$

$k(M_n) = 2^n - 1$

$\alpha_i \dot{+} \alpha_j$, $\alpha_i, \alpha_j \in \{0,1\}$ ($\dot{+}$ addition mod. 2)

$\partial x^\alpha = \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $\alpha \in M_n$

$$b_i^{[o]} = 1, \quad b_i^{[1]} = b_i \ , \ i=1,\dots,n, \quad b_i \in L$$

$$1 \cdot b_2 = b_i \cdot 1 = b_i$$

$$\partial^o x_i = 1$$

$$\frac{\partial^o f_{b_i}}{\partial x_i^o} = 1, \quad \frac{\partial^2 f_{b_1 b_2}}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \left(\frac{\partial f_{b_2}}{\partial x_j} \right)_{b_1}$$

$$P_i(\bar{a}_k, \bar{a}_j) = P_i(x_1,\dots,x_{k-1},a_k,x_{k+1},\dots,x_{j-1},a_j,x_{j+1},\dots,x_n)$$

The following theorem has been proved (see |1|).

THEOREM 1. *The system of pseudo-Boolean functional equations*

$$(1) \qquad \frac{\partial f_{a_i}}{\partial x_i} = P_i(x), \quad i=1,\dots,n \ ,$$

has a solution if and only if

$$(2.i) \qquad P_i(x_1,\dots,x_{i-1},a,x_{i+1},\dots,x_n) = 0, \quad i=1,\dots,n \ ,$$

$$(2.ij) \qquad \frac{\partial P_{ja_i}}{\partial x_i} = \frac{\partial P_{ia_j}}{\partial x_j} \ , \quad i \neq j, \quad i,j=1,\dots,n.$$

If conditions (2.i) and (2.ij) are fulfilles all the functions f *are determined by the formula*

$$(3.i_1,\dots,i_n) \qquad f(x) = c - \sum_{k=1}^{n}{}^{\oplus} P_{i_k}(\bar{a}_{i_{k+1}},\dots,\bar{a}_{i_n}) \oplus P_{i_n}(x) \ ,$$

where $i_1 i_2 \dots i_n$ *are permutations of set* $\{1,2,\dots,n\}$ *and* c *is a constant from* P.

The functional equation

$$(4) \qquad a \frac{\partial f_{b_1}}{\partial x} \oplus b \frac{\partial f_{b_2}}{\partial y} \oplus c \frac{\partial^2 f_{b_1 b_2}}{\partial x \partial y} = d\, g(x,y)$$

where $f : L^2 \to P$ is an unknown generalized pseudo-Boolean function $g : L^2 \to P$ is a known generalized pseudo-Boolen function , a,b,c,d are constants from, P, is obtained in |2| , where the next theorem is proved.

THEOREM 2. *The functional equation (4) has a solution if and only if*

(i) $d = a\,b\,(a \oplus b - c) \neq 0$

(ii) $$\frac{\partial^2 g_{b_1 b_2}}{\partial x \partial y} \oplus \frac{\partial g_{b_1}}{\partial x} \oplus \frac{\partial g_{b_2}}{\partial y} \oplus g = 0.$$

Here we shall observe a generalization of Theorem 2. Let us consider a generalized pseudo-Boolen functional equation with constant coefficients

$$(5) \qquad \sum_{\substack{\oplus \\ \alpha \in M_n}} a_\alpha \frac{\partial^{m(\alpha)} f_{b_1^{\lceil \alpha_1 \rceil} \dots b_n^{\lceil \alpha_n \rceil}}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} = d\,R(x) \,,$$

where $f : L^n \to P$ is a unknown generalized pseudo-Boolen function $R : L^n \to P$ is a known generalized function, and constants a_α , $(\alpha \in M_n)$ d are from P.

For every generalized pseudo-Boolen function with constant coefficients and n variables the following property can be proved.

LEMMA 1. *If the left-hand side of the equation contains only the unknown function* f *and its partial derivatives , it can be written in the form of the left-hand side of (5).*

The proof follows directly from the properties of partial derivatives.

$$\frac{\partial^2 f_{ab}}{\partial x_i \partial y_j} = \frac{\partial^2 f_{ba}}{\partial x_j \partial x_i} \,, \quad i \neq j, \quad (i,j=1,\dots,n)$$

$$\frac{\partial^m f_{a_1 \dots a_m}}{\partial x_i^m} = (-1)^{m+1} \frac{\partial f_{a_m}}{\partial x_i} \,, \quad (1 \leq i \leq n)$$

$$\frac{\partial^m f_{a_1\ldots a_{k_1}\ldots a_{k_2}\ldots a_{k_p}}}{\partial x_{i_1}^{k_1}\ldots\partial x_{i_p}^{k_p}} = (-1)^{m+p}\,\frac{\partial^p f_{a_{k_1}\ldots a_{k_p}}}{\partial x_{i_1}\ldots\partial x_{i_p}}$$

