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EQUATION OF OSCILLATION OF A VISCOELASTIC BAR (II)

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INTRODUCTION

In paper |5| the following equation was analysed:

$$(1)\qquad \partial_t^2 u(t,x) - \partial_x^2 u(t,x) + \int_0^t \partial_t^2 u(t-\tau,x)G(\tau)d\tau = 0$$

with the initial conditions:

$$(2)\qquad (2') \ u(0,x) = 0, \qquad (2'') \ \partial_t u(0,x) = \delta(x)$$

where

$$(3)\qquad \{G(t)\} = \sum_{i=1}^{\infty} a_i \ell^{i\alpha} = \{ \sum_{i=1}^{\infty} a_i \frac{t^{i\alpha-1}}{\Gamma(i\alpha)} \}, \ a_1 > 0, \ 0 < \alpha < 1.$$

The solution of equation (1) with condition (2′), which is a continuous function for $t \geq 0$, $x \neq 0$, having a line of symmetry $x = 0$, is:

$$(4)\qquad u(x) = \frac{1}{2} \ell \exp(-|x| \sum_{i=o}^{\infty} c_i \ell^{i\alpha-1}) \ ; \ c_o = 1$$

the coefficients c_i can be obtained in the following way:

$$c_1 = \frac{1}{2} a_1$$

(5)
$$c_2 = \frac{1}{2} (a_2 - (a_1/2)^2)$$
$$\cdots\cdots\cdots\cdots$$
$$c_i = \frac{1}{2} (a_i - \sum_{j=1}^{n-1} c_{i-j} c_j).$$

Solution (4) satisfies condition (2") too, if we have $i_1\alpha - 1 > 0$, where i_1 is the index of coefficients c_i such that $i_1 > 1$, $c_i = 0$ for $1 < i < i_1$ and $c_{i_1} \neq 0$.

This result is a generalization of the result from |3| in which problem (1), (2) was observed for:

(6)
$$G(t) = 2\lambda \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \lambda^2 \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + R(t), \quad \lambda > 0,\ 0 < \alpha < 1,$$

where either $R(t) \equiv 0$ or $R(0) = R'(0) = 0$ and $R''(t)$ is "sufficiently small" for $t \in [0,T]$, $R(t) \in C^\infty$.

This paper is a continuation of paper |5|. Our aim is to estimate the difference between the solution of equation (1) with conditions (2) when $G(t)$ is given by (6) and the solution when in the expression for $G(t)$ the remainder $R(t)$ is omitted (Proposition 3.). In Proposition 4 we observe a more general expression for $G(t)$; we construct the approximation of the solution of problem (1), (2) and estimate the distance from solution (4).

ESTIMATION OF THE COEFFICIENTS c_i

In order to obtain the error's estimation which has been named before, let us observe a relation between the coefficients c_i from (4) and a_i from (3).

PROPOSITION 1. *Let us suppose that the following inequalities for coefficients* a_i *hold:*

(7)
$$r^{-i}(-1)^{i-1}\binom{r}{i} \leq (-1)^{i-1} a_i \leq (-1)^{i-1} s^{-i}\binom{s}{i}$$

for two real numbers r,s, *such that* $0 < s < r < 1$. *Then for coefficients* c_i *we have:*

$$(8)\quad r^{-1}(-1)^{i-1}\binom{r/2}{i} \le (-1)^{i-1}c_i \le (-1)^{i-1}\binom{s/2}{i}s^{-i} .$$

P r o o f. The function $(-1)^{i-1}\frac{1}{x^i}\binom{x}{i} = \frac{(1-x)(2-x)\ldots(i-1-x)}{i!\,x^{i-1}}$ is monotonically decreasing for each $i \ge 2$ in the interval $0 < x < 1$, hence relation (7) makes sense. For $i=1$ (8) holds; this follows at once from (5). Suppose that (8) holds for $i = n-1$, $n \ge 2$, then

$$(-1)^{n-1}c_n = \frac{1}{2}\Big[(-1)^{n-1}a_n + \sum_{j=1}^{n-1}(-1)^{n-j-1}c_{n-j}(-1)^{j-1}c_j\Big] \ge$$

$$\ge \frac{1}{2}\Big[(-1)^{n-1}r^{-n}\binom{r}{n} + (-1)^n r^{-n}\sum_{j=0}^{n}\binom{r/2}{n-j}\binom{r/2}{j} + \frac{2(-1)^{n-1}}{r^n}\binom{r/2}{n}\Big]$$

$$\ge (-1)^{n-1}r^{-n}\binom{r/2}{n} .$$

In the same manner one proves the other inequality in (8).

