

Boundedness of solutions of nonlinear retarded differential equations by using new integral inequalities

Tazeen Athar¹, Usman Mustafa²,
Mahvish Samar³ and Abdul Shakoor^{4,5}

Abstract. The purpose of this work is to present several new retarded nonlinear integral inequalities which can act as powerful tools for studying qualitative properties such as existence, uniqueness, boundedness, stability, and asymptotics of solutions of nonlinear retarded differential and integral equations. These inequalities generalize some former famous inequalities on the subject. From applications perspective, two examples will be provided to demonstrate the strength of our inequalities in estimating the existence and boundedness of solutions for initial value problems of nonlinear retarded differential equations.

AMS Mathematics Subject Classification (2010): 39B72, 26D10, 34A34

Key words: Gronwall inequality; Boundedness; Retarded nonlinear differential equation

1. Introduction

It is a well known truth that there exists a class of mathematical models described by differential equations and a lot of differential equations do not possess the exact solutions or existence of solutions or the boundedness of solutions. On the other hand, integral inequalities have always been of great importance for the development of many branches of mathematics. Indeed, this importance increased during the last several years and especially the theory of integral inequalities nowadays may be regarded as an independent branch of mathematics. As a response to the needs of diverse applications, a large variety of integral inequalities have been proposed and studied in ([15],[18],[19] and references therein). Some integral inequalities for differential and integral equations are established by Gronwall [14], Bellman [8] and Pachpatte ([15],[20]) which provide explicit bounds on solutions of differential and integral equations.

¹Department of Computer Science, Shaheed Zulfikar Ali Bhutto Institute of Science and Technology, Islamabad 44000, Pakistan,

e-mail: tazeen.athar@gmail.com, ORCID iD: orcid.org/0009-0002-7020-6517

²Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan,

e-mail: usmanmustafa9595@gmail.com, ORCID iD: orcid.org/0009-0009-0970-3304

³School of Mathematical Sciences, Zhejiang Normal University, Jinhua 321004, China,

e-mail: mahvishsamar@hotmail.com, ORCID iD: orcid.org/0009-0005-1098-1054

⁴Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan,

e-mail: ashakoor313@iub.edu.pk, ORCID iD: orcid.org/0000-0002-8723-4621

⁵Corresponding author

Throughout this article, \mathbb{R} presents the set of real numbers and $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_1 = (0, \infty)$ are subsets of \mathbb{R} . Moreover, the class of all continuous functions and continuously differentiable functions defined on set S_1 with range in the set S_2 are denoted by $\zeta(S_1, S_2)$ and $\zeta'(S_1, S_2)$, respectively.

An important integral inequality was first time discovered by Gronwall [14] in 1919, which can be used as an effective tool in the study of existence, uniqueness and other properties of solutions of certain nonlinear differential and difference equations. The Gronwall inequality [14] is stated as follows: Let $y, f \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$ be nonnegative functions with a positive constant y_0 , then

$$(1.1) \quad y(r) \leq y_0 + \int_0^r f(\lambda)y(\lambda)d\lambda, \quad \forall r \in \mathbb{R}_+,$$

implies that

$$y(r) \leq y_0 \exp\left(\int_0^r f(\lambda)d\lambda\right), \quad \forall r \in \mathbb{R}_+.$$

A very important generalization of Gronwall inequality was given by Bellman [8] in 1943 by introducing a function $g(t)$ instead of constant y_0 , which was later named Gronwall-Bellman inequality [8]. He established that: Let $y, f, g \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$ and g be nondecreasing, then

$$(1.2) \quad y(r) \leq g(t) + \int_0^r f(\lambda)y(\lambda)d\lambda, \quad \forall r \in \mathbb{R}_+,$$

gives that

$$y(r) \leq g(t) \exp\left(\int_0^r f(\lambda)d\lambda\right), \quad \forall r \in \mathbb{R}_+.$$

Many useful generalizations, refinements, and extensions of the above inequalities and their applications can be found in ([4],[10],[17],[22],[2],[3],[1],[21],[5],[9],[23],[11],[28],[13],[12],[24],[26],[25]). However, most of these inequalities are only restricted to a particular topics such as Tian and Fan [28] established nonlinear integral inequality with power and gave its application in delay integro-differential equations in 2020 while El-Deeb and Rashid [13] studied new double dynamic inequalities associated with Leibniz integral rule on time scales in 2021. Recent articles were published on delay nonlinear dynamic inequalities of Gronwall-Bellman-Pachpatte type ([12],[24],[26]) in 2022. But there is a need to establish more general results for retarded nonlinear integral inequalities which can act as powerful tools for studying qualitative properties such as existence, uniqueness, boundedness, stability and asymptotics of solutions of initial value problems for nonlinear retarded differential and integral equations.

So, our main focus is to present several new retarded nonlinear integral inequalities that will help us to study explicit bounds and the existence of solutions of nonlinear retarded differential equations. This research work will ensure to open new opportunities for studying of nonlinear dynamic inequalities on time scale structures of varying nature, for example, we refer the interested reader to ([27],[6],[7]) for contributions, and the references cited therein.

The article is structured in the following way: Section 2 gives several new retarded nonlinear integral inequalities while boundedness and existence of solutions of nonlinear retarded differential equations will be discussed in Section 3 to explore the effectiveness of this work which can be seen by graphs. Section 4 states the conclusion of this article.

2. New retarded integral inequalities

This Section present several new retarded nonlinear integral inequalities which can act as powerful tools for studying qualitative properties such as existence, uniqueness, boundedness, stability and asymptotics of solutions of nonlinear retarded differential and integral equations. These results extend a few existing inequalities which were proved in ([8],[20],[4],[10]). Before proceeding to our main results, we will introduce the following inequality that will be used in the proof of upcoming inequalities:

Lemma 2.1. [16] *Let $a \geq 0$ and $m \geq n \geq 0$ and $m \neq 0$, then*

$$a^{\frac{n}{m}} \leq \frac{n}{m}a + \frac{m-n}{m}.$$

We begin with following theorem.

Theorem 2.2. *Let $y, f_i \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2, 3, 4, 5$ and $\gamma \in \zeta'(\mathbb{R}_+, \mathbb{R}_+)$ with $\gamma(r) \leq r, \gamma(0) = 0$ and $m \geq 1, n \in (0, 1]$. If*

$$(2.1) \quad \begin{aligned} y^m(r) &\leq f_1(r) + f_2(r) \int_0^{\gamma(r)} \left(f_3(s)y^m(s) + f_4(s)y(s) \right)^n ds \\ &+ f_2(r) \int_0^{\gamma(r)} f_5(s)y^m(s)ds, \forall r \in \mathbb{R}_+, \end{aligned}$$

holds, then

$$\begin{aligned}
 & y(r) \\
 & \leq \left\{ f_1(r) + f_2(r) \left(\exp \left(\int_0^{\gamma(r)} f_2(\lambda) \left(n f_3(\lambda) + \frac{n}{m} f_4(\lambda) + f_5(\lambda) \right) d\lambda \right) \right. \right. \\
 & \quad \times \int_0^{\gamma(r)} \left(n f_3(s) f_1(s) + n f_4(s) \left(\frac{1}{m} f_1(s) + \frac{m-1}{m} \right) + 1 - n + f_5(s) f_1(s) \right) \\
 (2.2) \quad & \left. \left. \times \exp \left(- \int_0^s f_2(\lambda) \left(n f_3(\lambda) + \frac{n}{m} f_4(\lambda) + f_5(\lambda) \right) d\lambda \right) ds \right) \right\}^{\frac{1}{m}}, \forall r \in \mathbb{R}_+.
 \end{aligned}$$

Proof. Assume that

$$\begin{aligned}
 V(r) &= \int_0^{\gamma(r)} \left(f_3(s) y^m(s) + f_4(s) y(s) \right)^n ds \\
 (2.3) \quad &+ \int_0^{\gamma(r)} f_5(s) y^m(s) ds, \quad \forall r \in \mathbb{R}_+,
 \end{aligned}$$

since $V(r) \geq 0$ is nondecreasing on \mathbb{R}_+ with $v(0) = 0$. Then (2.1) implies

$$(2.4) \quad y^m(r) \leq f_1(r) + f_2(r) V(r), \quad \forall r \in \mathbb{R}_+.$$

From (2.4) and using Lemma 2.1, we get

$$\begin{aligned}
 y(r) &\leq \left(f_1(r) + f_2(r) V(r) \right)^{\frac{1}{m}} \\
 (2.5) \quad &\leq \left(\frac{1}{m} f_1(r) + \frac{1}{m} f_2(r) V(r) + \frac{m-1}{m} \right), \quad \forall r \in \mathbb{R}_+.
 \end{aligned}$$

Applying Lemma 2.1 on (2.5), we have

$$\begin{aligned}
 V(r) &\leq \int_0^{\gamma(r)} \left(n f_3(s) y^m(s) + n f_4(s) y(s) + 1 - n \right) ds \\
 (2.6) \quad &+ \int_0^{\gamma(r)} f_5(s) y^m(s) ds, \quad \forall r \in \mathbb{R}_+.
 \end{aligned}$$

