

Characterization of almost \ast -Ricci-Yamabe solitons isometric to a unit sphere

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Abstract. The aim of this work is to characterize almost \ast -Ricci-Yamabe solitons on a Sasakian manifold where we proved that the manifold is isometric to the unit sphere \mathbb{S}^{2n+1} if its metric represents a complete almost \ast -Ricci-Yamabe solitons with $\alpha \neq 0$. Certain conditions under which the soliton reduces to \ast -Ricci-Yamabe soliton and when it becomes steady are also obtained.

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1. Introduction

Poincaré in 1904 wondered if there is a way to recognize a three dimensional sphere while the necessary measurements are operated from inside the shape, which later becomes the famous Poincaré conjecture. Since then, to prove or disprove the conjecture has become the core agenda of many researchers. The study of geometric flows amongst other methods is one of the main attraction for geometers due to its application in mathematical physics and it helps us to understand shapes in three or more dimensional spaces. One such significant flow is the Ricci flow introduced by Hamilton [12] which later become the heart of proof of the Poincaré conjecture by Perelman.

Many geometers [5, 18, 21, 24], over the past two years, studied and analyzed a new geometric flow which is a scalar combination of the Ricci [12] and Yamabe flows known as the Ricci-Yamabe flow introduced by Güler and Crasmareanu [11] and it is defined as the following:

Definition 1.1. [11] A Riemannian flow on M is a smooth map:

$$g : I \subset \mathbb{R} \rightarrow \text{Riem}(M),$$

where I is a given open interval.

Definition 1.2. [11] The map $RY^{(\alpha,\beta,g)} : I \rightarrow T_2^s(M)$ given by

$$RY^{(\alpha,\beta,g)}(t) := \frac{\partial g}{\partial t}(t) + 2\alpha Ric(t) + \beta R(t)g(t),$$

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is called the (α, β) -Ricci-Yamabe map of the Riemannian flow (M, g) . If $RY^{(\alpha, \beta, g)} \equiv 0$, then $g(\cdot)$ will be called an (α, β) -Ricci-Yamabe flow.

The self-limiting solution to this flow is called Ricci-Yamabe soliton which generalizes the well known Ricci soliton (see for instance, [22, 23]) and ρ -Einstein soliton. The Ricci-Yamabe soliton is defined as

$$\mathfrak{L}_U g + 2\alpha Ric = (2\Lambda - \beta S)g,$$

where Ric is the Ricci tensor, S is the scalar curvature and $\Lambda, \alpha, \beta \in \mathbb{R}$. If Λ is allowed to be a smooth function, then the soliton is called almost Ricci-Yamabe soliton.

Recently, Dwivedi and Patra [7] introduced the notion of almost $*$ -Ricci-Bourguignon soliton and studied its geometric characterization on Sasakian manifold where such manifold is introduced by Sasaki [17]. Sasakian manifold attracts geometers and physicists due to its application in complex geometry and string theory (for details, see [8, 13]).

One of the most interesting geometric properties for a soliton which represents a metric of a manifold is its isometry, may it be spheres or hyperbolic space. Obata [15] has shown that "In order for a complete Riemannian manifold of dimension $n \geq 2$ to admit a non-constant function ϕ with $\nabla_X d\phi = -c^2 \phi X$ for any vector X , it is necessary and sufficient that the manifold be isometric with a sphere $S(c)$ of radius $1/c$ in the $(n + 1)$ -Euclidean space." Deshmukh [3] obtained certain conditions and bounds for an almost Ricci soliton to be isometric to spheres. Many other geometers also obtained conditions under which the almost Ricci-Bourguignon soliton, almost $*$ -Ricci-Bourguignon soliton, almost Ricci-Yamabe soliton are isometric to spheres (see for further details, [6, 7, 9, 14]). Inspired by these mentioned works, we pondered if we assume a complete Sasakian manifold and define its metric by gradient almost $*$ -Ricci-Yamabe soliton and almost $*$ -Ricci-Yamabe soliton, would it still be isometric to the unit sphere? And if it is, then what are the conditions it need to satisfy?