If n=2, equation (5) has the form (4), i.e.

$$a_{10}\,\frac{\partial f_{b_1}}{\partial x_1}\oplus a_{01}\,\frac{\partial f_{b_2}}{\partial x_2}\oplus a_{11}\,\frac{\partial^2 f_{b_1 b_2}}{\partial x_1\partial x_2} = d\,P(x_1,x_2)\ .$$

If n=2, equation (5) has the form

$$a_{100}\,\frac{\partial f_{b_1}}{\partial x_1}\oplus a_{010}\frac{\partial f_{b_2}}{\partial x_2}\oplus a_{001}\,\frac{\partial f_{b_3}}{\partial x_3}\oplus$$

$$\oplus\ a_{110}\,\frac{\partial^2 f_{b_1 b_2}}{\partial x_1\partial x_2}\oplus a_{101}\,\frac{\partial^2 f_{b_1 b_3}}{\partial x_1\partial x_3}\oplus$$

$$\oplus\ a_{011}\,\frac{\partial^2 f_{b_2 b_3}}{\partial x_2\partial x_3}\oplus a_{111}\,\frac{\partial^3 f_{b_1 b_2 b_3}}{\partial x_1\partial x_2\partial x_3} = d\,R(x_1,x_2,x_3)\ .$$

THEOREM 3. *Functional equation (5) has a solution if and only if*

(6.n)
$$d = \prod_{m(\alpha)=1} a_\alpha \prod_{k=2}^{n}\Big(\prod_{i=1}^{\binom{n}{n-k}}\big(a_{\alpha_1^i}\oplus\ldots\oplus a_{\alpha_k^i} -$$

$$- a_{\alpha_1^i\dot{+}\alpha_2^i} - a_{\alpha_1^i\dot{+}\alpha_3^i} - \ldots - a_{\alpha_{k-1}^i\dot{+}\alpha_k^i}\oplus$$

$$\oplus\ a_{\alpha_1^i\dot{+}\alpha_2^i\dot{+}\alpha_3^i}\oplus a_{\alpha_1^i\dot{+}\alpha_2^i\dot{+}\alpha_4^i}\oplus\ldots\oplus a_{\alpha_{k-2}^i\dot{+}\alpha_{k-1}^i\dot{+}\alpha_k^i}\oplus$$

$$\oplus\ldots\oplus(-1)^{k+1}\,a_{\alpha_1^i\dot{+}\ldots\dot{+}\alpha_k^i}\big)\Big)\neq 0\ ,$$

where $m(\alpha_1^i)=\ldots= m(\alpha_k^i) = 1,\quad \alpha_j^i\in M_n\ ,$

and

$$\sum_{\alpha\in M_n}{}_{\oplus} \frac{\partial^{m(\alpha)} R_{b_1^{[\alpha_1]}\dots b_n^{[\alpha_n]}}}{\partial x_1^{\alpha_1}\dots \partial x_n^{\alpha_n}} \oplus R = 0 . \tag{7.n}$$

P r o o f. The partial derivatives

$$\frac{\partial^{m(p)} F_{b_1\dots b_n}}{\partial x_1^{\beta_1}\dots \partial x_n^{\beta_n}}, \quad \beta=\beta_1\beta_2,\dots,\beta_n \in M_n\setminus\{11\dots 1\},$$

of functional equation (5) from a system of functional equations

$$\sum_{\alpha\in M_n}{}_{\oplus} a_\alpha \frac{\partial^{m(\alpha)+m(\beta)} f_{b_1^{\beta_1}\dots b_n^{\beta_n}}}{\partial x_1^{\alpha_1+\beta_1}\dots \partial x_n^{\alpha_n+\beta_n}} = d \frac{\partial^{m(\beta)} R_{b_1\dots b_n}}{\partial x_1^{\beta_1}\dots \partial x_n^{\beta_n}}$$

$\beta\in M_n\setminus\{11\dots 1\}$.

System (8) has unique solution

$$\frac{\partial f_{b_1}}{\partial x_1},\dots,\frac{\partial f_{b_n}}{\partial x_n} \quad \text{if and only if the rank}$$

of the augmented matrix A_n' of the system (8) is 2^n-1.

