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SOME PROPERTIES OF LINEAR OPERATORS OF DISCRETE FUNCTIONS

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Let $(P,+,\cdot)$ be a commutative ring with an identity element 1, and let $L \neq \emptyset$ be a finite set; and L^n its Cartesian product. Let us consider a set $F = \{f/f:L^n \to R\}$, where f is a discrete function. Some operators on L and F can be defined.

DEFINITION 1. $\phi_j : L \to L \quad , j \in I$

$$\partial : F \to F$$

where

$$\phi_k(\phi_j x) = \phi_k x \quad j,k \in I,\ x \in L$$

$$\partial(f_{\phi_j}(x))_{x_i} = f(x_1,\dots,x_{i-1},\phi_j x_i, x_{i+1},\dots,x_n) - f(x), \quad x \in L.$$

THEOREM 1. *All operators* ϕ_j, ϕ_i *, and* $\partial\,(j,i \in I)$ *and all discrete functions* f *and* g *satisfy the following properties:*

(1) $\partial(f_{\phi_j})x_i = 0 \iff f = f(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$

(2) $\partial((f+g)_{\phi_j})x_i = \partial(f_{\phi_j})x_i + \partial(f_{\phi_j})x_i \ , \quad 1 \le i \le n$

(3) $\partial((kf)_{\phi_j})x_i = k\partial(f_{\phi_j})x_i, \quad k \in P,\ 1 \le i \le n$

(4) $\partial(\partial(f_{\phi_j})x_i)_{\phi_k})x_i = -\partial(f_{\phi_j})x_i, \quad 1 \le i \le n$

(5) $\partial((f\cdot g)\phi_j)x_i = g\cdot\partial(f\phi_j)x_i + f\cdot\partial(g\phi_j)x_i +$

$$+\ \partial(f\phi_j)x_i \cdot \partial(g\phi_j)x_i\ ,\quad 1 \le i \le n$$

(6) $\partial((\partial f\phi_j)x_i)\phi_k)x_j = \partial((\partial f\phi_k)x_j)x_i\ ,\quad i \neq j,\ 1 \le i \le n,\ 1 \le j \le n$

(7) $\partial(\partial(\ldots\partial(f\phi_j)x_i\ \ldots)\phi_j)x_i = (-1)^{m+1}\ \partial(f\phi_j)x_i,\quad 1 \le i \le n,$

∂ *is applied* m-*times.*

P r o o f. Relations (1), (3), (4) and (5) result immediately from Definition 1. Further, proving (2) let us use

$$(\phi_j\bar{x}_i) = (x_1,\ldots,x_{i-1},\phi_j x_i,\ldots,x_{i+1},\ldots,x_n)$$

thus

$$\partial((f+g)\phi_j)x_i = f(\phi_j\bar{x}_i)+g(\phi_j\bar{x}_i)-f(x)-g(x) =$$

$$= \partial(f\phi_j)x_i + \partial(g\phi_j)x_i\ ,\quad 1 \le i \le n.$$

The proof of (5)

$$\partial((f\cdot g)\phi_j)x_i = f(\phi_j\bar{x}_i)\cdot g(\phi_j\bar{x}_i) - f(x)\cdot g(x) =$$

$$= f(x)\cdot[g(\phi_j\bar{x}_i)-g(x)]+g(x)\cdot[f(\phi_j\bar{x}_i)-f(x)] +$$

$$+\ [f(\phi_j\bar{x}_i)-f(x)]\cdot[g(\phi_j\bar{x}_i)-g(x)]\quad .$$

Finally (7) will be proved by induction. For m=1 from (5) follows

$$\partial((\partial(f\phi_j)x_i)\phi_j)x_i = -\partial(f_{\phi_j})x_i\quad .$$

This equality is true according to (4).

Let us suppose that (7) is satisfied for each n. Now, using Definition 1. again in (7) we shall get

$$\partial(((-1)^{m+1}\partial(f_{\phi_j})x_i)\phi_j)x_i = (-1)^{m+2}\ \partial(f_{\phi_j})x_i\quad .$$

And so relation (7) is proved.