Differentiating (2.6) and utilizing (2.4) and (2.5), we obtain

$$\begin{aligned}
 & V'(r) \\
 & \leq \gamma'(r) \left(\left(n f_3(\gamma(r)) \left(f_1(r) + f_2(r) V(r) \right) + n f_4(\gamma(r)) \left(\frac{1}{m} f_1(r) \right. \right. \right. \\
 & \quad \left. \left. \left. + \frac{1}{m} f_2(r) V(r) + \frac{m-1}{m} \right) + 1 - n \right) + f_5(\gamma(r)) \left(f_1(r) + f_2(r) V(r) \right) \right) \\
 & \leq \gamma'(r) \left(n f_3(\gamma(r)) + \frac{n}{m} f_4(\gamma(r)) + f_5(\gamma(r)) \right) f_2(r) V(r) \\
 & \quad + \gamma'(r) \left(n f_3(\gamma(r)) f_1(r) + n f_4(\gamma(r)) \left(\frac{1}{m} f_1(r) + \frac{m-1}{m} \right) \right. \\
 (2.7) \quad & \left. + f_5(\gamma(r)) f_1(r) + 1 - n \right), \quad \forall r \in \mathbb{R}_+.
 \end{aligned}$$

The estimation of $V(r)$ can be obtained from (2.7) as follows:

$$\begin{aligned}
 & V(r) \\
 & \leq \exp \left(\int_0^{\gamma(r)} f_2(\lambda) \left(n f_3(\lambda) + \frac{n}{m} f_4(\lambda) + f_5(\lambda) \right) d\lambda \right) \\
 & \quad \times \int_0^{\gamma(r)} \left(n f_3(s) f_1(s) + n f_4(s) \left(\frac{1}{m} f_1(s) + \frac{m-1}{m} \right) + 1 - n + f_5(s) f_1(s) \right) \\
 (2.8) \quad & \times \exp \left(- \int_0^s f_2(\lambda) \left(n f_3(\lambda) + \frac{n}{m} f_4(\lambda) + f_5(\lambda) \right) d\lambda \right) ds, \quad \forall r \in \mathbb{R}_+.
 \end{aligned}$$

Putting (2.8) in (2.4) implies (2.1). The proof of Theorem 2.2 is completed. \square

Remark 2.3. It is observed that if we relax the assumptions of inequality (2.1), then we get many existing inequalities as special cases of Theorem 2.2, such as:

1. If we replace $f_5(r) = 0$ and $n = 1$, then Theorem 2.2 reduces to [[4], Theorem 2.2].
2. When we put $f_5(r) = 0$, $n = 1$ and $\gamma(r) = r$, then Theorem 2.2 becomes [[20], Theorem 1 part (a₁)].
3. If we take $f_3(r) = f_4(r) = 0$, $f_2(r) = 1$ and $n = m = 1$ in Theorem 2.2, then we obtain the famous Gronwall-Bellman inequality [10].

Here, we give another retarded nonlinear integral inequality of this Section that will generalize the inequalities in the literature established in ([20],[4]).

Corollary 2.4. *Let $y, f_i \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$, $i = 2, 3, 4, 5$ and $\gamma \in \zeta'(\mathbb{R}_+, \mathbb{R}_+)$ with $\gamma(r) \leq r, \gamma(0) = 0, g \in \zeta(\mathbb{R}_+, \mathbb{R}_1)$ be a real valued positive continuous, nonde-*

creasing and $m \geq 1$, $n \in (0, 1]$. If

$$(2.9) \quad y^m(r) \leq g^m(r) + f_2(r) \int_0^{\gamma(r)} \left(f_3(s)y^m(s) + f_4(s)y(s) \right)^n ds \\ + f_2(r) \int_0^{\gamma(r)} f_5(s)y^m(s) ds, \quad \forall r \in \mathbb{R}_+,$$

holds, then

$$(2.10) \quad y(r) \leq g(r) \left\{ 1 + f_2(r) \left(\exp \left(\int_0^{\gamma(r)} f_2(\lambda) \left(n f_3(\lambda) + \frac{n}{m} f_4(\lambda) g^{1-m}(\lambda) + f_5(\lambda) \right) d\lambda \right) \right. \right. \\ \times \int_0^{\gamma(r)} \left(n f_3(s) + n f_4(s) g^{1-m}(r) + f_5(s) + g^{-m_1}(s)(1-n) \right) \\ \left. \left. \times \exp \left(- \int_0^s f_2(\lambda) \left(n f_3(\lambda) + \frac{n}{m} f_4(\lambda) g^{1-m}(\lambda) + f_5(\lambda) \right) d\lambda \right) ds \right) \right\}^{\frac{1}{m}},$$

$\forall r \in \mathbb{R}_+$.

Proof. As we know that $g(r)$ is a real valued positive continuous, nondecreasing function on \mathbb{R}_+ and rearranging the inequality (2.9), we obtain

$$\left(\frac{y(r)}{g(r)} \right)^m \leq 1 + f_2(r) \int_0^{\gamma(r)} \left(n f_3(s) \left(\frac{y(r)}{g(r)} \right)^m + n f_4(s) g^{1-m}(s) \left(\frac{y(r)}{g(r)} \right) \right. \\ \left. + f_5(s) \left(\frac{y(r)}{g(r)} \right)^m + g^{-m_1}(s)(1-n) \right) ds, \quad \forall r \in \mathbb{R}_+.$$

Thus as an application of Theorem 2.2, implies (2.10). Proof is completed. \square

Remark 2.5. It is observed that if we relax the assumptions of inequality (2.9), we get many existing inequalities as special cases of Corollary 2.4, such as:

1. If we replace $f_5(r) = 0$ and $n = 1$, then Corollary 2.4 reduces to [[4], Theorem 2.2].
2. When we put $f_5(r) = 0$, $n = 1$ and $\gamma(r) = r$, then Corollary 2.4 converts to [[20], Theorem 1 part (a₂)].

Here we provide another inequality of this Section which extends the existing results in ([20],[4],[10]).

Theorem 2.6. Let $y, f_i \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2, 3, 4, 5$ and $\gamma \in \zeta'(\mathbb{R}_+, \mathbb{R}_+)$ with $\gamma(r) \leq r, \gamma(0) = 0$ and $m_1 \geq m_2 \geq 1, n \in (0, 1]$. If

$$(2.11) \quad \begin{aligned} y^{m_1}(r) &\leq f_1(r) + f_2(r) \int_0^{\gamma(r)} \left(f_3(s)y^{m_1}(s) + f_4(s)y^{m_2}(s) + f(s) \right)^n ds \\ &\quad + f_2(r) \int_0^{\gamma(r)} f_5(s)y^{m_1}(s) ds, \quad \forall r \in \mathbb{R}_+, \end{aligned}$$

holds, then

$$(2.12) \quad \begin{aligned} y(r) &\leq \left\{ f_1(r) + f_2(r) \left(\exp \left(\int_0^{\gamma(r)} f_2(\lambda) \left(n f_3(\lambda) + \frac{nm_2}{m_1} f_4(\lambda) + f_5(\lambda) \right) d\lambda \right) \right. \right. \\ &\quad \times \int_0^{\gamma(r)} \left(n f_3(s) f_1(s) + n f_4(s) \left(\frac{m_2}{m_1} f_1(s) + \frac{m_1 - m_2}{m_1} \right) + n f(s) + 1 \right. \\ &\quad \left. \left. - n + f_5(s) f_1(s) \right) \times \exp \left(- \int_0^s f_2(\lambda) \left(n f_3(\lambda) + \frac{nm_2}{m_1} f_4(\lambda) \right. \right. \right. \\ &\quad \left. \left. \left. + f_5(\lambda) \right) d\lambda \right) ds \right\}^{\frac{1}{m_1}}, \quad \forall r \in \mathbb{R}_+. \end{aligned}$$

Proof. Assume that

$$(2.13) \quad \begin{aligned} V_1(r) &= \int_0^{\gamma(r)} \left(f_3(s)y^{m_1}(s) + f_4(s)y^{m_2}(s) + f(s) \right)^n ds \\ &\quad + \int_0^{\gamma(r)} f_5(s)y^{m_1}(s) ds, \quad \forall r \in \mathbb{R}_+. \end{aligned}$$

Since $V_1(r) \geq 0$ is nondecreasing on \mathbb{R}_+ , with $V_1(0) = 0$. Then (2.11) implies

$$(2.14) \quad y^{m_1}(r) \leq f_1(r) + f_2(r)V_1(r), \quad \forall r \in \mathbb{R}_+,$$

or equivalently

$$(2.15) \quad y(r) \leq \left(f_1(r) + f_2(r)V_1(r) \right)^{\frac{1}{m_1}}, \quad \forall r \in \mathbb{R}_+.$$

From (2.15) and using Lemma 2.1, we have

$$(2.16) \quad \begin{aligned} y^{m_2}(r) &\leq \left(f_1(r) + f_2(r)V_1(r) \right)^{\frac{m_2}{m_1}} \\ &\leq \left(\frac{m_2}{m_1} f_1(r) + \frac{m_2}{m_1} f_2(r)V_1(r) + \frac{m_1 - m_2}{m_1} \right), \quad \forall r \in \mathbb{R}_+. \end{aligned}$$