PROPOSITION 2. *Let us suppose that for the coefficients* a_i *from (3) the following inequalities hold* : $|a_i| \le b_i$, $i=1,2,\ldots$, $b_i \ge 0$. *Also, we suppose that there exists* $r > 0$ *such that for each* z, $|z| \le r$ *the following conditions are satisfied:*

a) the series $\sum_{i=1}^{\infty} b_i z^i = K(z)$ *converges ;*

b) $\tilde{G}(z) = \sum_{i=1}^{\infty} a_i z^i \ne -1$

Then $c_i \le M/r^i$, $M = \sqrt{1 + K(r)}$

P r o o f. It follows from the fact that the function $1 + \tilde{G}(z)$ is regular, different from zero in the circle $\{z; |z| \le r\}$ and from Cauchy inequalities.

COROLLARY. *Provided that* $|a_i| \le C\gamma^i$ *for* $i \ge 1$ *and that there exists* $r > 0$ *such that* $\gamma < 1/r$ *and* $C < (1-\gamma r)/\gamma r$, *then* $|c_i| \le \sqrt{2}/r^i$.

P r o o f. Let us show that the conditions of Proposition 2 are satisfied for $|z| \le r$:

a) $|K(z)| = |\sum_{i=1}^{\infty} b_i z^i| \le C\gamma r \sum_{i=o}^{\infty} (\gamma r)^i \le C\gamma r/(1-\gamma r)$;

b) $|\tilde{G}(z)| \le K(|z|) \le C\gamma r/(1-\gamma r) < 1$, $\tilde{G}(0) = 0$ and it follows

$\tilde{G}(z) \ne -1$;

c) $M = \sqrt{1 + K(|z|)} \le \sqrt{2}$.

APPROXIMATION OF THE SOLUTION OF EQUATION (1) AND THE MEASURE OF APPROXIMATION

It is known from |5| that the solution of problem (1),(2) for G(t) given by (6) and $R(t) \equiv 0$ is:

(9) $u_\lambda(x) = \frac{1}{2} \ell \; \exp(-|x|s)\exp(-\lambda|x|s^{1-\alpha})$ or

(10) $u_\lambda(x,t) = \frac{1}{2} \ell \begin{cases} (t-|x|)^{-1}\phi(0,-(1-\alpha),-\lambda|x|(t-|x|)^{-(1-\alpha)}), & t > |x| \\ 0, & t \le |x| \end{cases}$

where ϕ is Wright's function |6|.

For two elements f and g from C let us denote

$f \le_T g \Leftrightarrow f(t) \le g(t)$, $0 \le t \le T$.

PROPOSITION 3. *Let us suppose that:*

a) G(t) *is given by relation (3) where* $a_1 = 2\lambda$, $a_2 = \lambda^2$, $\lambda > 0$;

b) The coefficients a_i, $i \ge 1$, *satisfy the conditions of the Corollary to Proposition 2 and* *c)* $\alpha > 1/k$ *, where* $k > 2$ *such that* $c_i = 0$ *for* $2 \leq i < k$ *and* $c_k \neq 0$.

Then

$$(11) \qquad u(x) = u_\lambda(x)\exp(-|x| \sum_{i=k}^{\infty} c_i \ell^{i\alpha-1})$$

is the solution of equation (1) with conditions (2). For $|x| \leq t \leq T\ (\lambda|x|)^{1/\sigma} + |x| \equiv T_1(|x|)$ *and* $0 \leq t \leq T$, $\sigma = 1-\alpha$, *we have*

$$(12) \qquad |u(x)-u_\lambda(x)| \leq \frac{\nu|x|^{1-1/\sigma}\lambda^{-1/\sigma}T_1^{\delta+1}}{2\delta(\delta+1)}\left(\frac{T^{2n}}{(2n)!}C(2n+1) + \right.$$

$$\left. + \frac{T^{2n+1}}{(2n+1)!}C(2n+2)\right)\cdot \sum_{k=o}^{\infty} \frac{(|x|\ \nu\ T_1^{\delta})^k}{\Gamma(k+2)\Gamma((k+1)\delta)}$$

where

$$(13) \qquad C(k) = \frac{1}{\sigma\pi}\,\Gamma\left(\frac{k}{\sigma}\right)\cdot\cos^{-k/\sigma}\frac{\sigma\pi}{2}\ ;\ \delta = k\alpha-1;\ \nu \geq \frac{M}{r^k}\sum_{j\geq o}\frac{T_1^{j\alpha}}{r^j\Gamma(j\alpha+1)} .$$

P r o o f. The assumptions of Proposition 3 imply the conditions of Proposition 2 from |5|, hence there exists a solution of form (4) which is a continuous function for $t \geq 0$, $x \neq 0$ having a line of symmetry $x = 0$ and satisfying conditions (2). The coefficients c_i are given by (5). Since $c_2 = \frac{1}{2}(a_2 - -(a_1/2)^2) = 0$, it follows that $k \geq 3$. The coefficients a_i satisfy the conditions of the Corollary to Proposition 2 and so $|c_i| \leq \sqrt{2}/r^i$.