THEOREM 2. *For all discrete functions* f *and all operators* ϕ_j *(* $j \in I$ *) the following equality is held*

$$\partial((\ldots\partial(f_{\phi_{j_1}})_{x_{i_1}}\ldots)_{\phi_{j_m}})_{x_{i_m}} =$$

$$(8) \quad = \sum_{k=1}^{m} (-1)^{m-k} \sum_{i_1,\ldots,i_k}^{j_1,\ldots,j_m} f((\phi_{j_1}\bar{x}_{i_1}),\ldots,(\phi_{j_k}\bar{x}_{i_k})) + (-1)^m f(x),$$

where $\{i_1,\ldots,i_k\}$ *is a subset of* $\{j_1,\ldots,j_m\}$, $1 \le m \le n$.

P r o o f. For m=1 equality (8) becomes

$$\partial(f_{\phi_{j_1}})_{x_{i_1}} = f(\phi_{j_1}\bar{x}_{i_1}) - f(x) ,$$

it is true according to Definition 1.

Assume that equality (8) is true for every $m (m < n)$. Applying Definition 1. on (7) we get

$$\partial((\partial((\ldots\partial(f_{\phi_{j_1}})_{x_{i_1}}\ldots)_{\phi_{j_m}})_{x_{i_m}})_{\phi_{j_{m+1}}})_{x_{i_{m+1}}} =$$

$$= \sum_{k=1}^{m+1} (-1)^{m+1-k} \sum_{i_1,\ldots,i_k}^{j_1,\ldots,j_{m+1}} f((\phi_{j_i}\bar{x}_{i_1}),\ldots,(\phi_{i_k}\bar{x}_{i_k})) +$$

$$+ (-1)^{m+1} f(x) ,$$

thus, the theorem is proved.

THEOREM 3. *If*

$$f(\phi_j^n(x_n)) = f(\phi_j x_1,\ldots,\phi_{j_n} x_n)$$

$$f(\phi_i^n(x_n)) = f(\phi_i x_1,\ldots,\phi_{i_n} x_n) ,$$

then for all operators ϕ_j, ϕ_i, $i,j \in I$ *and all discrete functions the following equality holds:*

$$(9) \quad f(\phi_j^n(x_n) - f(\phi_i^n(x_n) = \sum_{m=1}^{n} \sum_{i_1,\ldots,i_m}^{1,\ldots,n} (\partial(\ldots\partial(f\,\phi_{j_1}(\phi_i x_1)_{x_{i_1}}\ldots$$

$$\ldots)_{\phi_{j_m}})_{x_{i_m}}) \quad (i_1 < i_2 < \ldots < i_m).$$

P r o o f. For m=1 (8) becomes

$$(10) \quad f(\phi_j^1 x_1) - f(\phi_i^1 x_1) = \partial(f_{\phi_j}(\phi_1 x_1))_{x_1}$$

and it is true according to Definition 1.

Let us suppose that (9) is true for n-1, i.e.

$$(11) \quad f(\phi_j^{n-1}(x_{n-1}),x_n)-f(\phi_i^{n-1}(x_{n-1}),x_n) = \partial(f_{\phi_j}(x_n))_{x_{n-1}}$$

Applying Definition 1. on (11) it will be transformed into equality (9).

Thus theorem 3. is proved.

In the next examples we shall show that these linear operators, given by Definition 1., cover partial derivatives of pseudo-Boolean functions and some parts of the following operators: partial derivatives of Boolean functions, Newton differences and lattice derivatives of discrete functions.