Applying Lemma 2.1 on (2.13), we have

$$(2.17) \quad \begin{aligned} V_1(r) &\leq \int_0^{\gamma(r)} \left(n f_3(s) y^{m_1}(s) + n f_4(s) y^{m_2}(s) + n f(s) + 1 - n \right) ds \\ &\quad + \int_0^{\gamma(r)} f_5(s) y^{m_1}(s) ds, \end{aligned}$$

for all $r \in \mathbb{R}_+$. Differentiating (2.17) and using (2.17) and (2.16), we obtain

$$(2.18) \quad \begin{aligned} &V_1'(r) \\ &\leq \gamma'(r) \left((n f_3(\gamma(r))) \left(f_1(r) + f_2(r) V_1(r) \right) + n f_4(\gamma(r)) \left(\frac{m_2}{m_1} f_1(r) + \frac{m_2}{m_1} f_2(r) \right. \right. \\ &\quad \left. \left. V_1(r) + \frac{m_1 - m_2}{m_1} \right) + n f(\gamma(r)) + 1 - n + f_5(\gamma(r)) \left(f_1(r) + f_2(r) V_1(r) \right) \right) \\ &\leq \gamma'(r) \left(n f_3(\gamma(r)) + \frac{n m_2}{m_1} f_4(\gamma(r)) + f_5(\gamma(r)) \right) f_2(r) V_1(r) \\ &\quad + \gamma'(r) \left(n f_3(\gamma(r)) f_1(r) + n f_4(\gamma(r)) \left(\frac{m_2}{m_1} f_1(r) + \frac{m_1 - m_2}{m_1} \right) \right. \\ &\quad \left. + n f(\gamma(r)) + 1 - n + f_5(\gamma(r)) f_1(r) \right), \quad \forall r \in \mathbb{R}_+. \end{aligned}$$

The estimation of $V_1(r)$ can be obtained from (2.18) as follows:

$$(2.19) \quad \begin{aligned} &V_1(r) n \\ &\leq \exp \left(\int_0^{\gamma(r)} f_2(\lambda) \left(n f_3(\lambda) + \frac{n m_2}{m_1} f_4(\lambda) + f_5(\lambda) \right) d\lambda \right) \int_0^{\gamma(r)} \left(n f_3(s) f_1(s) \right. \\ &\quad \left. + n f_4(s) \left(\frac{m_2}{m_1} f_1(s) + \frac{m_1 - m_2}{m_1} \right) + n f_6(s) + 1 - n + f_5(s) f_1(s) \right) \\ &\quad \times \exp \left(- \int_0^s f_2(\lambda) \left(n f_3(\lambda) + \frac{n m_2}{m_1} f_4(\lambda) + f_5(\lambda) \right) d\lambda \right) ds, \quad \forall r \in \mathbb{R}_+. \end{aligned}$$

Substituting (2.19) in (2.15), we obtain the result of (2.12). This concludes the proof. \square

Remark 2.7. It is observed that if we relax the assumption of inequality (2.11), we obtain many existing inequalities as special cases of Theorem 2.6, such as:

1. If we put $f_5(r) = 0$, $n = 1$ and $f(r) = 1$, then Theorem 2.6 reduces to [[4], Theorem 2.3].
2. When we replace $f_5(r) = 0$, $n = 1$, $f(r) = 1$, $m_2 = 1$ and $\gamma(r) = r$, then Theorem 2.6 converts to [6, Theorem 1 part (a₁)].

3. If we take $f_5(r) = 0$ and $n = 1$, then Theorem 2.6 becomes [4] Theorem 2.1.

Now, we present the following retarded nonlinear integral inequality of Gronwall-Bellmen type of this Section that will generalize the inequalities given in ([20],[4],[10]).

Corollary 2.8. *Let $y, f_i \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$, $i = 2, 3, 4, 5$ and $\gamma \in \zeta'(\mathbb{R}_+, \mathbb{R}_+)$ with $\gamma(r) \leq r, \gamma(0) = 0$, $g \in \zeta(\mathbb{R}_+, \mathbb{R}_1)$ be a real valued positive continuous, nondecreasing and $m_1 \geq m_2 \geq 1$, $n \in (0, 1]$. If*

$$(2.20) \quad \begin{aligned} y^{m_1}(r) \leq & g^{m_1}(r) + f_2(r) \int_0^{\gamma(r)} \left(f_3(s)y^{m_1}(s) + f_4(s)y^{m_2}(s) + f(s) \right)^n ds \\ & + f_2(r) \int_0^{\gamma(r)} f_5(s)y^{m_1}(s)ds, \quad \forall r \in \mathbb{R}_+, \end{aligned}$$

holds, then

$$(2.21) \quad \begin{aligned} y(r) \leq & g(r) \left\{ 1 + f_2(r) \left(\exp \left(\int_0^{\gamma(r)} f_2(\lambda) \left(n f_3(\lambda) + \frac{nm_2}{m_1} f_4(\lambda) g^{m_2-m_1}(\lambda) \right. \right. \right. \right. \\ & \left. \left. \left. + f_5(\lambda) \right) d\lambda \right) \times \int_0^{\gamma(r)} \left(n f_3(s) + n f_4(s) g^{m_2-m_1}(s) + f_5(s) \right. \right. \\ & \left. \left. + n g^{-m_1}(s) f(s) + g^{-m_1}(s)(1-n) \right) \times \exp \left(- \int_0^s f_2(\lambda) \left(n f_3(\lambda) \right. \right. \right. \\ & \left. \left. \left. + \frac{nm_2}{m_1} f_4(\lambda) g^{m_2-m_1}(\lambda) + f_5(\lambda) \right) d\lambda \right) ds \right\}^{\frac{1}{m_1}}, \end{aligned}$$

$$\forall r \in \mathbb{R}_+.$$

Proof. As we know that $g(r)$ is a real valued positive continuous, nondecreasing function on \mathbb{R}_+ and rearranging the inequality 2.8, we obtain

$$\begin{aligned} & \left(\frac{y(r)}{g(r)} \right)^{m_1} \\ & \leq 1 + f_2(r) \int_0^{\gamma(r)} \left(n f_3(s) \left(\frac{y(r)}{g(r)} \right)^{m_1} + n f_4(s) g^{m_2-m_1}(s) \left(\frac{y(r)}{g(r)} \right)^{m_2} \right. \\ & \quad \left. + f_5(s) \left(\frac{y(r)}{g(r)} \right)^{m_1} + n g^{-m_1}(r) f(s) + g^{-m_1}(r)(1-n) \right) ds, \quad \forall r \in \mathbb{R}_+. \end{aligned}$$

Thus as an application of inequality of Theorem 2.6 implies (2.20). □

Remark 2.9. It is observed that if we relax the assumption of inequality (2.8), we obtain many existing inequalities as special cases, such as:

1. If we put $f_5(r) = 0$, $n = 1$ and $f(r) = 1$, then Corollary (2.2) reduces to Theorem (2.4) becomes [4].
2. When we replace $f_5(r) = 0$, $m_2 = 1$, $n = 1$, $f(r) = 1$ and $\gamma(r) = r$, then Corollary 2.2 converts to [[20], Theorem 1 part (a₂)].
3. If we take $f_5(r) = 0$ and $n = 1$, then Corollary (2.2) becomes [10] Theorem (2.2).

Here we state and prove another retarded nonlinear integral inequality of this Section which extends the existing inequalities given in ([20],[4],[10]).

Theorem 2.10. *Let $y, f_i \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2, 3, 4, 5$ and $\gamma \in \zeta'(\mathbb{R}_+, \mathbb{R}_+)$ with $\gamma(r) \leq r$, $\gamma(0) = 0$ and $k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ with $\frac{\partial}{\partial r}k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ and $m_1 \geq m_2 \geq 1$. If*

$$(2.22) \quad \begin{aligned} y^{m_1}(r) &\leq f_1(r) + f_2(r) \int_0^{\gamma(r)} k(r, s) \left(f_3(s)y^{m_1}(s) + f_4(s)y^{m_2}(s) \right) ds \\ &+ f_2(r) \int_0^{\gamma(r)} f_5(s)y^{m_1}(s) ds, \quad \forall r \in \mathbb{R}_+, \end{aligned}$$

holds, then

$$(2.23) \quad \begin{aligned} y(r) &\leq \left\{ f_1(r) + f_2(r) \left(\exp \left(\int_0^{\gamma(r)} A_1(r) dr \right) \int_0^{\gamma(r)} \right. \right. \\ &\left. \left. B_1(s) \exp \left(- \int_0^s A_1(r) dr \right) ds \right) \right\}^{\frac{1}{m_1}}, \end{aligned}$$

$\forall r \in \mathbb{R}_+$, where

$$(2.24) \quad \begin{aligned} A_1(r) &= \left(\left(f_5(\gamma(r)) + k(r, \gamma(r)) \left(f_3(\gamma(r)) + \frac{m_2}{m_1} f_4(\gamma(r)) \right) \right) f_2(r) \gamma'(r) \right. \\ &\left. + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(f_3(s) + \frac{m_2}{m_1} f_4(s) \right) f_2(s) ds \right), \quad \forall r \in \mathbb{R}_+. \end{aligned}$$

$$(2.25) \quad \begin{aligned} B_1(r) &= \gamma'(r) \left(f_5(\gamma(r)) f_1(r) + k(r, \gamma(r)) \left(f_3(\gamma(r)) f_1(r) + f_4(\gamma(r)) \left(\frac{m_2}{m_1} f_1(r) \right. \right. \right. \\ &\left. \left. \left. + \frac{m_1 - m_2}{m_1} \right) \right) \right) + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(f_3(s) f_1(s) + f_4(s) \left(\frac{m_2}{m_1} f_1(s) \right. \right. \\ &\left. \left. + \frac{m_1 - m_2}{m_1} \right) \right) ds, \quad \forall r \in \mathbb{R}_+. \end{aligned}$$