To answer the questions that we asked, we introduce the notion of almost $*$ -Ricci-Yamabe soliton as

$$(1.1) \quad \mathfrak{L}_U g + 2\alpha Ric^* = (2\Lambda - \beta S^*)g,$$

where Λ is a smooth function, $\alpha, \beta \in \mathbb{R}$, Ric^* is the Ricci tensor and S^* , the $*$ -scalar curvature. If $U = \nabla f$, where ∇ denotes the gradient in (1.1), then (1.1) reduces to gradient almost $*$ -Ricci-Yamabe soliton as

$$(1.2) \quad \nabla^2 f + \alpha Ric^* = \left(\Lambda - \frac{\beta S^*}{2}\right)g,$$

where $\nabla^2 f = Hess f$ is the Hessian of a smooth function f .

In this paper, following the method used by Dwivedi and Patra [7] and extending their results, we have found conditions under which the metric g of a complete Sasakian manifold admits gradient almost and almost $*$ -Ricci-Yamabe soliton ($*$ -RYS, in short), being isometric to a unit sphere provided

$\alpha \neq 0$. Also, we have shown that the soliton is \ast -Ricci flat with constant scalar curvature $4n^2$ and certain conditions for the soliton to reduce to \ast -RYS and its steadiness are also obtained. Lastly, we make use of an example constructed in [9] to prove the existence of our results.

2. Preliminaries

In this section, we give some basic notions which will be useful for proving our results.

A contact manifold is a $(2n + 1)$ dimensional manifold which admits a contact 1-form η satisfying $\eta \wedge (d\eta)^n \neq 0$. Therefore, for a contact structure, there exists a characteristic vector field ξ satisfying $d\eta(\xi, \cdot) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact sub-bundle defined by $\eta = 0$ by \mathfrak{D} gives a $(1, 1)$ -tensor field ϕ and a Riemannian metric g such that the following relations hold:

$$(2.1) \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta = g(X, \xi),$$

$$(2.2) \quad d\eta(X, Y) = g(X, \phi Y),$$

for any vector fields X, Y on M . This structure is called a contact metric structure and the manifold equipped with such a structure is called a contact metric manifold M of dimension $(2n + 1)$ with an associated metric g . We know that from the foregoing equations we have

$$(2.3) \quad \eta \circ \phi = 0, \quad \phi(\xi) = 0, \quad \text{and } \text{rank}(\phi) = 2n.$$

The Riemannian curvature tensor R of g is given by the formula

$$(2.4) \quad R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, \quad X, Y, Z \in \chi(M)$$

where ∇ and $\chi(M)$ are the Levi-Civita connection of g and the Lie algebra of vector fields on the manifold, respectively. The Ricci operator Q is a $(1, 1)$ -tensor field defined by

$$g(QX, Y) = \text{Ric}(X, Y), \quad X, Y \in \chi(M)$$

and the scalar curvature S and the gradient of the scalar curvature ∇S (see [4]) are, respectively, the smooth function defined by $S = \text{tr } Q$ and

$$(2.5) \quad \frac{1}{2}g(X, \nabla S) = (\text{div } Q)(X), \quad X \in \chi(M).$$

We call such manifold a Sasakian manifold if any of the following three equivalent conditions hold (see [1, 2]):

1. The metric cone $(C(M), \bar{g}) = (\mathbb{R}_+ \times M, dr^2 \oplus r^2 g)$ is Kähler.
2. The Riemann curvature tensor R of g satisfies the identity

$$(2.6) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad X, Y \in \chi(M)$$

