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SUBALGEBRAS OF COMMUTATIVE SEMIGROUP SATISFYING THE LAW $x^r = x^{r+m}$

Georgi Čupona
Matematički fakultet p.f. 504 Skopje
Siniša Crvenković, Gradimir Vojvodić
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul. dr Ilije Djuričića 4, Jugoslavija

ABSTRACT

An algebra with a type Ω and a carrier A is an Ω-subalgebra of a semigroup S if $A \subseteq S$ and if there is a mapping $\omega \mapsto \bar{\omega}$ of Ω into S such that $\omega(a_1,\dots,a_n) = \bar{\omega}a_1 \dots a_n$, for every n-ary operator $\omega \in \Omega$ and the sequence of elements $a_1,\dots,a_n$ of A. If $\underline{C}$ is a class of semigroups then by $\underline{C}(\Omega)$ is denoted the class of Ω-algebras (i.e. algebras of the type Ω) which are subalgebras of semigroups belonging to $\underline{C}$. It is well known (see |1| p. 185 or |4| p. 78) that SEM(Ω) is the class of all Ω-algebras. It is also known (|5|) that ABSEM(Ω) is a variety. The object of our investigations is the set V of varietis $\underline{V}$ of semigroups such that $\underline{V}(\Omega)$ is also a variety. In Theorem 1. of this paper we show that $\underline{C}_{r,m}(\Omega)$ is a variety only if $r = 1$ or Ω does not contain n -ary operators for $n \geq 2$, where $\underline{C}_{r,m}$ is the class of commutative semigroups which satisfy the law $x^r = x^{r+m}$.

0. MAIN RESULTS

First, we note that if Ω is a set of finitary operators then $\Omega(n) = \{\omega \in \Omega | \omega$ is an n-ary operator$\}$. Obviously an Ω-algebra is an Ω-subalgebra of a semigroup S iff the corresponding restriction $\Omega \setminus \Omega(0)$-algebra is an $\Omega \setminus \Omega(0)$-subalgebra of S. Thus, we can assume that $\Omega(0) = \emptyset$ i.e. that Ω does not contain nullary operators.

THEOREM 1. $\underline{C}_{r,m}(\Omega)$ *is a variety iff* r=1 *or* $\Omega = \Omega(1)$.

THEOREM 2. *Let* A *be a nonempty set,* r *and* m *two positive integers, and* L *a subsemigroup of the semigroup* T_A *of all transformations of* A*, such that* $L \in \underline{C}_{r,m}$. *Then, there exists a semigroup* $M \in \underline{C}_{r,m}$ *with the following properties:*

(i) L *is a subsemigroup of* M*;*

(ii) $A \subseteq M$ *;*

(iii) $(\forall a \in A, \exists \phi \in L)\phi(a) = \phi a$. (ϕa *is the "product" of* ϕ *and* a *in* M)

Before giving the formulation of the last theorem, we have to give some preliminary definitions. Namely, if A is a nonempty set, then by O(A) is denoted the set of finitary (not nullary) operations on A, i.e. $O(A) = \bigcup_{n=1}^{\infty} O_n(A)$, where $O_n(A) = A^{A^n}$ consists of all n-ary operations on A. If $L \subseteq O(A)$, then $L(n) = L \cap O_n(A)$. An infinite collection $\{\overset{i}{+} \mid i=1,2,\dots\}$ of partial binary operations can be defined on O(A) by

$$(1)\quad \phi \in O_n(A),\ \psi \in O_m(A),\ i \leq n \ \Rightarrow$$
$$\phi \overset{i}{+} \psi(x_1,\dots,x_{m+n-1}) = \phi(x_1,\dots,x_{i-1},\psi(x_i,\dots,x_{i+m-1}),$$
$$,x_{i+m},\dots,x_{m+n-1})$$

(See for example |6| p. 7-49 or |3| p. 9). We have that $(O(A),\overset{1}{+})$ is a monoid. Further on, for the operation $\overset{1}{+}$ a usual multiplicative notation will be used. An operation $\phi \in O_n(A)$ is called commutative if

$$(2)\qquad \phi(a_1,\dots,a_n) = \phi(a_{i_1},\dots,a_{i_n})$$

for every sequence $a_1,\dots,a_n \in A$ and permutation $\nu \mapsto i_\nu$ of $N_n = \{1,2,\dots,n\}$.