Proof. Assume that

$$(2.26) \quad \begin{aligned} V_2(r) &= \int_0^{\gamma(r)} k(r,s) \left(f_3(s)y^{m_1}(s) + f_4(s)y^{m_2}(s) \right) ds \\ &+ \int_0^{\gamma(r)} f_5(s)y^{m_1}(s) ds, \quad \forall r \in \mathbb{R}_+, \end{aligned}$$

since $V_2(r) \geq 0$ is nondecreasing on \mathbb{R}_+ with $V_2(0) = 0$. Then (2.22) implies

$$(2.27) \quad y^{m_1}(r) \leq f_1(r) + f_2(r)V_2(r), \quad \forall r \in \mathbb{R}_+.$$

or equivalently

$$(2.28) \quad y(r) \leq \left(f_1(r) + f_2(r)V_2(r) \right)^{\frac{1}{m_1}}, \quad \forall r \in \mathbb{R}_+.$$

From (2.28) and using Lemma 2.1, we have

$$(2.29) \quad \begin{aligned} y^{m_2}(r) &\leq \left(f_1(r) + f_2(r)V_2(r) \right)^{\frac{m_2}{m_1}} \\ &\leq \left(\frac{m_2}{m_1} f_1(r) + \frac{m_2}{m_1} f_2(r)V_2(r) + \frac{m_1 - m_2}{m_1} \right), \quad \forall r \in \mathbb{R}_+. \end{aligned}$$

Differentiating (2.26), using (2.28) and (2.29), we obtain

$$\begin{aligned}
 & V_2'(r) \\
 & \leq \gamma'(r) \left\{ f_5(\gamma(r)) \left(f_1(r) + f_2(r)V_2(r) \right) + k(r, \gamma(r)) \left(f_3(\gamma(r)) \left(f_1(r) \right. \right. \right. \\
 & \quad \left. \left. \left. + f_2(r)V_2(r) \right) + f_4(\gamma(r)) \left(\frac{m_2}{m_1} f_1(r) + \frac{m_2}{m_1} f_2(r)V_2(r) + \frac{m_1 - m_2}{m_1} \right) \right) \right\} \\
 & \quad + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) f_3(s) \times \left(f_1(s) + f_2(s)V_2(s) \right) + f_4(s) \left(\frac{m_2}{m_1} f_1(s) \right. \\
 & \quad \left. + \frac{m_2}{m_1} f_2(s)V_2(s) + \frac{m_1 - m_2}{m_1} \right) \\
 & \leq \left(\left(f_5(\gamma(r)) + k(r, \gamma(r)) \left(f_3(\gamma(r)) + \frac{m_2}{m_1} f_4(\gamma(r)) \right) \right) f_2(r) \gamma'(r) \right. \\
 & \quad \left. + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(f_3(s) + \frac{m_2}{m_1} f_4(s) \right) f_2(s) ds \right) V_2(r) + \gamma'(r) \left(f_5(\gamma(r)) f_1(r) \right. \\
 & \quad \left. + k(r, \gamma(r)) \left(f_3(\gamma(r)) f_1(r) + f_4(\gamma(r)) \left(\frac{m_2}{m_1} f_1(r) + \frac{m_1 - m_2}{m_1} \right) \right) \right) \\
 & \quad + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(f_3(s) f_1(s) + f_4(s) \left(\frac{m_2}{m_1} f_1(s) + \frac{m_1 - m_2}{m_1} \right) \right) ds \\
 & \tag{2.30} \qquad \qquad \qquad = A_1(r)V_2(r) + B_1(r), \quad \forall t \in \mathbb{R}_+,
 \end{aligned}$$

where $A_1(r)$ and $B_1(r)$ are defined in (2.24) and (2.25), respectively. The inequality (2.30) gives us the following estimation for $V_2(r)$

$$\begin{aligned}
 V_2(r) & \leq \exp \left(\int_0^{\gamma(r)} A_1(r) dr \right) \int_0^{\gamma(r)} B_1(s) \\
 & \tag{2.31} \qquad \qquad \qquad \times \exp \left(- \int_0^s (A_1(r) dr) \right) ds, \quad \forall r \in \mathbb{R}_+.
 \end{aligned}$$

Putting (2.31) in (2.28) implies (2.23). Proof is completed. \square

Remark 2.11. It is observed that if we relax the assumptions of inequality (2.22), we obtain many existing inequalities as special cases of Theorem 2.10, such as:

1. If we replace $f_5(r) = 0$ and $m_2 = 1$, then Theorem (2.3) reduces to Theorem (2.5) [4].
2. When we put $f_5(r) = 0$, $m_2 = 1$ and $\gamma(r) = r$, then Theorem (2.3) converts to [6, Theorem 1 part (a₃)].
3. If we take $f_5(r) = 0$ and $k(r, s) = 1$, then Theorem (2.3) becomes Theorem (2.3) [4].
4. When $f_5(r) = 0$, $m_2 = 1$ and $k(r, s) = 1$, then Theorem (2.3) reduces to [6, Theorem 1 part (a₁)].
5. If we replace $f_5(r) = 0$, then Theorem (2.3) converts to Theorem (2.3) [10].

Now, we give another inequality of this Section that will generalize the inequalities in the literature given in ([20],[4],[10]).

Corollary 2.12. *Let $y, f_i \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$, $i = 2, 3, 4, 5$ and $\gamma \in \zeta'(\mathbb{R}_+, \mathbb{R}_+)$ with $\gamma(r) \leq r, \gamma(0) = 0$, $g \in \zeta(\mathbb{R}_+, \mathbb{R}_1)$ be a real valued positive continuous, nondecreasing function and $k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ with $\frac{\partial}{\partial r}k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ and $m_1 \geq m_2 \geq 1$. If*

$$\begin{aligned}
 (2.32) \quad y^{m_1}(r) &\leq g^{m_1}(r) + f_2(r) \int_0^{\gamma(r)} k(r, s) \left(f_3(s)y^{m_1}(s) + f_4(s)y^{m_2}(s) \right) ds \\
 &+ f_2(r) \int_0^{\gamma(r)} f_5(s)y^{m_1}(s) ds, \quad \forall r \in \mathbb{R}_+,
 \end{aligned}$$

holds, then

$$\begin{aligned}
 (2.33) \quad y(r) &\leq g(r) \left\{ 1 + f_2(r) \exp \left(\int_0^{\gamma(r)} A_2(r) dr \right) \int_0^{\gamma(r)} B_2(s) \right. \\
 &\left. \times \exp \left(- \int_0^s A_2(r) dr \right) ds \right\}^{\frac{1}{m_1}},
 \end{aligned}$$

$\forall r \in \mathbb{R}_+$, where

$$\begin{aligned}
 A_2(r) &= \left(f_5(\gamma(r)) + k(r, \gamma(r)) \left(f_3(\gamma(r)) + \frac{m_2}{m_1} f_4(\gamma(r)) g^{m_2-m_1}(\gamma(r)) \right) \right) f_2(r) \\
 &+ \gamma'(r) + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(f_3(s) + \frac{m_2}{m_1} f_4(s) g^{m_2-m_1}(s) \right) f_2(s) ds, \quad \forall r \in \mathbb{R}_+.
 \end{aligned}$$

$$\begin{aligned}
B_2(r) &= \gamma'(r) \left(f_5(\gamma(r)) + k(r, \gamma(r)) \left(f_3(\gamma(r)) + f_4(\gamma(r)) g^{m_2-m_1}(\gamma(r)) \right) \right) \\
&\quad + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(f_3(s) + f_4(s) g^{m_2-m_1}(s) \right), \quad \forall r \in \mathbb{R}_+.
\end{aligned}$$

Proof. As we know that $g(r)$ is a real valued positive continuous and nondecreasing function on \mathbb{R}_+ and rearranging the inequality (2.32), we obtain

$$\begin{aligned}
&\left(\frac{y(r)}{g(r)} \right)^{m_1} \\
&\leq 1 + f_2(r) \int_0^{\gamma(r)} k(r, s) \left(f_3(s) \left(\frac{y(r)}{g(r)} \right)^{m_1} + f_4(s) g^{m_2-m_1}(s) \left(\frac{y(r)}{g(r)} \right)^{m_2} \right. \\
&\quad \left. + f_5(s) \left(\frac{y(r)}{g(r)} \right)^{m_1} \right) ds, \quad \forall r \in \mathbb{R}_+.
\end{aligned}$$

Thus as an application of inequality of Theorem 2.10 implies (2.33). Proof is completed. \square

Remark 2.13. It is observed that if we relax the assumptions of inequality (2.32), we get many existing inequalities as special cases of Corollary 2.12, such as:

1. When we replace $f_5(r) = 0$ and $m_2 = 1$, then Corollary 2.12 reduces to Theorem 2.6 [4].
2. If we put $f_5(r) = 0$ and $k(r, s) = 1$, then Corollary 2.12 converts to Theorem 2.4 [4].
3. When we take $f_5(r) = 0$, $m_2 = 1$, $\gamma(r) = t$ and $k(r, s) = 1$, then Corollary 2.12 becomes [6, Theorem 1 part (a₂)].
4. If $f_5(r) = 0$, then Corollary 2.12 becomes Theorem 2.4 [10].

Now, we give the following result that will generalize the inequalities presented in ([20],[4]).