3. The structural tensor field ϕ satisfies the identity

$$(2.7) \quad (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in \chi(M)$$

We know that M is a K -contact manifold if ξ is Killing. Also, a Sasakian manifold is a K -contact manifold but a K -contact manifold is not Sasakian for $dim > 3$. The following relations are valid on a Sasakian manifold (see [1]):

$$(2.8) \quad \nabla_X \xi = -\phi X, \quad \nabla_\xi \xi = 0, \quad Q\xi = 2n\xi, \quad X \in \chi(M).$$

Further, setting $X = \xi$ on the last term of the above equation and then taking covariant derivative along $X \in \chi(M)$, we get

$$(2.9) \quad (\nabla_X Q)\xi = Q\phi X - 2n\phi X.$$

A contact metric manifold is said to be η -Einstein if

$$(2.10) \quad Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \quad X, Y \in \chi(M)$$

where a, b are smooth functions on M . However, by Okumura [16], if the manifold is a K -contact manifold with $dim > 3$, then a, b become constants.

From the geometric point of view, preserving the structure upon transformation is important. Therefore, such preservation of a K -contact and Sasakian structures can be obtained by a D -homothetic deformation which is described as

$$(2.11) \quad \bar{\eta} = \nu\eta, \quad \bar{\xi} = \frac{1}{\nu}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = \nu g + \nu(\nu - 1)\eta \otimes \eta,$$

where $\nu \in \mathbb{R}_+$. Now, we recall the following for later use:

Definition 2.1. [9] A K -contact η -Einstein manifold with $a = -2$ is D -homothetically fixed.

Following [19], we define an infinitesimal contact transformation as follows:

Definition 2.2. A potential vector field U is an infinitesimal contact transformation on an almost contact metric manifold if $\mathfrak{L}_U \eta = \psi\eta$ for some function ψ . In particular, if $\mathfrak{L}_U \eta = 0$, then U is said to be strict infinitesimal contact transformation. Moreover, U is called an infinitesimal automorphism if it leaves all the structure tensors invariant.

Lemma 2.3. [10] Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold and $\{E_i\}_{1 \leq i \leq 2n+1}$ be a local orthonormal frame on M . Then, for $Y \in \chi(M)$, we have

$$\sum_i g((\nabla_{\phi Y} Q)\phi E_i, E_i) = 0, \quad \sum_i g((\nabla_{\phi E_i} Q)\phi Y, E_i) = \frac{1}{2}X(S).$$

Lemma 2.4. [7] *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold and f be a smooth function on M . If $\{E_i\}_{1 \leq i \leq 2n+1}$ is a local orthonormal frame on M , then for $Y \in \chi(M)$, we have*

$$\begin{aligned} \sum_i g(Y, \nabla_{\phi E_i} \nabla f) g(\xi, E_i) &= 0 \\ \sum_i g(\xi, \nabla_{E_i} \nabla f) g(\phi Y, E_i) &= g(\phi Y, \nabla_{\xi} \nabla f) \\ \sum_i g(\phi Y, \nabla_{E_i} \nabla f) g(\xi, E_i) &= g(\xi, \nabla_{\phi Y} \nabla f) \\ \sum_i g(\xi, R(E_i, Y) \nabla f) g(\xi, E_i) &= Y(f) - \eta(\nabla f) \eta(Y). \end{aligned}$$

Also, we recall the expression of \ast -Ricci tensor on a Sasakian manifold by the lemma:

Lemma 2.5. [9] *The expression of \ast -Ricci tensor Ric^* on a Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is*

$$(2.12) \quad Ric^*(X, Y) = Ric(X, Y) - (2n-1)g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \chi(M).$$

As a direct consequence of the above lemma, we have the following:

Corollary 2.6. *The \ast -scalar curvature S^* on a Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is given by $S^* = S - 4n^2$.*

3. Main Results

In this section, we will prove our main results and all the vector fields we considered here are on the Sasakian manifold M .