THEOREM 3. *Let* L *be a commutative subsemigroup of the semigroup* O(A) *such that all the operations belonging to* L *are commutative and* $\phi \overset{i}{+} \psi = \phi\psi$, *for any* $\phi,\psi \in L$ *and* $i \in \{1,\dots$ $\dots,n\}$ *where* $\phi \in L(n)$. *Let* m *be a positive integer and assume that* L *satisfies the following statement:*

(*) If $\phi_1,\ldots,\phi_p \in L$, $\phi_\nu \in L(n_\nu+1)$ *and* $i_1,\ldots,i_p, j_1,\ldots,j_p,\alpha_1,\ldots$ $\ldots, \alpha_q,\beta_1,\ldots,\beta_q$ *are positive integers such that:*

(3) $i_\nu \equiv j_\nu (\mathrm{mod} m)$, $\alpha_\lambda \equiv \beta_\lambda (\mathrm{mod} m)$

and

(4) $1+i_1n_1+\ldots+i_pn_p = \alpha_1+\ldots+\alpha_q$

$1+j_1n_1+\ldots+j_pn_p = \beta_1+\ldots+\beta_q$,

then

(5) $\phi_1^{i_1}\ldots\phi_p^{i_p}(x_1^{\alpha_1},\ldots,x_q^{\alpha_q}) = \phi_1^{j_1}\ldots\phi_p^{j_p}(x_1^{\beta_1},\ldots,x_q^{\beta_q})$

is an identity equation on A. *Then, there exists a semigroup* $M \in \underline{C}_{1,m}$ *and a homomorphism* $\phi \mapsto \bar{\phi}$ *from* L *into* M *such that the following statements are satisfied:*

(i) $(\forall\phi \in L(1))\ \bar{\phi} = \phi$;

(ii) $A \subseteq M$;

(iii) $(\forall a_1,\ldots,a_n \in A,\ \phi \in L(n))\phi(a_1,\ldots,a_n) = \bar{\phi}a_1\ldots a_n$.

Obviously, the part of Theorem 1. for r=1, is a special case of Theorem 2.. But the corresponding generalization for $r \geq 2$ is not true, for, by Theorem 1., $\underline{C}_{r,m}(\Omega)$ is not a variety if $r \geq 2$ and $\Omega \neq \Omega(1)$. It should also be noticed that if $L \neq L(1)$ then the homomorphism $\phi \mapsto \bar{\phi}$ is not a monomorphism, for $\bar{\bar{\phi}} = \phi^{m+1}$ but if $\phi \in L(n)$ $n \geq 2$, then $\phi^{m+1} \neq \phi$. This suggests the problem of finding the set of varieties $\underline{V}$ of semigroups such that every subsemigroup $L \in \underline{V}$ of O(A) (or more special of $\mathcal{T}_A = O_1(A)$) can be embedded in a semigroup $M \in \underline{V}$.

1. P r o o f of Theorem 1. Identities in $\underline{C}_{r,m}(\Omega)$. Obviously, if A is an Ω-algebra belonging to $\underline{C}_{r,m}(\Omega)$ then A satisfies the following identity equations:

(**) $\phi(x_1,\ldots,x_n) = \phi(x_{i_1},\ldots,x_{i_n})$

for every $\phi \in \Omega(n)$ and permutation $\nu \mapsto i_\nu$ of N_n, i.e. the all operations of the algebra are commutative,

(***) $\quad \phi\psi = \psi\phi = \phi \overset{i}{+} \psi$

for any $\phi, \psi \in \Omega$ and $i \in \{1,2.,,,.n\}$ where $\phi \in \Omega(n)$;

and (*´) which is obtained from (*) in Theorem 3., replacing L with Ω and (3) with

(3´) $\quad i_\nu = j_\nu$ or $(i_\nu, j_\nu \geq r$ and $i_\nu \equiv j_\nu (\mathrm{mod}\, m))$

$\alpha_\lambda = \beta_\lambda$ or $(\alpha_\lambda, \beta_\lambda \geq r$ and $\alpha_\lambda \equiv \beta_\lambda (\mathrm{mod}\, m))$