Theorem 2.14. Let $y, f_i \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2, 3, 4, 5$ and $\gamma \in \zeta'(\mathbb{R}_+, \mathbb{R}_+)$ with $\gamma(r) \leq r$, $\gamma(0) = 0$ and $k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ with $\frac{\partial}{\partial r} k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ and $m \geq 1$, $n \in (0, 1]$. If

$$\begin{aligned}
(2.34) \quad y^m(r) &\leq f_1(r) + f_2(r) \int_0^{\gamma(r)} k(r, s) \left(f_3(s) y^m(s) + f_4(s) y(s) \right)^n ds \\
&\quad + f_2(r) \int_0^{\gamma(r)} f_5(s) y^m(s) ds,
\end{aligned}$$

$\forall r \in \mathbb{R}_+$, holds, then

$$(2.35) \quad y(r) \leq \left\{ f_1(r) + f_2(r) \left(\exp \left(\int_0^{\gamma(r)} A_3(r) dr \right) \int_0^{\gamma(r)} B_3(s) \right. \right. \\ \left. \left. \times \exp \left(- \int_0^s A_3(r) dr \right) ds \right) \right\}^{\frac{1}{m}},$$

for all $r \in \mathbb{R}_+$, where

$$(2.36) \quad A_3(r) = \left(f_5(\gamma(r)) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) + \frac{n}{m} f_4(\gamma(r)) \right) \right) f_2(r) \gamma'(r) \\ + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n f_3(s) + \frac{n}{m} f_4(s) \right) f_2(s) ds, \quad \forall r \in \mathbb{R}_+,$$

$$(2.37) \quad B_3(r) = \gamma'(r) \left\{ f_5(\gamma(r)) f_1(r) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) f_1(r) + n f_4(\gamma(r)) \left(\frac{1}{m} f_1(r) \right. \right. \right. \\ \left. \left. \left. + \frac{m-1}{m} \right) + 1 - n \right) \right\} + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n f_3(s) f_1(s) + n f_4(s) \right. \\ \left. \left(\frac{1}{m} f_1(s) + \frac{m-1}{m} \right) + 1 - n \right) ds, \quad \forall r \in \mathbb{R}_+.$$

Proof. Assume that

$$(2.38) \quad V_3(r) = \int_0^{\gamma(r)} k(r, s) \left(f_3(s) y^m(s) + f_4(s) y(s) \right)^n ds \\ + \int_0^{\gamma(r)} f_5(s) y^m(s) ds, \quad \forall r \in \mathbb{R}_+,$$

since $V_3(r) \geq 0$ is nondecreasing on \mathbb{R}_+ with $V_3(0) = 0$. Then (2.34) implies

$$(2.39) \quad y^m(r) \leq f_1(r) + f_2(r) V_3(r), \quad \forall r \in \mathbb{R}_+.$$

From (2.39) and using Lemma 2.1, we get

$$(2.40) \quad y(r) \leq \left(f_1(r) + f_2(r) V_3(r) \right)^{\frac{1}{m}} \\ \leq \left(\frac{1}{m} f_1(r) + \frac{1}{m} f_2(r) V_3(r) + \frac{m-1}{m} \right), \quad \forall r \in \mathbb{R}_+.$$

Applying Lemma 2.1 on (2.38), we have

$$(2.41) \quad \begin{aligned} V_3(r) &= \int_0^{\gamma(r)} k(r, s) \left(n f_3(s) y^m(s) + n f_4(s) y(s) + 1 - n \right) ds \\ &+ \int_0^{\gamma(r)} f_5(s) y^m(s) ds, \forall r \in \mathbb{R}_+. \end{aligned}$$

Differentiating (2.41), and using (2.39) and (2.40), we obtain

$$\begin{aligned} &V_3'(r) \\ &\leq \gamma'(r) \left(f_5(\gamma(r)) \left(f_1(r) + f_2(r) V_3(r) \right) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) \right. \right. \\ &\quad \left. \left. \left(f_1(r) + f_2(r) V_3(r) \right) + n f_4(\gamma(r)) \left(\frac{1}{m} f_1(r) + \frac{1}{m} f_2(r) V_3(r) \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{m-1}{m} \right) + 1 - n \right) \right) + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \times \left\{ n f_3(s) \left(f_1(s) \right. \right. \\ &\quad \left. \left. + f_2(s) V_3(s) \right) + n f_4(s) \left(\frac{1}{m} f_1(r) + \frac{1}{m} f_2(s) V_3(s) + \frac{m-1}{m} \right) + 1 - n \right\} \\ &\leq \left(\left(f_5(\gamma(r)) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) + \frac{n}{m} f_4(\gamma(r)) \right) \right) f_2(r) \gamma'(r) \right. \\ &\quad \left. + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n f_3(s) + \frac{n}{m} f_4(s) \right) f_2(s) \right) V_3(r) + \gamma'(r) \left\{ f_5(\gamma(r)) f_1(r) \right. \\ &\quad \left. + k(r, \gamma(r)) \left(n f_3(\gamma(r)) f_1(r) + n f_4(\gamma(r)) \left(\frac{1}{m} f_1(r) + \frac{m-1}{m} \right) + 1 - n \right) \right\} \\ &\quad \left. + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n f_3(s) f_1(s) + n f_4(s) \left(\frac{1}{m} f_1(s) + \frac{m-1}{m} \right) + 1 - n \right), \right. \end{aligned}$$

or equivalently,

$$(2.42) \quad V_3'(r) \leq A_3(r) V_3(r) + B_3(r), \quad \forall r \in \mathbb{R}_+,$$

where $A_3(r)$ and $B_3(r)$ are defined in (2.36) and (2.37), respectively. The inequality (2.42) gives us the following estimation for $V_3(r)$

$$(2.43) \quad \begin{aligned} V_3(r) &\leq \exp \left(\int_0^{\gamma(r)} A_3(r) dr \right) \int_0^{\gamma(r)} B_3(s) \\ &\times \exp \left(- \int_0^s A_3(r) dr \right) ds, \quad \forall r \in \mathbb{R}_+. \end{aligned}$$

Substituting (2.43) in (2.39), after rearranging we get (2.35). Proof is done. \square

Remark 2.15. It is observed that if we relax the assumption of inequality (2.34), we obtain many existing inequalities as special cases of Theorem 2.14, such as:

1. If we replace $f_5(r) = 0$ and $n = 1$, then Theorem 2.14 reduces to Theorem 2.5 [4].
2. When we put $f_5(r) = 0$, $n = 1$ and $\gamma(r) = r$, then Theorem 2.14 converts to [[20], Theorem 1 part (a₃)].
3. If we take $f_5(r) = 0$, $n = 1$ and $k(r, s) = 1$, then Theorem 2.14 becomes Theorem 2.3 [4].
4. When $f_5(r) = 0$, $n = 1$ and $\gamma(r) = r$, $k(r, s) = 1$, then Theorem 2.14 reduces to [[20], Theorem 1 part (a₁)].

Here, we give the following corollary that will generalize the inequalities in ([20],[4]).

Corollary 2.16. *Let $y, f_i \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$, $i = 2, 3, 4, 5$ and $\gamma \in \zeta'(\mathbb{R}_+, \mathbb{R}_+)$ with $\gamma(r) \leq r$, $\gamma(0) = 0$, $g \in \zeta(\mathbb{R}_+, \mathbb{R}_1)$ be a real valued positive continuous, nondecreasing function and $k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ with $\frac{\partial}{\partial r}k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ and $m \geq 1$, $n \in (0, 1]$. If*

$$(2.44) \quad y^m(r) \leq g^m(r) + f_2(r) \left(\int_0^{\gamma(r)} k(r, s) \left(f_3(s)y^m(s) + f_4(s)y(s) \right)^n ds + \int_0^{\gamma(r)} f_5(s)y^m(s)ds \right),$$

$\forall r \in \mathbb{R}_+$ holds, then

$$(2.45) \quad y(r) \leq g(r) \left\{ 1 + f_2(r) \left(\exp \left(\int_0^{\gamma(r)} A_4(r)dr \right) \int_0^{\gamma(r)} B_4(s) \times \exp \left(- \int_0^s A_4(r)dr \right) ds \right) \right\}^{\frac{1}{m}},$$

$\forall r \in \mathbb{R}_+$, where

$$A_4(r) = \left(f_5(\gamma(r)) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) + \frac{n}{m} f_4(\gamma(r)) g^{1-m}(\gamma(r)) \right) \right) f_2(r) \\ \gamma'(r) + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n f_3(s) + \frac{n}{m} f_4(s) g^{1-m}(s) \right) f_2(s) ds, \quad \forall r \in \mathbb{R}_+,$$

$$\begin{aligned}
 B_4(r) = & \gamma'(r) \left\{ f_5(\gamma(r)) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) + n f_4(\gamma(r)) g^{1-m}(\gamma(r)) \right. \right. \\
 & \left. \left. + g^{-m_1}(s)(1-n) \right) \right\} + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n f_3(s) + n f_4(s) g^{1-m}(s) \right. \\
 & \left. + g^{-m_1}(s)(1-n) \right) ds, \quad \forall r \in \mathbb{R}_+.
 \end{aligned}$$

Proof. As we know that $g(r)$ is a real valued positive continuous, nondecreasing function on \mathbb{R}_+ and rearranging the inequality (2.44), we obtain

$$\begin{aligned}
 \left(\frac{y(r)}{g(r)} \right)^m \leq & 1 + f_2(r) \int_0^{\gamma(r)} k(r, s) \left(n f_3(s) \left(\frac{y(r)}{g(r)} \right)^m + n f_4(s) g^{1-m}(s) \left(\frac{y(r)}{g(r)} \right) \right. \\
 & \left. + f_5(s) \left(\frac{y(r)}{g(r)} \right)^m + g^{-m_1}(s)(1-n) \right) ds, \quad \forall r \in \mathbb{R}_+.
 \end{aligned}$$

Thus as an application of inequality of Theorem 2.14 implies (2.45). Proof is completed. □

Remark 2.17. It is observed that if we relax the assumptions of inequality (2.44), we obtain many existing inequalities as special cases of Corollary 2.16, such as:

1. If we replace $f_5(r) = 0$ and $n = 1$, then Corollary 2.16 reduces to Theorem 2.6 [4].
2. When we put $f_5(r) = 0$, $n = 1$ and $k(r, s) = 1$, then Corollary 2.16 converts to Theorem 2.4 [4]
3. If we take $f_5(r) = 0$, $n = 1$ and $\gamma(r) = r$, $k(r, s) = 1$, then Corollary 2.16 becomes [[20], Theorem 1 part (a₂)].