First of all, let us make use of (2.12) in (1.1), we get

$$(3.1) \quad \begin{aligned} (\mathfrak{L}_U g)(X, Y) + 2\alpha Ric(X, Y) &= (2\Lambda - \beta S^* + 2\alpha(2n-1))g(X, Y) \\ &\quad + 2\alpha \eta(X)\eta(Y). \end{aligned}$$

Taking Lie derivative of $R(X, \xi)\xi = X - \eta(X)\xi$ along U gives

$$(3.2) \quad (\mathfrak{L}_U R)(X, \xi)\xi + R(X, \xi)\mathfrak{L}_U \xi + g(X, \mathfrak{L}_U \xi)\xi + (\mathfrak{L}_U g)(X, \xi)\xi + \eta(\mathfrak{L}_U \xi)X = 0.$$

Using the fact that $Q\xi = 2n\xi$ and (3.1), we get

$$(3.3) \quad (\mathfrak{L}_U g)(X, \xi) = (2\Lambda - \beta S^*)\eta(X).$$

Again, taking Lie derivative of $\eta(X) = g(X, \xi)$ and $g(\xi, \xi) = 1$ along U respectively provides

$$(3.4) \quad (\mathfrak{L}_U \eta)X - g(X, \mathfrak{L}_U \xi) = (2\Lambda - \beta S^*)\eta(X),$$

and

$$(3.5) \quad \eta(\mathfrak{L}_U\xi) = -\frac{1}{2}(\mathfrak{L}_Ug)(\xi, \xi).$$

From (3.3), we get

$$(3.6) \quad \eta(\mathfrak{L}_U\xi) = -\left(\Lambda - \frac{\beta S^*}{2}\right).$$

Again, from (3.4), we obtain

$$(3.7) \quad (\mathfrak{L}_U\eta)\xi = \left(\Lambda - \frac{\beta S^*}{2}\right).$$

Making use of these foregoing equations in (3.2) results in the following lemma.

Lemma 3.1. *If a Sasakian metric g represents an almost $*$ -RYS with $\alpha \neq 0$, then the relation*

$$(\mathfrak{L}_UR)(X, \xi)\xi = (2\Lambda - \beta S^*)(X - \eta(X)\xi), \quad X \in \chi(M)$$

holds.

Now, using Lemma 2.5, we can write (1.2) as

$$(3.8) \quad \nabla_X\nabla f + \alpha QX = \left(\Lambda - \frac{\beta S^*}{2} + \alpha(2n - 1)\right)X + \alpha\eta(X)\xi.$$

Applying covariant derivative in

$$(3.9) \quad R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z$$

to obtain the curvature tensor expression

$$\begin{aligned} R(X, Z)\nabla f &= \nabla_X\nabla_Z\nabla f - \nabla_Z\nabla_X\nabla f - \nabla_{[X, Z]}\nabla f \\ &= X\left(\Lambda - \frac{\beta S^*}{2}\right)Z - \alpha\eta(Z)\phi X + 2\alpha g(X, \phi Z)\xi \\ &\quad - Z\left(\Lambda - \frac{\beta S^*}{2}\right)X - \alpha(\nabla_X Q)Z + \alpha(\nabla_Z Q)X + \alpha\eta(X)\phi Z. \end{aligned}$$

Taking inner product with respect to the vector field Y in the above equation, we obtain

$$\begin{aligned} (3.10) \quad g(R(X, Z)\nabla f, Y) &= X\left(\Lambda - \frac{\beta S^*}{2}\right)g(Y, Z) - \alpha\eta(Z)g(Y, \phi X) \\ &\quad + 2\alpha g(X, \phi Z)\eta(Y) - Z\left(\Lambda - \frac{\beta S^*}{2}\right)g(X, Y) \\ &\quad - \alpha g((\nabla_X Q)Z, Y) + \alpha g((\nabla_Z Q)X, Y) \\ &\quad + \alpha\eta(X)g(Y, \phi Z). \end{aligned}$$