It can be easily seen that all identity equations, which hold in all Ω-algebras belonging to $\underline{C}_{r,m}(\Omega)$, are consequences of (*´), (**) and (***). Namely, let ξ be an Ω-term (a term with operational symbols from Ω) with i_ν occurrences of the operator ω_ν, and α_λ occurrences of the variable x_λ. Then, by a finite number of applications of (**) and (***) we can obtain that

$$\xi = \omega_1^{i_1} \ldots \omega_p^{i_p}(x_1^{\alpha_1}, \ldots, x_q^{\alpha_q})$$

is an identity in $\underline{C}_{r,m}(\Omega)$. We have to show that if (3´) is not satisfied then (5) is not an identity in $\underline{C}_{r,m}(\Omega)$. Let F be a semigroup in $\underline{C}_{r,m}$ which is freely generated by $\Omega \cup \{e_1, e_2, \ldots$ $\ldots, e_k, \ldots\}$, where $e_\nu \notin \Omega$. By putting

$$\omega(u_1, \ldots, u_n) = \omega u_1 \ldots u_n ,$$

for every $\omega \in \Omega(n)$ and $u_1, \ldots, u_n \in F$ we obtain an Ω-algebra F, which, obviously, belongs to $\underline{C}_{r,m}(\Omega)$. If (3´) is not satisfied, then

$$\omega_1^{i_1} \ldots \omega_p^{i_p} e_1^{\alpha_1} \ldots e_q^{\alpha_q} \neq \omega_1^{j_1} \ldots \omega_p^{j_p} e_1^{\beta_1} \ldots e_q^{\beta_q} ,$$

in the semigroup F, i.e.

$$\omega_1^{i_1} \ldots \omega_p^{i_p}(e_1^{\alpha_1}, \ldots, e_q^{\alpha_q}) \neq \omega_1^{j_1} \ldots \omega_p^{j_p}(e_1^{\beta_1}, \ldots, e_q^{\beta_q})$$

in the Ω-algebra F.

This proves that (*´), (**) and (***) is an axiom system for the set of identities which are satisfied in all Ω-algebras belonging to $\underline{C}_{r,m}(\Omega)$.

1.2 $r \geq 2$ and $\Omega \neq \Omega(1)$. We shall give an example of an Ω-algebra which does not belong to $\underline{C}_{r,m}(\Omega)$, although it satisfies all the identities $(*')$, $(**)$ and $(***)$.

Let $\omega \in \Omega(n+1)$, where $n \geq 1$, and let i be the least positive integer such that $in+1-r=p \geq 0$. Thus, $1 \leq i < r$. Let $E = \{e_1,\dots,e_{rn},e\}$ be a set with $rn+1$ distinct elements and let A be the Ω-algebra with the presentation

$$\langle E;\omega^i(e_1,\dots,e_p,e^r) = \omega^r(e_1,\dots,e_{rn},e) \rangle_{(*'),(**),(***)}$$

where the indices $(*')$, $(**)$, $(***)$ mean that A satisfies all the identities $(*')$, $(**)$, $(***)$.

In algebra A the following inequality holds:

$$\omega^i(e_1,\dots,e_p,e^r) \neq \omega^{r+m}(e_1,\dots,e_{rn},e^{1+mn}) \quad , \tag{6}$$

for neither the left nor right hand side allows a proper transformation by $(*')$ and, by applying defining relation on $\omega^i(e_1,\dots,e_p,e^r)$ we get $\omega^r(e_1,\dots,e_{rn},e)$, so we can only turn to $\omega^i(e_1,\dots,e_p,e^r)$. But, if we assume that $A \in \underline{C}_{r,m}(\Omega)$, i.e. that A is an Ω subalgebra of semigroup $\mathbf{S} \in \underline{C}_{r,m}$, then we would have:

$$\begin{aligned}\omega^i(e_1,\dots,e_p,e^r) &= \bar{\omega}^i e_1\dots e_p e^r = \\ &= \bar{\omega}^i e_1\dots e_p e^{r+mn} = \omega^i(e_1,\dots,e_p e^r)e^{mn} = \\ &= \omega^r(e_1,\dots,e_p,e)e^{mn} = \bar{\omega}^r e_1\dots e_p e^{1+mn} = \\ &= \bar{\omega}^{r+m} e_1 \dots e_p e^{1+mn} = \omega^{r+m}(e_1,\dots,e_p,e^{1+mn}) \ .\end{aligned}$$

This example shows that, if $r \geq 2$ and $\Omega \neq \Omega(1)$, then $\underline{C}_{r,m}(\Omega)$ is a proper quasi variety.

1.3 $r = 1$. Let A be an Ω-algebra, and let Ω' be a subset of Ω such that different operators of Ω induce different operations on A, and for every $\omega \in \Omega(n)$, there is an $\omega' \in \Omega'(n)$ such that ω and ω' induce the same operation on A. Then, the Ω-algebra A is an Ω-subalgebra of a semigroup S iff the corresponding restricted

Ω' -algebra is an Ω'-subalgebra of S. Morover, (A,Ω) satisfies the identity $(*')$, $(**)$ and $(***)$ iff (A,Ω') satisfies the same identities. Therefore, we can assume that Ω is a set of finitary operations on A, i.e. $\Omega \subseteq O(A)$.