Here, we give the following theorem of this Section that will generalize the inequalities in the literature given in ([20],[4],[10]).

Theorem 2.18. *Let $y, f_i \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2, 3, 4, 5$ and $\gamma \in \zeta'(\mathbb{R}_+, \mathbb{R}_+)$ with $\gamma(r) \leq r$, $\gamma(0) = 0$ and $k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ with $\frac{\partial}{\partial r} k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ and $m_1 \geq m_2 \geq 1$, $n \in (0, 1]$. If*

$$\begin{aligned}
 y^{m_1}(r) \leq & f_1(r) + f_2(r) \int_0^{\gamma(r)} k(r, s) \left(f_3(s) y^{m_1}(s) + f_4(s) y^{m_2}(s) \right)^n ds \\
 (2.46) \quad & + f_2(r) \int_0^{\gamma(r)} f_5(s) y^{m_1}(s) ds, \quad \forall r \in \mathbb{R}_+,
 \end{aligned}$$

holds, then

$$(2.47) \quad y(r) \leq \left\{ f_1(r) + f_2(r) \left(\exp \left(\int_0^{\gamma(r)} A_5(r) dr \right) \int_0^{\gamma(r)} B_5(s) \exp \left(- \int_0^s A_5(r) dr \right) ds \right)^{\frac{1}{m_1}} \right\},$$

$\forall r \in \mathbb{R}_+$, where

$$(2.48) \quad A_5(r) = \left\{ f_5(\gamma(r)) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) + \frac{nm_2}{m_1} f_4(\gamma(r)) \right) \right\} f_2(r) \gamma'(r) + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n f_3(s) + \frac{nm_2}{m_1} f_4(s) \right) f_2(s) ds, \quad \forall r \in \mathbb{R}_+,$$

(B40)

$$(2.50) \quad = \gamma'(r) \left(f_5(\gamma(r)) f_1(r) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) f_1(r) + n f_4(\gamma(r)) \left(\frac{m_2}{m_1} f_1(r) + \frac{m_1 - m_2}{m_1} \right) + 1 - n \right) \right) + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n f_3(s) f_1(s) + n f_4(s) \left(\frac{m_2}{m_1} f_1(s) + \frac{m_1 - m_2}{m_1} \right) + 1 - n \right) ds, \quad \forall r \in \mathbb{R}_+.$$

Proof. Assume that

$$(2.51) \quad V_4(r) = \int_0^{\gamma(r)} k(r, s) \left(f_3(s) y^{m_1}(s) + f_4(s) u^{m_2}(s) \right)^n ds + \int_0^{\gamma(r)} f_5(s) y^{m_1}(s) ds, \quad \forall r \in \mathbb{R}_+,$$

since $V_4(r) \geq 0$ is nondecreasing on \mathbb{R}_+ . Then (2.46) can be written as

$$(2.52) \quad y^{m_1}(r) \leq f_1(r) + f_2(r) V_4(r), \quad \forall r \in \mathbb{R}_+.$$

or equivalently

$$(2.53) \quad y(r) \leq \left(f_1(r) + f_2(r) V_4(r) \right)^{\frac{1}{m_1}}, \quad \forall r \in \mathbb{R}_+.$$

From (2.53) and using Lemma 2.1, we obtain

$$(2.54) \quad y^{m_2}(r) \leq \left(f_1(r) + f_2(r) V_4(r) \right)^{\frac{m_2}{m_1}} \leq \left(\frac{m_2}{m_1} f_1(r) + \frac{m_2}{m_1} f_2(r) V_4(r) + \frac{m_1 - m_2}{m_1} \right), \quad \forall r \in \mathbb{R}_+.$$

Applying Lemma 2.1 on (2.51), we have

$$(2.55) \quad V_4(r) = \int_0^{\gamma(r)} k(r, s) \left(n f_3(s) y^{m_1}(s) + n f_4(s) y^{m_2}(s) + 1 - n \right) ds + \int_0^{\gamma(r)} f_5(s) y^{m_1}(s) ds,$$

for all $r \in \mathbb{R}_+$. Differentiating (2.55), and using (2.53) and (2.54), we obtain

$$\begin{aligned} V_4'(r) \leq & \gamma'(r) \left\{ f_5(\gamma(r)) \left(f_1(r) + f_2(r) V_4(r) \right) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) \left(f_1(r) \right. \right. \right. \\ & \left. \left. \left. + f_2(r) V_4(r) \right) + n f_4(\gamma(r)) \left(\frac{m_2}{m_1} f_1(r) + \frac{m_2}{m_1} f_2(r) V_4(r) + \frac{m_1 - m_2}{m_1} \right) \right. \right. \\ & \left. \left. + \frac{m_2}{m_1} f_2(s) V_4(s) + \frac{m_1 - m_2}{m_1} \right) + 1 - n \right\} ds, \end{aligned}$$

or equivalently,

$$\begin{aligned} (2.56) \quad V_4'(r) \leq & \left(\left\{ f_5(\gamma(r)) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) + \frac{nm_2}{m_1} f_4(\gamma(r)) \right) \right\} f_2(r) \gamma'(r) \right. \\ & \left. + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \times \left(n f_3(s) + \frac{nm_2}{m_1} f_4(s) \right) f_2(s) ds \right) V_4(r) \\ & + \gamma'(r) \left(f_5(\gamma(r)) f_1(r) + k(r, \gamma(r)) \times \left(n f_3(\gamma(r)) f_1(r) \right. \right. \\ & \left. \left. + n f_4(\gamma(r)) \left(\frac{m_2}{m_1} f_1(r) + \frac{m_1 - m_2}{m_1} \right) + 1 - n \right) \right) + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \\ & \times \left(n f_3(s) f_1(s) + n f_4(s) \left(\frac{m_2}{m_1} f_1(s) + \frac{m_1 - m_2}{m_1} \right) + 1 - n \right) ds \\ & = A_5(r) V_4(r) + B_5(r), \quad \forall r \in \mathbb{R}_+, \end{aligned}$$

where $A_5(r)$ and $B_5(r)$ are defined in (2.48) and (2.49), respectively. The inequality (2.56) gives us the following estimation for $V_4(r)$

$$(2.57) \quad V_4(r) \leq \exp \left(\int_0^{\gamma(r)} A_5(r) dr \right) \int_0^{\gamma(r)} B_5(s) \times \exp \left(- \int_0^s A_5(r) dr \right) ds, \quad \forall r \in \mathbb{R}_+.$$

Substituting (2.57) in (2.53), which gives (2.47). The proof of Theorem 2.18 is done. \square

Remark 2.19. It is observed that if we relax the assumptions of inequality (2.46), we obtain many existing inequalities as special cases of Theorem 2.18, such as:

1. If we put $f_5(r) = 0$, $n = 1$ and $m_2 = 1$, then Theorem 2.18 reduces to Theorem 2.5 [4].
2. When we replace $f_5(r) = 0$, $n = 1$, $m_2 = 1$ and $\gamma(r) = r$, then Theorem 2.18 converts to [[20], Theorem 1 part (a₃)].
3. If we take $f_5(r) = 0$, $n = 1$, $m_2 = 1$ and $k(r, s) = 1$, then Theorem 2.18 becomes Theorem 2.3 [4].
4. When we put $f_5(r) = 0$, $n = 1$, $m_2 = 1$, $\gamma(r) = r$ and $k(r, s) = 1$, then Theorem 2.18 reduces to [[20], Theorem 1 part (a₁)].
5. If we replace $f_5(r) = 0$ and $n = 1$, then Theorem 2.18 becomes Theorem 2.3 [10].

Now, we present last retarded nonlinear integral inequality of Gronwall-Bellmen type of this Section that will generalize the inequalities in the literature given in ([20],[4],[10]).