EXAMPLE 1. Let $f : L^n \to P$ be a generalized pseudo-Boolean function, and operators ϕ_j , $j \in I$

$$\phi_j x_i = a \;, \; j \in I \;, \; a \in L \;,$$

where x_i, $1 \leq i \leq n$ are variables of generalized pseudo-Boolean functions. Operators ∂ are generalized pseudo-Boolean functions.

$$\partial(f_{\phi_j}(x))_{x_i} = f(x_1,\ldots,x_{i-1}, a, x_{i+1},\ldots,x_n) - f(x)$$

These operators ∂ are partial derivatives of generalized pseudo-Boolean functions with respect to the variables x_i, $1 \leq i \leq n$ (see |2|).

EXAMPLE 2. A binary operation $\oplus$ with the following properties is defined on L: for every a, b, c $\in$ L

$$a \oplus b = b \oplus a$$

$$(a\oplus b)\oplus c = a\oplus(b\oplus c)$$

$$(\exists e \in L)\; e\oplus a = a\oplus e = a$$

$$(\forall a \in L)(\exists a' \in L)\; a\oplus a' = a' \oplus a = e$$

$$a\oplus a = a.$$

$L \neq \emptyset$ is a finite set, R is the set of real numbers, $f : L^n \to R$ is a real function. If the operators ϕ_j, $j \in I$ are defined in the following way

$$\phi_j x_i = x_i + h \ , \quad h \in L,$$

then operators ∂ are Newton differences

$$\partial(f_{\phi_j}(x))_{x_i} = f(x_1,\dots,x_{i-1},x_2 \oplus h,x_{i+1},\dots,x_n)-f(x).$$

EXAMPLE 3. Let (P,V,Λ) be a distributive lattice, $L \neq \emptyset$ a finite subset of P, and $f : L^n \to P$ a discrete function. If operators ϕ_j, $j \in I$ are defined

$$\phi_j x_i = x_i \vee a, \ a \in L,$$

where x_i, $1 \leq i \leq n$ are variables of discrete functions, then operators ∂ are discrete functions on the lattice

$$\partial(f_{\phi_j}(x))_{x_i} = f(x_1,\dots,x_{i-1},x_i \vee a,x_{i+1},\dots,x_n) \vee f(x) \ ,$$

$1 \leq i \leq n$

(see |3|).

EXAMPLE 4. Let $(P,+,\cdot)$ be a commutative ring with identity element 1. $L \neq \emptyset$ a finite subset of P, and $f : L^n \to P$. If operator ϕ_j, $j \in I$ are given by

$$\phi_j x_i = x_i + 1 \quad (\text{where } 1 + 1 = 1)$$

x_i, $1 \leq i \leq n$ are variables of f, then operators ∂ are

$$\partial(f_{\phi_j}(x))_{x_i} = f(x_1,\dots,x_{i-1},x_i+1,x_{i+1},\dots,x_n) - f(x)$$

(see |3|).

REFERENCES

|1| Akers, Jr., *On a theory of Boolean functions, Siam J. 487-498, 1956.*

|2| K. Gilezan, *Les dérivées partielles des fonctions pseudo-Booléennes genéralisée, Discrete Applied Mathematics, 4 (1982) 37-45, North-Holland Publishing Company.*

|3| M.Stojaković, *Sur L'algébre distentielle, Mat. vesnik 2(27) Sv. 3 1965.*

|4| A.Thayse et M. Davio, *Boolean Differential Calculus and its Application to Switching Theory, IEEE Transactions on Computers, v. c-22, No. 4, IV 409-420, 1973.*

|5| S.Rudeanu, *Boolean functions and equations, North-Holland, 1974.*

|6| M.Davio, J.P. Deschamps, A. Thayse, *Discrete and switching functions, Gr. Publishing Company, 1978.*

REZIME

NEKA SVOJSTVA LINEARNIH OPERATORA DISKRETNIH FUNKCIJA

U ovom radu data je jedna nova definicija linearnih operatora koji pokrivaju parcijalne izvode generalisanih pseudo-Bulovih funkcija i delove sledećih operatora: parcijalne izvode Bulovih funkcija, neke Njutnove razlike, kao i neke latisne izvode diskretnih funkcija.