Let L be the subsemigroup of O(A) generated by Ω and let an Ω-algebra satisfy $(*')$, $(**)$, $(***)$. Then, the L-algebra A satisfy the same propositions and by the Theorem 3. the L-algebra is an L-subalgebra of a semigroup $M \in \underline{C}_{1,m}$, hence, we obtain that the given Ω-algebra A is an Ω-subalgebra of M.

1.4 $r \geq 2$ and $\Omega = \Omega(1)$. In this case an Ω-algebra satisfies all the idenitities $(*')$, $(**)$ and $(***)$ iff the semigroup L of transformations (generated by Ω) belongs to $\underline{C}_{r,m}$. By the Theorem 2. we have that if an Ω-algebra satisfies all the identities $(*')$, $(**)$ and $(***)$, then it is an Ω-subalgebra of a semigroup $S \in \underline{C}_{r,m}$. Therefore, $\underline{C}_{r,m}(\Omega)$ is, in this case, a variety.

Thus, the proof of Theorem 1. is completed, i.e. it is reduced to Theorems 2. and 3..

2. P r o o f of Theorem 2. If r=1, then Theorem 2. is a corollary of Theorem 3.. Thus, we have to consider only the case $r \geq 2$.

We may assume that L is a submonoid of $T_A = O_1(A)$, for if it is not we can add to L the identity transformation $\varepsilon_A : a \mapsto a$.

Let B be the monoid in the veriety $\underline{C}_{r,m}$, which is freely generated by A, i.e. the elements of S are "commutative product of powers" $a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_q^{\alpha_q}$, where $a_1, \ldots, a_q \in A$, $a_i \neq a_j$ for $i \neq j$ and $\alpha_\nu \geq 0$.

$$a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_q^{\alpha_q} = a_1^{\beta_1} a_2^{\beta_2} \ldots a_q^{\beta_q}$$

iff

$$(\forall \nu \in \{1,2,\ldots,q\})(\alpha_\nu = \beta_\nu \text{ or } (\alpha_\nu, \beta_\nu \geq r \text{ and } \alpha_\nu \equiv \beta_\nu (\mathrm{mod}\, m))).$$

Let C be the direct product of L and B. If $u = (\phi, a_1^{\alpha_1} \ldots a_q^{\alpha_q})$ then we denote u by $\phi \underline{a}$, where $\underline{a} = a_1^{\alpha_1} \ldots a_q^{\alpha_q}$.

If $u=u'a$, $v=\phi u'a'$, $u'\in C$, $a'=\phi(a)$, then we say that (u,v) and (v,u) are two pairs of neighbours. Two elements u,v from C are called equivalent, which is denote by $u \approx v$, iff there is a sequence $u_o,u_1,\dots,u_k$ of elements of C such that $u=u_o$, $v=u_k$, $k\geq 0$ and (u_{i-1},u_i) is a pair of neighbours for each $i\in\{1,\dots,k\}$. Obviously, $\approx$ is a congruence on C. Denote by M the corresponding factor monoid $C_{/\approx}$.

We can assume that L is a submonoid of M, for we have:

(i') $\phi,\psi\in L \Rightarrow (\phi\approx\psi \Rightarrow \phi=\psi)$.

If $a=\phi(a')$ then $a\approx\phi a'$, and thus the proof will be completed if we show that the following proposition is satisfied

(ii') $a,a'\in A \Rightarrow (a\approx a' \Rightarrow a=a')$.

Let $a\in A$ and $u_o,u_1,\dots,u_k$ be a sequence of elements of C such that $a=u_o$ and (u_{i-1},u_i) is a pair of neighbours for each $i\in\{1,2,\dots,k\}$. We are going to show that each u_i has a form $u_i=\phi_i a_i$ where $\phi_i(a_i)=a$. First, this is true for $i=0$, $a=u_o=$ $=\varepsilon a$, $\varepsilon(a)=a$. Assume that $u_{k-1}=\phi_{k-1}a_{k-1}$ and $\phi_{k-1}(a_{k-1})=a$. Then, we have

(I) $u_k=\phi\phi_{k-1}a_k$, $\phi(a_k)=a_{k-1}$, and thus $\phi_{k-1}\phi(a_k)=a$

or

(II) $u_k=\phi_{k-1}a_k$, $\phi_{k-1}=\phi\phi_{k-1}$ $\phi(a_{k-1})=a_k$ and then

$\phi_{k-1}(a_k)=\phi_{k-1}\phi(a_{k-1}) = \phi_{k-1}(a_{k-1}) = a$.