Corollary 2.20. *Let $y, f_i \in \zeta(\mathbb{R}_+, \mathbb{R}_+)$, $i = 2, 3, 4, 5$ and $\gamma \in \zeta'(\mathbb{R}_+, \mathbb{R}_+)$ with $\gamma(r) \leq r$, $\gamma(0) = 0$, $g \in \zeta(\mathbb{R}_+, \mathbb{R}_1)$ be a real valued positive continuous and nondecreasing function, $k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ with $\frac{\partial}{\partial r} k(r, s) \in \zeta(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_1)$ and $m_1 \geq m_2 \geq 1$, $n \in (0, 1]$. If*

$$\begin{aligned}
 (2.58) \quad y^{m_1}(r) &\leq g^{m_1}(r) + f_2(r) \int_0^{\gamma(r)} k(r, s) \left(f_3(s)y^{m_1}(s) + f_4(s)y^{m_2}(s) \right)^n ds \\
 &\quad + f_2(r) \int_0^{\gamma(r)} f_5(s)y^{m_1}(s) ds, \quad \forall r \in \mathbb{R}_+,
 \end{aligned}$$

holds, then

$$\begin{aligned}
 (2.59) \quad y(r) &\leq g(r) \left\{ 1 + f_2(r) \left(\exp \left(\int_0^{\gamma(r)} A_6(r) dr \right) \int_0^{\gamma(r)} B_6(s) \right. \right. \\
 &\quad \left. \left. \exp \left(- \int_0^s A_6(r) dr \right) ds \right) \right\}^{\frac{1}{m_1}},
 \end{aligned}$$

$\forall r \in \mathbb{R}_+$, where

$$A_6(r) = \left\{ f_5(\gamma(r)) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) + \frac{nm_2}{m_1} f_4(\gamma(r)) g^{m_2-m_1}(\gamma(r)) \right) \right\} \\ f_2(r) \gamma'(r) + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n f_3(s) \right. \\ \left. + \frac{nm_2}{m_1} f_4(s) g^{m_2-m_1}(s) \right) f_2(s) ds, \quad \forall r \in \mathbb{R}_+,$$

$$B_6(r) = \gamma'(r) \left(f_5(\gamma(r)) + k(r, \gamma(r)) \left(n f_3(\gamma(r)) + n f_4(\gamma(r)) g^{m_2-m_1}(\gamma(r)) \right. \right. \\ \left. \left. + g^{-m_1}(\gamma(r))(1-n) \right) \right) + \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n f_3(s) + n f_4(s) \right. \\ \left. \times g^{m_2-m_1}(s) + g^{-m_1}(s)(1-n) \right) ds, \quad \forall r \in \mathbb{R}_+.$$

Proof. As we know that $g(r)$ is a real valued positive continuous, nondecreasing function on \mathbb{R}_+ and rearranging the inequality (2.58), we obtain

$$\left(\frac{y(r)}{g(r)} \right)^{m_1} \\ \leq 1 + f_2(r) \int_0^{\gamma(r)} k(r, s) \left(n f_3(s) \left(\frac{y(r)}{g(r)} \right)^{m_1} + n f_4(s) g^{m_2-m_1}(s) \left(\frac{y(r)}{g(r)} \right)^{m_2} \right. \\ \left. + f_5(s) \left(\frac{y(r)}{g(r)} \right)^{m_1} + g^{-m_1}(r)(1-n) \right) ds, \quad \forall r \in \mathbb{R}_+.$$

Thus as an application of inequality of Theorem 2.18 implies (2.59). Proof is completed. \square

Remark 2.21. It is observed that if we relax assumptions of inequality (2.59), we get many existing results as special cases of Corollary 2.20, such as:

1. If we put $f_5(r) = 0$, $n = 1$ and $m_2 = 1$, then Corollary 2.20 reduces to Theorem 2.6 [4].
2. When we replace $f_5(r) = 0$, $n = 1$, $m_2 = 1$ and $k(r, s) = 1$, then Corollary 2.20 converts to Theorem 2.4 [4].
3. If we take $f_5(r) = 0$, $n = 1$, $m_2 = 1$ and $k(r, s) = 1$, then Corollary 2.20 becomes [[20], Theorem 1 part (a₂)].
4. When $f_5(r) = 0$, and $n = 1$, then Corollary 2.20 reduces to Theorem 2.4 [10].

Thus, as an application of Theorem 2.2 with $f_2(r) = 1$, implies

$$\begin{aligned}
 & |y(r)| \\
 & \leq \left\{ (r^3 + 1) + \exp \left(\int_0^{\gamma(r)} \left(n \frac{f_3(\gamma^{-1}(t))}{\gamma'(\gamma^{-1}(t))} + \frac{n}{m} \frac{f_4(\gamma^{-1}(t))}{\gamma'(\gamma^{-1}(t))} + \frac{f_5(\gamma^{-1}(t))}{\gamma'(\gamma^{-1}(t))} \right) dr \right) \right. \\
 & \quad \times \int_0^{\gamma(r)} \left(\frac{n f_3(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} |y_0|^m + n \frac{f_4(\gamma^{-1}(t))}{\gamma'(\gamma^{-1}(t))} \left(\frac{1}{m} |y_0|^m + \frac{m-1}{m} \right) + 1 - n \right) \\
 & \quad + \frac{f_5(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} |y_0|^m \Big) dr \exp \left(- \int_0^s \left(n \frac{f_3(\gamma^{-1}(t))}{\gamma'(\gamma^{-1}(t))} + \frac{n}{m} \frac{f_4(\gamma^{-1}(t))}{\gamma'(\gamma^{-1}(t))} \right. \right. \\
 (3.6) \quad & \left. \left. + \frac{f_5(\gamma^{-1}(t))}{\gamma'(\gamma^{-1}(t))} \right) ds \right) \Big\}^{\frac{1}{m}}, \quad \forall r \in \mathbb{R}_+.
 \end{aligned}$$

Thus, the estimation in inequality (3.6) implies the boundedness and existence of the solution of (3.1).

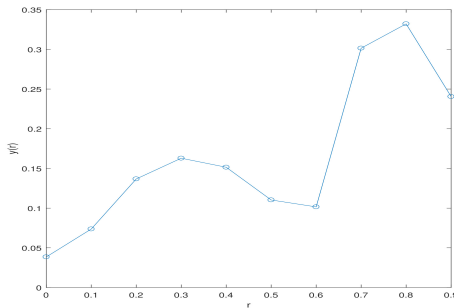


Figure 1:

Particularly, if we substitute $n = \frac{1}{2}$, $m = 4$, $\gamma(r) = \sqrt{r}$, $f_2(r) = 1$, $f_3(r) = r^3$, $f_4(r) = r^4$, $f_5(r) = r^5$, we obtain

$$\begin{aligned}
 y(r) & \leq \left\{ (r^3 + 1) + \left(\exp \left(\int_0^{\sqrt{r}} \left(\frac{1}{2} r^3 + \frac{1}{2} \frac{r^4}{4} + r^5 \right) d\lambda \right) \right. \right. \\
 & \quad \times \int_0^{\sqrt{r}} \left(\frac{1}{2} r^3 (r^3 + 1) + \frac{1}{2} r^4 \left(\frac{1}{4} (r^3 + 1) + \frac{4-1}{4} \right) + 1 - \frac{1}{2} + r^5 (r^3 + 1) \right) \\
 (3.7) \quad & \left. \left. \times \exp \left(- \int_0^s \left(\frac{1}{2} r^3 + \frac{1}{2} \frac{r^4}{4} + r^5 \right) d\lambda \right) ds \right) \right\}^{\frac{1}{4}}, \quad \forall r \in \mathbb{R}_+.
 \end{aligned}$$

The estimated boundedness and existence of unknown y for $0 \leq r \leq 1$ is shown in Figure 1.

Example 3.2. Consider the following retarded nonlinear differential equation

$$(3.8) \quad \begin{cases} my^{m-1}(r) \frac{dy}{dr} = f_1(r) + k(r, s)M\left((r, y^m(\gamma(r)), H_1(r, y(\gamma(r))))^n \right. \\ \qquad \qquad \qquad \left. + H_2(r, y^m(\gamma(r))), \right. \\ y(0) = 1, \end{cases}$$

$\forall r \in \mathbb{R}_+$, where M is real-valued nonnegative continuous function defined on $[\mathbb{R}_+^3, \mathbb{R}_+]$, and H_1 and H_2 defined on $[\mathbb{R}_+^2, \mathbb{R}_+]$, satisfy the following conditions:

$$(3.9) \quad |f_1(r)| \leq 3r^2 \quad \forall r \in \mathbb{R}_+,$$

$$(3.10) \quad |H_1(r, y)| \leq f_4(r)|y(r)| \quad \forall r \in \mathbb{R}_+,$$

$$(3.11) \quad |H_2(r, y^m)| \leq f_5(r)|y^m(r)| \quad \forall r \in \mathbb{R}_+,$$

$$(3.12) \quad |M(r, y, H_1)|^n \leq \left(f_3(r)|y^m(r)| + f_4(r)|y(r)| \right)^n \quad \forall r \in \mathbb{R}_+,$$

where $y(r)$, $f_1(r)$, $f_2(r)$, $f_3(r)$, $f_4(r)$, $f_5(r)$, $\gamma(r)$ and $n, m, k(r, s)$ are defined as in Theorem 2.14, integrate both sides of the equation (3.8) from 0 to r and using the inequalities (3.9), (3.10), (3.11) and (3.12), we get

$$\begin{aligned} & |y(r)|^m \\ & \leq (r^3 + 1) + \int_0^r k(r, s) \left(f_3(s)y^m(\gamma(r)) + f_4(s)y(\gamma(r)) \right)^n dr \\ & \quad + \int_0^r f_5(s)y^m(\gamma(r))dr \\ & \leq (r^3 + 1) + \int_0^{\gamma(r)} k(r, s) \left(\frac{f_3(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} y^m(\gamma(r)) + \frac{f_4(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} y(\gamma(r)) \right)^n ds \\ & \quad + \int_0^{\gamma(r)} \frac{f_5(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} y^m(\gamma(r)) ds, \quad \forall r \in \mathbb{R}_+. \end{aligned}$$