Since Q and ϕ commute on a Sasakian manifold, we have (see Lemma 2.1, [9])

$$(3.11) \quad \nabla_{\xi}Q = Q\phi - \phi Q \text{ and } \nabla_{\xi}Q = 0.$$

Also, utilizing the symmetric and anti-symmetric properties of Q and ϕ , respectively, and thus setting $Y = Z = \xi$ in (3.10) and using (2.9), (3.11) and the fact that $\phi(\xi) = 0$, we obtain

$$(3.12) \quad X \left(\Lambda - \frac{\beta S^*}{2} + f \right) = \xi \left(\Lambda - \frac{\beta S^*}{2} + f \right) \eta(X).$$

Putting $\zeta = \Lambda - \frac{\beta S^*}{2} + f$, (3.12) becomes

$$(3.13) \quad X(\zeta) = \xi(\zeta)\eta(X)$$

Now, taking covariant derivative of (3.13) along $Y \in \chi(M)$ and using (2.8) gives

$$(3.14) \quad g(\nabla_Y \nabla \zeta, X) = Y(\xi(\zeta))\eta(X) + \xi(\zeta)g(\phi X, Y).$$

Utilizing (3.14) in the symmetric property of $Hess_{\zeta}$, we get

$$2\xi(\zeta)g(\phi X, Y) = X(\xi(\zeta))\eta(Y) - Y(\xi(\zeta))\eta(X).$$

Hence, we choose $X, Y \perp \xi$ to get $\xi(\zeta) = 0$ on the manifold as $d\eta$ is non-zero on the manifold. This further implies that $\nabla \zeta = 0$ on M . Thus, we conclude that $\zeta = \Lambda - \frac{\beta S^*}{2} + f$ is constant on M . On the contrary, replacing X by ξ in (3.10) and using (3.11) and (3.11) results in

$$(3.15) \quad \begin{aligned} g(R(\xi, Z)\nabla f, Y) &= g(Q\phi Z, Y) - (2n-1)g(\phi Z, Y) \\ &+ \xi \left(\Lambda - \frac{\beta S^*}{2} \right) g(Y, Z) - Z \left(\Lambda - \frac{\beta S^*}{2} \right) \eta(Y). \end{aligned}$$

Also, the relation

$$(3.16) \quad R(\xi, Z)\nabla f = Z(f)\xi - \xi(f)Z$$

follows directly from (2.6). Substituting the above equation in (3.15), we obtain

$$(3.17) \quad g(Q\phi Z, Y) - (2n-1)g(\phi Z, Y) = Z(\zeta)\eta(Y) - \xi(\zeta)g(Y, Z).$$

Since ζ is constant, (3.17) yields $Q\phi Z = (2n-1)\phi Z$. Then, setting $Z = \phi Z$ and using (2.1), (2.8), we obtain

$$(3.18) \quad Ric(Y, Z) = (2n-1)g(Y, Z) + \eta(Y)\eta(Z), \quad Y, Z \in \chi(M).$$

Hence, M becomes an η -Einstein manifold. Further, suppose that M is complete, then (3.18) shows that M is compact and positive Sasakian. Again, plugging in (3.18) into (3.8) yields

$$(3.19) \quad \nabla_X \nabla f = \left(\Lambda - \frac{\beta S^*}{2} \right) X, \quad X \in \chi(M).$$

As $\zeta = \left(\Lambda - \frac{\beta S^*}{2} + f\right)$ is a constant, the foregoing expression can be written as

$$(3.20) \quad \nabla_X D\gamma = \gamma X,$$

where $\gamma = \Lambda - \frac{\beta S^*}{2}$, D being a gradient operator and Λ is a non-constant smooth function on the manifold. Therefore, by applying Obata's theorem [15] to our results, we conclude with the following theorem.