This completes the proof of Theorem 2..

3. P r o o f of Theorem 3. L satisfies the assumptions of Theorem 3. iff $L\cup\{\varepsilon\}$ satisfies them, and thus we can assume that L is a submonoid of O(A).

3.1. Let $\equiv$ be the least congruence on L such that $\bar{L}=L_{/\equiv}\in \underline{C}_{1,m}$. More explicitly, $\equiv$ is defined in the following way:

Let $\phi,\psi\in L$, then, $\phi\equiv\psi$ iff there exist $\phi_1,\dots,\phi_p\in L$ and nonnegetive integers $i_{\nu\lambda}$, $j_{\nu\lambda}$ such that:

(3.1) $i_{\nu\lambda}=j_{\nu\lambda}=0$ or $(i_{\nu\lambda},j_{\nu\lambda}\geq 0$ and $i_{\nu\lambda}\equiv j_{\nu\lambda}(\mathrm{mod}\, m))$

and the following equalities are satisfied:

$$\phi = \phi_1^{i_{11}} \phi_2^{i_{12}} \dots \phi_p^{i_{1p}}$$

$$(3.2) \quad \begin{array}{l} \phi_1^{j_{11}} \phi_2^{j_{12}} \dots \phi_p^{j_{1p}} = \phi_1^{i_{21}} \phi_2^{i_{22}} \dots \phi_p^{i_{2p}} \\ \dots\dots\dots\dots\dots\dots \\ \phi_1^{j_{q-11}} \phi_2^{j_{q-12}} \dots \phi_p^{j_{q-1p}} = \phi_1^{i_{q1}} \phi_2^{i_{q2}} \dots \phi_p^{i_{qp}} \\ \phi^{j_{q1}} \phi^{j_{q2}} \dots \phi^{j_{qp}} = \psi. \end{array}$$

From the given definition immediately follows

3.1.1. $\phi \in L(n')$, $\psi \in L(n'')$, $\phi \equiv \psi \Rightarrow n' \equiv n'' (\mathrm{mod}\, m)$.

3.1.2. $\phi \in L(1)$, $\psi \in L$, $\phi \equiv \psi \Rightarrow \phi = \psi$ (Thus, we assume that $L(1) \subseteq \bar{L} = L_{/\equiv}$).

Now, we are going to show that if $\phi \equiv \psi$, then ϕ and ψ have the same action to "similar sequences".

3.1.3. Let $\phi \equiv \psi$, $\phi \in L(n'+1)$, $\psi \in L(n''+1)$ and $\alpha_1, \dots, \alpha_q, \beta_1, \dots$ $\dots, \beta_q$ are such that $\alpha_\nu, \beta_\nu > 0$, $\alpha_\nu \equiv \beta_\nu$ (modm)

$$(3.3) \qquad \alpha_1 + \dots + \alpha_q = n'+1, \ \beta_1 + \dots + \beta_q = n'' + 1.$$

Then,

$$(3.4) \qquad \phi(x_1^{\alpha_1}, \dots, x_q^{\alpha_q}) = \psi(x_1^{\beta_1}, \dots, x_q^{\beta_q})$$

is an identity equation on A.

P r o o f. If $n'=0$ or $n''=0$, then by 3.1.2. $\phi=\psi$. Thus, we can assume that $n' > 0$ and $n'' > 0$. Let (3.2) be satisfied and let $\phi_\nu \in L(n_\nu+1)$. From $n' > 0$ and $n'' > 0$ it follows that for each μ there exists a λ such that $j_{\mu\lambda} > 0$ and $n_\lambda > 0$. We can assume that $j_{11} > 0$, $n_1 > 0$. Let s_1 be the least nonnegative integer such that

$$(3.5) \qquad 1 + j_{11}n_1 + s_1 m n_1 + j_{12}n_2 + \dots + j_{1p}n_p - (\beta_1 + \beta_2 + \dots + \beta_q) = t_1 \geq 0 .$$

Then

$$t_1 \equiv 1 + j_{11}n_1 + j_{12}n_2 + \dots + j_{1p}n_p - (\beta_1 + \dots + \beta_q) \, (\mathrm{mod}\, m)$$

$$\equiv 1+i_{11}n_1+i_{12}+n_2+\ldots+i_{1p}n_p-(\alpha_1+\ldots+\alpha_q)\ (\mathrm{mod}\, m) \equiv 0\ (\mathrm{mod}\, m).$$