The functions $u(x)$ and $u_\lambda(x)$ are continuous, so we can use the inequality of the form $\leq_T$. From (11) follows:

$$|u(x) - u_\lambda(x)| \leq_T |u_\lambda(x)|\,|\exp(-|x| \sum_{i=k}^{\infty} c_i \ell^{i\alpha-1}) - I|$$

Using (10) and |1| we have

$$(14) \qquad u_\lambda(t,x) = \tfrac{1}{2}\ell(\lambda|x|)^{-1/\sigma}|[(\lambda|x|)^{-1/\sigma}(t-|x|)]^{-1}\phi(0,-\sigma, -$$

$$- (\lambda|x|)^{-1/\sigma}(t-|x|)^{-\sigma})| \leq \frac{1}{2}\,\ell^2(\lambda|x|)^{-1/\sigma}\left(\frac{T^{2n}}{(2n)!}C(2n+1) + \right.$$

$$\left. + \frac{T^{2n+1}}{(2n+1)!}\,C(2n+2)\right) = \frac{1}{2}\,\ell^2\Omega(\lambda)$$

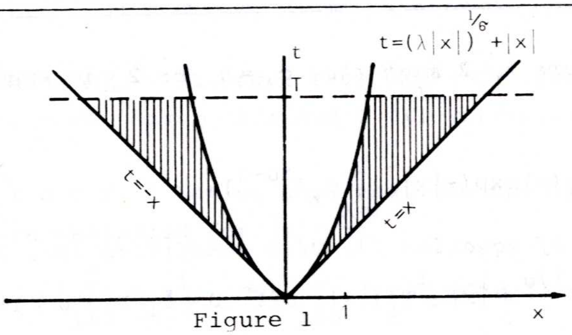

Figure 1

for $|x| \le t \le T(\lambda|x|)^{1/\sigma} + |x| = T_1$ or (Fig. 1) $0 \le (\lambda|x|)^{-1/\sigma} \cdot (t-|x|) \le T$ and $C(k)$ is given by (13).

Applying the results from |2|, we have

$$|\exp(-|x| \sum_{i=k}^{\infty} c_i \ell^{i\alpha-1}) - I| \le_T \Omega_{\delta,\nu} \ell^{\delta} \quad , \quad \text{where}$$

$$(15) \qquad \Omega_{\delta,\nu} = |x| \nu \Gamma(\delta) \sum_{k=o}^{\infty} \frac{(\nu|x|T_1^{\delta})^k}{\Gamma(k+2)\Gamma((k+1)\delta)}$$

for $0 \le t \le T$.

REMARK

This approximation is suitable for $0 \le T \le 1$. If $T > 1$, one can use the following estimation for the function ϕ (|4|):

$$(16) \qquad t^{-1}\phi(0,-\sigma,t^{-\sigma}) \le \frac{1}{\sigma\pi} \cos^{-1/\sigma}(\frac{\sigma\pi}{2})\Gamma(1/\sigma) \equiv K(\sigma) \ .$$

The coefficients a_1 and a_2 are not always supposed to have such a special form as in Proposition 3. We kept them in this form in order to show the influence of the neglecting of addend $R(t)$ in function $G(t)$ on the solution of equation (1), as was done in |3|.

We are going to show only one statement of a more general type:

PROPOSITION 4. *Let us suppose that:*

a) $G(t)$ *is given by (3), where* $a_1 > 0, a_2 > (a_1/2)^2$ *;*

b) *Coefficients* a_i , $i \geq 1$, *satisfy the conditions of the Corollary to Proposition 2;*

c) $1/3 < \alpha < 1/2$.

Then $\tilde{u}_{k-1}(x) = \frac{1}{2}\ell \exp(-|x| \sum_{i=o}^{k-1} c_i \ell^{i\alpha-1})$ *is a solution of problem (1), (2) and*

$$(17) \quad |u(x)-\tilde{u}_{k-1}(x)| \leq_T \frac{1}{2}\ell^3 \Omega(c_1) \frac{K(1-2\alpha)}{(|x|c_2)^{1/(2\alpha-1)}} \prod_{i=3}^{k-1} (I + \ell^{i\alpha-1}\Omega_{i\alpha-1',|x|c_i}) \cdot \Omega_{\delta,\nu}\ell^{\delta}$$

for $|x| \leq t \leq T(\lambda|x|)^{1/\sigma} + |x| \leq T_1(|x|)$ *and* $0 \leq t \leq T$.