Thus, as an application of Theorem 2.14 with $f_2(r) = 1$, implies

$$(3.13) \quad \begin{aligned} y(r) & \leq \left\{ (r^3 + 1) + \exp \left(\int_0^{\gamma(r)} A_7(r) dr \right) \int_0^{\gamma(r)} B_7(s) \right. \\ & \quad \left. \times \exp \left(- \int_0^s A_7(r) dr \right) ds \right\}^{\frac{1}{m}}, \end{aligned}$$

for all $r \in \mathbb{R}_+$, where

$$\begin{aligned}
 A_7(r) &= \left(\frac{f_5(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} + k(r, \gamma(r)) \left(n \frac{f_3(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} + \frac{n}{m} \frac{f_4(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} \right) \right) \gamma'(r) \\
 (3.14) \quad &+ \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n \frac{f_3(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} + \frac{n}{m} \frac{f_4(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} \right) ds, \quad \forall r \in \mathbb{R}_+,
 \end{aligned}$$

$$\begin{aligned}
 B_7(r) &= \gamma'(r) \left\{ \frac{f_5(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} (r^3 + 1) + k(r, \gamma(r)) \left(n \frac{f_3(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} \right) (r^3 + 1) \right. \\
 &+ n \frac{f_4(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} \left(\frac{1}{m} (r^3 + 1) + \frac{m-1}{m} \right) + 1 - n \left. \right\} \\
 &+ \int_0^{\gamma(r)} \frac{\partial}{\partial r} k(r, s) \left(n \frac{f_3(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} (r^3 + 1) + n \frac{f_4(\gamma^{-1}(s))}{\gamma'(\gamma^{-1}(s))} \left(\frac{1}{m} (r^3 + 1) \right. \right. \\
 (3.15) \quad &\left. \left. + \frac{m-1}{m} \right) + 1 - n \right) ds, \quad \forall r \in \mathbb{R}_+.
 \end{aligned}$$

Thus, the estimation in inequality (3.15) implies the boundedness and existence of the solution of (3.8). The estimated boundedness and existence of unknown y for $0 \leq r \leq 1$ is shown in Figure 2.

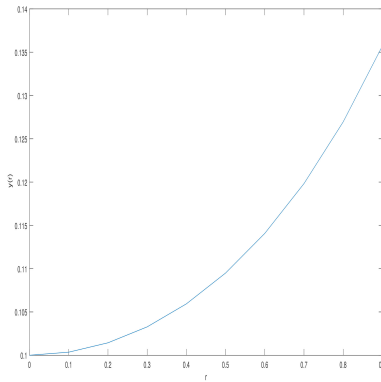


Figure 2:

4. Conclusion

It is of great interest to derive the most general inequality for retarded nonlinear integral inequality, this would be rather complicated. The compact perspective of this article is to establish several new retarded nonlinear integral inequalities, and many fresh and existing famous inequalities might be achieved

by making an appropriate selection of parameters. Moreover, these inequalities permit us to study the qualitative properties such as existence, uniqueness, boundedness, stability and asymptotics of solutions of classes of more general retarded nonlinear differential, and integral equations. Many renowned and existing important special cases can be explored based on different choices of parameters (see remarks 2.3, 2.5, 2.7, 2.9, 2.11, 2.13, 2.15, 2.17, 2.19, 2.21) from our integral inequalities of this article. So, these inequalities can handle the problems of nonlinear partial differential equations in applied sciences. This research work will ensure to open the new opportunities for studying nonlinear dynamic inequalities on time scale structure of varying nature.

Acknowledgement

The authors would like to thank the editor and the reviewers for their generous time to review and giving valuable comments/suggestions which improve the quality of this article.

References

- [1] ABDELDAIM, A., AND EL-DEEB, A. A. Some new retarded nonlinear integral inequalities with iterated integrals and their applications in retarded differential equations and integral equations. *J. Fract. Calc. Appl.* 5, suppl. 3S (2014), Paper No. 9, 9.
- [2] ABDELDAIM, A., AND EL-DEEB, A. A. On generalized of certain retarded nonlinear integral inequalities and its applications in retarded integro-differential equations. *Appl. Math. Comput.* 256 (2015), 375–380.
- [3] ABDELDAIM, A., AND EL-DEEB, A. A. On some generalizations of certain retarded nonlinear integral inequalities with iterated integrals and an application in retarded differential equation. *J. Egyptian Math. Soc.* 23, 3 (2015), 470–475.
- [4] ABDELDAIM, A., EL-DEEB, A. A., AND AHMED, R. G. On retarded nonlinear integral inequalities of Gronwall and applications. *J. Math. Inequal.* 13, 4 (2019), 1023–1038.
- [5] AGARWAL, R. P., DENG, S., AND ZHANG, W. Generalization of a retarded Gronwall-like inequality and its applications. *Appl. Math. Comput.* 165, 3 (2005), 599–612.
- [6] AKI N, L. On innovations of n -dimensional integral-type inequality on time scales. *Adv. Difference Equ.* (2021), Paper No. 148, 10.
- [7] AKI N, L. On generalized weighted dynamic inequalities for diamond – α integral on time scales calculus. *Indian J. Pure Appl. Math.* 55, 1 (2024), 363–376.
- [8] BELLMAN, R. The stability of solutions of linear differential equations. *Duke Math. J.* 10 (1943), 643–647.
- [9] BIHARI, I. A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations. *Acta Math. Acad. Sci. Hungar.* 7 (1956), 81–94.

- [10] EL-DEEB, A., AND AHMED, R. G. On some explicit bounds on certain retarded nonlinear integral inequalities with applications. *Adv. Inequal. Appl.* 2016 (2016), Article-ID.
- [11] EL-DEEB, A. A., AND AHMED, R. G. On some generalizations of certain nonlinear retarded integral inequalities for Volterra-Fredholm integral equations and their applications in delay differential equations. *J. Egyptian Math. Soc.* 25, 3 (2017), 279–285.
- [12] EL-DEEB, A. A., AND BALEANU, D. Some new dynamic Gronwall-Bellman-Pachpatte type inequalities with delay on time scales and certain applications. *J. Inequal. Appl.* (2022), Paper No. 45, 19.
- [13] EL-DEEB, A. A., AND RASHID, S. On some new double dynamic inequalities associated with Leibniz integral rule on time scales. *Adv. Difference Equ.* (2021), Paper No. 125, 22.
- [14] GRONWALL, T. H. Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *Ann. of Math. (2)* 20, 4 (1919), 292–296.
- [15] HILDEBRANDT, T. H. *Introduction to the theory of integration*. Pure and Applied Mathematics, Vol. XIII. Academic Press, New York-London, 1963.
- [16] LI, Z., AND WANG, W.-S. Some nonlinear Gronwall-Bellman type retarded integral inequalities with power and their applications. *Appl. Math. Comput.* 347 (2019), 839–852.
- [17] LIPOVAN, O. A retarded Gronwall-like inequality and its applications. *J. Math. Anal. Appl.* 252, 1 (2000), 389–401.
- [18] MITRINOVIĆ, D. S. *Analiitičke nejednakosti*, vol. 2 of *University of Belgrade Monographs*. Gradjevinska Knjiga, Belgrade, 1970. With the collaboration of Petar M. Vasić.
- [19] PACHPATTE, B. G. *Inequalities for differential and integral equations*, vol. 197 of *Mathematics in Science and Engineering*. Academic Press, Inc., San Diego, CA, 1998.
- [20] PACHPATTE, B. G. On some new inequalities related to a certain inequality arising in the theory of differential equations. *J. Math. Anal. Appl.* 251, 2 (2000), 736–751.
- [21] PACHPATTE, B. G. *Integral and finite difference inequalities and applications*, vol. 205 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, 2006.
- [22] SHAKOOR, A., ALI, I., AZAM, M., REHMAN, A., AND ZAHID IQBAL, M. Further nonlinear retarded integral inequalities for Gronwall-Bellman type and their applications. *Iran. J. Sci. Technol. Trans. A Sci.* 43, 5 (2019), 2559–2568.
- [23] SHAKOOR, A., ALI, I., WALI, S., AND REHMAN, A. Some generalizations of retarded nonlinear integral inequalities and its applications. *J. Math. Inequal.* 14, 4 (2020), 1223–1235.
- [24] SHAKOOR, A., SAMAR, M., ATHAR, T., AND SADDIQUE, M. Results for retarded nonlinear integral inequalities with mixed powers and their applications to delay integrodifferential equations. *Ukrainian Math. J.* 75, 3 (2023), 478–493. Reprint of *Ukrain. Mat. Zh.* 74 (2023), no. 3, 416–429.

- [25] SHAKOOR, A., SAMAR, M., WALI, S., AND SALEEM, M. Retarded nonlinear integral inequalities of Gronwall-Bellman-Pachpatte type and their applications. *Honam Math. J.* 45, 1 (2023), 54–70.
- [26] SONG, H., AND MENG, F. Some generalizations of delay integral inequalities of gronwall-bellman type with power and their applications. *Mathematical Foundations of Computing* 5, 1 (2022).
- [27] TANG, C., WU, Q., AND XU, J. Estimates of fractional integral operators on variable exponent Lebesgue spaces. *J. Funct. Spaces* (2016), Art. ID 2438157, 7.
- [28] TIAN, Y., AND FAN, M. Nonlinear integral inequality with power and its application in delay integro-differential equations. *Adv. Difference Equ.* (2020), Paper No. 142, 11.

Received by the editors December 19, 2023

First published online October 5, 2024