Now, by (*) we have:

$$\phi(x_1^{\alpha_1},\ldots,x_q^{\alpha_q}) = \phi_1^{i_{11}}\ldots\phi_p^{i_{1p}}(x_1^{\alpha_1},\ldots,x_q^{\alpha_q})$$
$$= \phi_1^{j_{11}+s_1 m}\,\phi_2^{j_{12}}\ldots\phi_p^{j_{1p}}(x_1^{\beta_1+t_1},x_2^{\beta_2},\ldots,x_q^{\beta_q}).$$

If $j_{2\lambda_2}$, $n_{\lambda_2} > 0$, then in the same way we obtain:

$$\phi(x_1^{\alpha_1},\ldots,x_q^{\alpha_q}) = \phi_1^{j_{21}}\ldots\phi_p^{j_{2p}}\phi_{\lambda_2}^{s_2 m}(x_1^{\beta_1+t_2},x_2^{\beta_2},\ldots,x_q^{\beta_q})$$

where s_2 is chosen in a similar way as s_1. Finally, we should obtain

$$\phi(x_1^{\alpha_1},\ldots,x_q^{\alpha_q}) = \psi(x_1^{\beta_1},\ldots,x_q^{\beta_q}).$$

3.1.4. If $\phi \equiv \psi$ and $\phi,\psi \in L(n)$ then $\phi=\psi$.

P r o o f. This is an immediate corollary from 3.1.3..

Further on, if $\phi \in L$ then by $\bar{\phi}$ shall be denoted the element of $\bar{L} = L_{/\equiv}$ such that $\phi \in \bar{\phi}$.

3.2. As in 2, denote by B the monoid in the variety $\underline{C}_{1,m}$ which is freely generated by A, and by C the direct product $\bar{L} \times B$. An element $u=(\bar{\phi},a_1^{\alpha_1}\ldots a_q^{\alpha_q})$ shall be $\bar{\phi}\underline{a}$. The relation of neighbourhoodness shall also be defined in the same way. Namely, if

$$u=u'a,\quad v=\bar{\phi}u'a_1a_2\ldots a_n,\quad \phi \in L(n) \text{ and } a=\phi(a_1,\ldots,a_n),$$

then (u,v) and (v,u) are the pairs of neighbours generated by ϕ. The relation $\approx$ is the reflexive and transitive extension of the relation of neighbourhoodness; $\approx$ is a congruence on C. Denote the factor monoid by M.

If $\phi,\psi \in L$ then $\bar{\phi} \approx \bar{\psi}$ iff $\phi \equiv \psi$, and thus $\bar{L} \subseteq M$. By 3.1.2. we have $L(1) \subseteq M$. In further considerations, we are going to prove the following statement:

(ii') $a,a' \in A \Rightarrow (a \approx a' \Leftrightarrow a = a')$, which implies (ii), and as we have $a=\phi(a_1,\dots,a_n) \Rightarrow a \approx \bar{\phi} a_1 \dots a_n$ this will complete the proof of Theorem 2.

3.3. In order to prove statement (ii´), we shall consider a special subset T of C, and a mapping $u \mapsto [u]$ from T into A. If $u \in T$ then u is called a "term", and $[u]$ the "value" of u.

Let $u=\bar{\phi} a_o a_1 \dots a_p \in C$, $\phi \in L(n+1)$, $a_\nu \in A$ be such that $n \equiv p(\mathrm{mod}\, m)$. Then, $u \in T$ iff a) $n \geq 1$, or b) $n=0$, and there is a decomposition $\phi=\phi_o \phi_1 \dots \phi_p$ such that

$$\phi_o(a_o) = \phi_1(a_1) = \dots = \phi_p(a_p) = a.$$

In case a), there exist nonnegative integers i,j such that

$$(im+1)n+1 = jm+p+1$$

and then we put

$$[u] = \phi^{im+1}(a_o^{jm+1}, a_1, \dots, a_p) .$$

In case b), value $[u]$ is defined by $[u]=a$.

The value $[u]$ of a term u of form a), does not depend on i,j or on ϕ by (*) and 3.1.3.. But, we have to show that the same is true for a term of a form b).