Theorem 3.2. *A complete Sasakian manifold admitting a gradient almost $*$ -Ricci-Yamabe soliton as its metric is $*$ -Ricci flat, compact positive-Sasakian and isometric to the unit sphere \mathbb{S}^{2n+1} provided $\alpha \neq 0$.*

By looking at the above theorem, a natural question arises as to whether the soliton would still behave as such without considering the gradient vector field of a smooth function f or not. Before giving an answer to this logical assumption, let us deduce some propositions which we will use later in the proofs.

Proposition 3.3. *For a Sasakian metric g admitting almost $*$ -RYS with $\alpha \neq 0$, the following formula holds:*

$$\begin{aligned} (\mathfrak{L}_U \nabla)(X, \xi) &= 2\alpha(2n-1)\phi X - 2\alpha\phi QX + X \left(\Lambda - \frac{\beta S^*}{2}\right) \xi \\ &\quad + \xi \left(\Lambda - \frac{\beta S^*}{2}\right) X - \eta(X)\nabla \left(\Lambda - \frac{\beta S^*}{2}\right), \end{aligned}$$

for any vector field X on M .

Proof. Taking covariant derivative of (3.1) along an arbitrary $Z \in \chi(M)$ and using (2.8), we get

$$\begin{aligned} (\nabla_Z \mathfrak{L}_U g)(X, Y) + 2\alpha(\nabla_Z Ric)(X, Y) &= Z(2\Lambda - \beta S^*)g(X, Y) \\ &\quad - 2\alpha[\eta(X)g(Y, \phi Z) \\ &\quad + \eta(Y)g(X, \phi Z)]. \end{aligned} \quad (3.21)$$

Now, we recall the formula given by Yano (see, [20])

$$\begin{aligned} (\mathfrak{L}_U \nabla_Z g - \nabla_Z \mathfrak{L}_U g - \nabla_{[U, Z]} g)(X, Y) &= -g((\mathfrak{L}_U \nabla)(Z, X), Y) \\ &\quad - g((\mathfrak{L}_U \nabla)(Z, Y), X), \end{aligned} \quad (3.22)$$

for all $X, Y, Z \in \chi(M)$. Since g is parallel, inserting (3.21) into (3.22), we obtain

$$\begin{aligned} g((\mathfrak{L}_U \nabla)(Z, X), Y) + g((\mathfrak{L}_U \nabla)(Z, Y), X) + 2\alpha(\nabla_Z Ric)(X, Y) \\ = 2Z \left(\Lambda - \frac{\beta S^*}{2}\right) g(X, Y) - 2\alpha[\eta(X)g(Y, \phi Z) \\ + \eta(Y)g(X, \phi Z)]. \end{aligned} \quad (3.23)$$

Interchanging cyclically X, Y, Z in the foregoing equation and using the symmetry $(\mathfrak{L}_U \nabla)(X, Y) = (\mathfrak{L}_U \nabla)(Y, X)$ results in

$$\begin{aligned}
 g((\mathfrak{L}_U \nabla)(X, Y), Z) &= \alpha[(\nabla_Z Ric)(X, Y) - (\nabla_X Ric)(Y, Z) \\
 &\quad - (\nabla_Y Ric)(Z, X)] + X \left(\Lambda - \frac{\beta S^*}{2} \right) g(Y, Z) + Y \left(\Lambda - \frac{\beta S^*}{2} \right) g(X, Z) \\
 (3.24) \quad &- Z \left(\Lambda - \frac{\beta S^*}{2} \right) g(X, Y) - 2\alpha[\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z)].
 \end{aligned}$$

Finally, setting $Y = \xi$ in (3.24) and utilizing (2.9) and (3.11), (3.11) completes the proof. \square

Proposition 3.4. *For a Sasakian metric g admitting almost \ast -RYS, the formula*

$$\begin{aligned}
 g \left(\xi, \nabla_X \nabla \left(\Lambda - \frac{\beta S}{2} \right) \right) \xi - \eta(X) \nabla_\xi \nabla \left(\Lambda - \frac{\beta S}{2} \right) &= 4\alpha QX \\
 - 2 \left(\Lambda - \frac{\beta S^*}{2} + 2\alpha(2n - 1) \right) X & \\
 + 2 \left(\Lambda - \frac{\beta S^*}{2} - 2\alpha \right) \eta(X) \xi + 2(\phi X) \left(\Lambda - \frac{\beta S}{2} \right) \xi & \\
 - \nabla_X \nabla \left(\Lambda - \frac{\beta S}{2} \right) &
 \end{aligned}$$

is valid provided that ξ leaves Λ invariant and $\alpha \neq 0$.