(The notations are as in Proposition 3.)

P r o o f. Let us observe that

$$|\exp(-|x|c_2)\ell^{2\alpha-1}| = |t^{-1}\phi(0,-(1-2\alpha),-|x|c_2t^{2\alpha-1})| \leq \frac{K(1-2\alpha)\ell}{(|x|c_2)^{1/2\alpha-1}}$$

and from |4|

$$|\ell\exp(-|x|c_i\ell^{i\alpha-1})| = |\ell + \ell \sum_{k=1}^{\infty} (-|x|c_i\ell^{i\alpha-1})^k|$$

$$\leq_T \ell + \Omega_{i\alpha-1,|x|c_i}\ell^{i\alpha} .$$

NUMERICAL EXAMPLE

Let us suppose that the coefficients a_i satisfy the conditions of Proposition 3 and the inequalities $|a_i| \leq C\gamma^i$, $i \geq 1$; (for $\lambda = \gamma < 1/r$ and $C < (1/\gamma r)-1$ the coefficients c_i from the Corollary to Proposition 3 satisfy $|c_i| < \sqrt{2}/r^i$). Then the measure of approximation, can be expressed for $|x|=1$:

	λ	$1/r$	$\lvert u(x) - u_\lambda(x)\rvert \leq$		
			$T_1 = 1$ $(T=10^{-5})$	$T_1=2$	$T_1=10$
	10^{-2}	10^{-1}	$3.5611 \cdot 10^{-7}$	$1.5570 \cdot 10^{1}$	$2.7395 \cdot 10^{2}$
	10^{-3}	10^{-2}	$3.4500 \cdot 10^{-8}$	$1.3772 \cdot 10^{0}$	$1.5594 \cdot 10^{1}$
$\alpha=1/2$	10^{-4}	10^{-3}	$3.4291 \cdot 10^{-9}$	$1.3639 \cdot 10^{-1}$	$1.5283 \cdot 10^{0}$
$k=3$	10^{-5}	10^{-4}	$3.2744 \cdot 10^{-10}$	$1.3654 \cdot 10^{-2}$	$1.5254 \cdot 10^{-1}$
	10^{-6}	10^{-5}	$3.2744 \cdot 10^{-11}$	$1.3654 \cdot 10^{-3}$	$1.5251 \cdot 10^{-2}$
	10^{-7}	10^{-6}	$3.2744 \cdot 10^{-12}$	$1.3654 \cdot 10^{-4}$	$1.5251 \cdot 10^{-3}$
	10^{-2}	10^{-1}	$3.6529 \cdot 10^{-3}$	$7.5038 \cdot 10^{0}$	
$\alpha=1/2$	10^{-3}	10^{-2}	$2.0021 \cdot 10^{-4}$	$6.1189 \cdot 10^{-1}$	$2.6787 \cdot 10^{2}$
$k=5$	10^{-4}	10^{-3}	$1.9884 \cdot 10^{-5}$	$6.1168 \cdot 10^{-2}$	$2.5642 \cdot 10^{1}$
	10^{-5}	10^{-4}	$1.9870 \cdot 10^{-6}$	$6.1091 \cdot 10^{-3}$	$2.5544 \cdot 10^{0}$
	10^{-6}	10^{-5}	$1.9870 \cdot 10^{-7}$	$6.1091 \cdot 10^{-4}$	$2.5544 \cdot 10^{-1}$
	10^{-7}	10^{-6}	$1.9870 \cdot 10^{-8}$	$6.1091 \cdot 10^{-5}$	$2.5544 \cdot 10^{-2}$

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REZIME

JEDNAČINA OSCILACIJE ŽILAVOELASTIČNOG ŠTAPA (II)

Ovaj rad je nastavak rada |5|. U radu je ocenjena razlika rešenja jednačine

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + \int_0^t \partial_t^2 u(t-\tau,x)G(\tau)d\tau = 0$$

sa početnim uslovima

$$u(0,x) = 0 \qquad \partial_t u(0,x) = \delta(x)$$

kada $G(t)$ ima oblik

$$G(t) = 2\lambda t^{\alpha-1}/\Gamma(\alpha) + \lambda t^{2\alpha-1}/\Gamma(2\alpha) + R(t)$$

i rešenja koje se dobija kada se u tom izrazu zanemari $R(t)$ (Tvrdjenje 3)

U tvrdjenju 4 posmatran je izraz $G(t)$ oblika:

$$G(t) = \sum_{i=1}^{\infty} a_i \ell^{i\alpha} \quad ;$$

formirana su aproksimativna rešenja koja takodje zadovoljavaju početni uslov i ocenjena je greška.