Namely, if it is possible for ϕ to have another decomposition $\phi=\psi_o\psi_1\dots\psi_q$ such that

$$\psi_o(b_o) = \psi_1(b_1) = \dots = \psi_q(b_q) = b ,$$

where $a_o a_1 \dots a_p = b_o b_1 \dots b_q$ in B, we have to show that $a=b$. First, we can assume that $p=q$ and that $a_\nu = b_\nu$. Then, we have

$$\begin{aligned} a &= \phi_o(a_o) = \phi_o^m \phi_o(a_o) = \phi_o^m \phi_1(a_1) = \dots = \phi_o^m \phi_1^m \dots \phi_p^m \phi_o(a_o) = \\ &= \psi_o^m \psi_1^m \dots \psi_p^m \phi_o^p \phi_o(a_o) = \phi_1 \psi_o^m \psi_1^m \dots \psi_p^m \phi_o^p(a_1) = \\ &= \phi_1 \psi_o^m \psi_1^{m-1} \dots \psi_p^m \phi_o^p \psi_o(a_o) = \phi_1 \psi_o \psi_1^{m-1} \dots \psi_p^m \phi_o^{p-1} \phi_o(a_o) = \\ &= \dots = \phi_1 \phi_2 \dots \phi_p \psi_o^p \psi_1^{m-1} \dots \psi_p^{m-1} \phi_o(a_o) = \psi_o^{p+1} \psi_1^m \dots \psi_p^m(a_o) = \\ &= \psi_o \psi_1^m \dots \psi_p^m(a_o) = \psi_1^{m+1} \psi_2^m \dots \psi_p^m(a_1) = \psi_2^{m+1} \psi_3^m \dots \psi_p^m(a_2) = \\ &= \psi_p^{m+1}(a_p) = \psi_p(a_p) = b. \end{aligned}$$

Thus, the value $[u]$ of a term u is uniquely determined.

Now, we shall state some propositions concerning terms and values of terms.

3.3.1. If $\overline{\phi\underline{a}} \in T$ and $\phi \in L(sm+1)$ for some $s \geq o$ then $\overline{\phi\psi\underline{a}} \in T$ and

$$[\,\bar{\phi}[\overline{\psi\underline{a}}]] = [\overline{\phi\psi\underline{a}}] \ .$$

3.3.2. If $\overline{\phi\underline{a}a} \in T$ and $a=\psi(a_1,\dots,a_n)$ then $\overline{\phi\psi\underline{a}a_1} \dots a_n \in T$ and

$$[\overline{\phi\psi\underline{a}}\ a_1 \dots a_n] = [\overline{\phi\underline{a}a}\,].$$

3.3.3. If $\overline{\phi\psi\underline{a}b_1 \dots b_n} \in T$ and $\psi(b_1,\dots,b_n) = a$ then $\overline{\phi\psi^m}\underline{a}a \in T$ and

$$[\,\overline{\phi\psi\underline{a}b_1 \dots b_n}] = [\overline{\phi\psi^m\underline{a}a}] \ .$$

The proofs of 3.3.1. and 3.3.2. are straightforward and will not be given explicitly. If $\phi\psi$ is not unary, then 3.3.3. is a corollary of 3.3.2., and we are going to consider only the case when $\phi,\psi \in L(1)$:

Assume that $\overline{\phi\psi^i\underline{a}b_1} \in T$ and $[\overline{\phi\psi^i\underline{a}b_1}] = d$, $i \geq 1$. Then,

$$\phi\psi^i = \phi_o\phi_1 \dots \phi_p, \quad \underline{a}b_1 = a_1 \dots a_p b_1, \quad p \equiv 0 \,(\mathrm{mod}\, m) \quad b_1, a_\nu \in A, \quad \psi(b_1) = a,$$

$$d = \phi_o(b_1) = \phi_1(a_1) = \dots = \phi_p(a_p) \ ,$$

and

$$\psi(d) = \phi_o(\psi(b_1)) = \psi\phi_1(a_1) = \dots = \psi\phi_p(a_p) \ ,$$

where we obtain

$$[\phi_o\phi_1 \dots \phi_p\psi^p\underline{a}a] = [\phi\psi^i\underline{a}a] = \psi(d)$$

for $a = \psi(b_1)$, and $\psi^{i+p} = \psi^i$. From $\overline{\phi\psi^i}\underline{a}a \in T$ and $a = \psi(b_1)$, by 3.3.2., it follows that $\phi\psi^{i+1}\underline{a}b_1 \in T$ and $[\phi\psi^i\underline{a}a] = [\bar{\phi}\psi^{i+1}\underline{a}b_1]$.