For n=2 the augmented matrix of system (8) is equivalent to the matrix

$$A_2' = \begin{bmatrix} a_{10} & a_{01} & a_{11} & R \\ 0 & a_{01} & a_{01} & R\oplus\frac{\partial R_{b_1}}{\partial x_1} \\ 0 & 0 & a_{01}\oplus a_{10}-a_{11} & R\oplus\frac{\partial R_{b_1}}{\partial x_1}\oplus\frac{\partial R_{b_2}}{\partial x_2} \\ 0 & 0 & 0 & R\oplus\frac{\partial R_{b_1}}{\partial x_1}\oplus\frac{\partial R_{b_2}}{\partial x_2}\oplus\frac{\partial^2 R_{b_1 b_2}}{\partial x_1\partial x_2} \end{bmatrix}$$

Thus, rank $A_2' = 3$ if and only if

$$a_{10}\, a_{01}(a_{01} \oplus a_{01} - a_{11}) \neq 0$$

$$R \oplus \frac{\partial R_{b_1}}{\partial x_1} \oplus \frac{\partial R_{b_2}}{\partial x_2} \oplus \frac{\partial^2 R_{b_1 b_2}}{\partial x_1 \partial x_2} = 0 .$$

This is proved in Theorem 1.

According to (6.n) for n=2 it follows

$$(6.2') \quad \prod_{m(\alpha)=1} a_\alpha \prod_{k=1}^{2} \Big(\prod_{i=1}^{\binom{2}{2-k}} (a_{\alpha_1^i} \oplus a_{\alpha_2^i} - a_{\alpha_1^i \dot{+} \alpha_2^i}) \Big) \neq 0 .$$

For n=3 the rank of the augmented matrix A_n' is 2^3-1 if and only if

$$\prod_{m(\alpha)=1} a_\alpha \prod_{k=2}^{3} \Big(\prod_{i=1}^{\binom{3}{3-k}} (a_{\alpha_1^i} \oplus a_{\alpha_2^i} \oplus \dots \oplus a_{\alpha_k^i} -$$

$$(7.3') \quad - a_{\alpha_1^i \dot{+} \alpha_2^i} - \dots - a_{\alpha_{k-1}^i \dot{+} \alpha_k^i} \oplus a_{\alpha_1^i \dot{+} \alpha_2^i \dot{+} \alpha_3^i} \oplus$$

$$\oplus \dots \oplus a_{\alpha_{k-2}^i \dot{+} \alpha_{k-1}^i \dot{+} \alpha_k^i} \oplus \dots \oplus (-1)^{k+1} a_{\alpha_1^i \dot{+} \dots \dot{+} \alpha_k^i})\Big) =$$

$$= \prod_{m(\alpha)=1} (a_{\alpha_i^1} \oplus a_{\alpha_2^1} - a_{\alpha_1^1 \dot{+} \alpha_2^1})(a_{\alpha_1^2} \oplus a_{\alpha_2^2} - a_{\alpha_1^2 \dot{+} \alpha_2^2})$$

$$(a_{\alpha_1^3} \oplus a_{\alpha_2^3} - a_{\alpha_1^3 \dot{+} \alpha_2^3})(a_{\alpha_1^1} \oplus a_{\alpha_2^1} \oplus a_{\alpha_3^1} - a_{\alpha_1^1 \dot{+} \alpha_2^1} -$$

$$- a_{\alpha_1^1 \dot{+} \alpha_3^1} - a_{\alpha_2^1 \dot{+} \alpha_3^1} \oplus a_{\alpha_1^1 \dot{+} \alpha_2^1 \dot{+} \alpha_3^1}) \neq 0,$$

$$m(\alpha_1^1) = m(\alpha_2^1) = m(\alpha_3^1) = 1.$$

According to (6.2') and (7.3') mathematical induction leads us to the proof of (6.n). (7.n) is proved in the same way according to (7.2) and (7.3). Thus, the theorem is proved.

REMARK. A new system of functional equations can be formed from the system of functional equations if it satisfies conditions (6.n) and (7.n)

$$\frac{\partial f_{b_i}}{\partial x_i} = P_i(x) \quad , \quad i=1,\dots,n. \tag{9}$$

System (9) has a solution if and only if conditions (2.i) and (2.ii) are fulfilled.

(2.i) and (2.ii) follows immediately from (6.n) and (7.n).

This, the system of functional equations (9) has a solution which is determined by the formula $(3.1_i \dots i_n)$.

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REZIME

GENERALISANE PSEUDO-BULOVE FUNKCIONALNE JEDNAČINE SA KONSTANTNIM KOEFICIJENTIMA I SA n PROMENLJIVIH

U ovom radu dati su potrebni uslovi (6.n) i (7.n) da generalisana funkcionalna jednačina (5) sa konstantnim koeficijentima i sa n promenljivih ima rešenja.