Thus we have

$$[\overline{\phi\psi^m}\underline{a}a] = [\overline{\phi\psi^{m+1}}\underline{a}b_1] = [\overline{\phi\psi}\underline{a}b_1].$$

4.3. Here statement (ii´) (from the end of 3.2.) will be shown, and this will complete the proof of Theorem 3.

First, we prove that

4.3.1. If $a=u_o,u_1,\dots,u_p$ is a sequence of elements of C such that $p\geq 0$ and u_{i-1},u_i is a pair of neighbours generated by ϕ_i for each $i\in\{1,\dots,p\}$, then $\phi_1^m\dots\phi_q^m\, u_i \in T$ for each $i\in\{1,\dots,p\}$ and:

$$(3.6) \qquad [\phi_i^m a] = [\phi_1^m\dots\phi_i^m u_i] = a.$$

P r o o f. Assume that (3.6) is true, and that $i<p$. Then:

(I) $u_i = ub, \quad u_{i+1} = \phi u b_1\dots b_n$,

or

(II) $u_i = \phi u b_1\dots b_n, \quad u_{i+1} = ub$,

where $\phi=\phi_{i+1}$, $b=\phi(b_1,,,.b_n)$, $u\in C$.

In case (I), by 3.3.2. we have that

$$[\phi_1^m\dots\phi_i^m\phi^m u_{i+1}] = [\phi_1^m\dots\phi_i^m\phi u b_1\dots b_n] = [\phi_1^m\dots\phi_i^m u_i] = a ,$$

and by 3.3.1.

$$[\phi^m a] = [\phi^m[\phi_1^m\dots\phi_i^m u_i]] = [\phi^m\phi_1^m\dots\phi_i^m ub] =$$

$$= [\phi^m\phi_1^m\dots\phi_i^m\phi u b_1\dots b_n] = [\phi^m\phi_1^m\dots\phi_i^m u_{i+1}] = a .$$

In case (II), we have:

$$a = [\phi_1^m\dots\phi_i^m u_i] = [\phi_1^m\dots\phi_i^m\phi u b_1\dots b_n]$$ and by 3.1.3. this implies that

$$a = [\phi_1^m\dots\phi_i^m\phi^m ub] = [\phi_1^m\dots\phi_i^m\phi^m u_{i+1}] .$$

We also have

$$[\phi^m a] = [\phi^m[\phi_1^m\dots\phi_i^m u_i]] = [\phi_1^m\dots\phi_i^m\phi^m\,\phi u b_1\dots b_n] =$$

$$= [\phi_1^m\dots\phi_i^m\phi u b_1\dots b_n] = [\phi_1^m\dots\phi_i^m u_i] = a$$

and this complete the proof of 4.3.1.

P r o o f. of 3.2. (ii´)

Assume that $a, a' \in A$ and $a \approx a'$. Then, there exists a sequence of elements $u_o, u_1, \ldots, u_p$ of C such that $a=u_o$, $a'=u_p$ and (u_{i-1}, u_i) is a pair of neighbours generated by $\phi_i \in L$. By 4.3.1. we have

$$a = [\phi_1^m \ldots \phi_p^m a'] \ , \quad a = [\phi_1^m a] = \ldots = [\phi_p^m a],$$

and also

$$a' = [\phi_p^m \ldots \phi_1^m a], \quad a' = [\phi_1 a'] = \ldots = [\phi_p a'],$$

which implies that $a=a'$.

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REZIME

PODALGEBRE KOMUTATIVNIH POLUGRUPA KOJE ZADOVOLJAVAJU ZAKON $x^r = x^{r+m}$

Algebra tipa Ω sa nosačem A naziva se Ω-podalgebra polugrupe S ako je $A \subset S$ i ako postoji preslikavanje $\omega \mapsto \bar{\omega}$ Ω u S takvo da je

$$\omega(a_1, \ldots, a_n) = \bar{\omega} a_1 \ldots a_n$$

za svaku n-arnu operaciju $\omega \in \Omega$ i niz elemenata $a_1, \dots, a_n$ iz A. Ako je K klasa polugrupa tada sa $K(\Omega)$ označavamo klasu Ω-algebri koje su podalgebre polugrupa koje pripadaju K. Ako je K varijetet polugrupa, tada je $K(\Omega)$ kvazivarijetet Ω-algebri.

U ovom radu daju se potrebni i dovoljni uslovi da $\underline{C}_{r,m}(\Omega)$ bude varijetet (Teorema 1.). U teoremama 2. i 3. dat je opis polugrupa operacija koje se mogu potopiti u polugrupe iz $\underline{C}_{r,m}$.