Proof. We know that ξ is Killing on a Sasakian manifold, this implies $\mathfrak{L}_\xi Ric = 0$, from which we obtain $\xi(S) = 0$. Also, from Corollary 2.6, we have $S^* = S - 4n^2$. This further implies $\xi(S^*) = 0$ and $X \left(\Lambda - \frac{\beta S^*}{2} \right) = X \left(\Lambda - \frac{\beta S}{2} \right)$ due to the fact that $\xi(\Lambda) = 0$. Thus, Proposition 3.3 reduces to

$$\begin{aligned}
 (\mathfrak{L}_U \nabla)(X, \xi) &= 2\alpha(2n - 1)\phi X - 2\alpha\phi QX \\
 (3.25) \quad &+ X \left(\Lambda - \frac{\beta S}{2} \right) \xi - \eta(X) \nabla \left(\Lambda - \frac{\beta S}{2} \right),
 \end{aligned}$$

where $X \in \chi(M)$. Setting $X = \xi$ in (3.25) and using (3.18) results in

$$(3.26) \quad (\mathfrak{L}_U \nabla)(\xi, \xi) = -\nabla \left(\Lambda - \frac{\beta S}{2} \right).$$

Taking the covariant derivative along X in (3.26) and using (2.8) yields

$$\begin{aligned}
 (\nabla_X \mathfrak{L}_U \nabla)(\xi, \xi) &= 4\alpha QX - 4\alpha(2n - 1)X - 4\alpha\eta(X)\xi \\
 (3.27) \quad &+ 2(\phi X) \left(\Lambda - \frac{\beta S}{2} \right) \xi - \nabla_X \nabla \left(\Lambda - \frac{\beta S}{2} \right).
 \end{aligned}$$

On the other hand, differentiating (3.25) along ξ gives

$$(3.28) \quad (\nabla_{\xi} \mathfrak{L}_U \nabla)(X, \xi) = g \left(X, \nabla_{\xi} \nabla \left(\Lambda - \frac{\beta S}{2} \right) \right) \xi - \eta(X) \nabla_{\xi} \nabla \left(\Lambda - \frac{\beta S}{2} \right),$$

where we have used $\nabla_{\xi} Q = \nabla_{\xi} \xi = \nabla_{\xi} \phi = 0$. Now, from the commutation formula by Yano (see, [20]) we have

$$(3.29) \quad (\mathfrak{L}_U R)(X, Y)Z = (\nabla_X \mathfrak{L}_U \nabla)(Y, Z) - (\nabla_Y \mathfrak{L}_U \nabla)(X, Z).$$

Finally, setting $Y = Z = \xi$ in the above expression and applying Lemma 3.1, (3.27) and (3.28) thus completes the proof. \square

Proposition 3.5. *For a Sasakian metric g representing almost $*$ -RYS, the formula*

$$\begin{aligned} Ric \left(Y, \nabla \left(\Lambda - \frac{\beta S}{2} \right) \right) &= (4n-1)Y \left(\Lambda - \frac{\beta S}{2} \right) + 4\alpha g \left(\phi Y, \nabla_{\xi} \nabla \left(\Lambda - \frac{\beta S}{2} \right) \right) \\ &\quad + \eta(Y) div \left(\nabla_{\xi} \nabla \left(\Lambda - \frac{\beta S}{2} \right) \right) \\ &\quad - g \left(\xi, \nabla_Y \nabla_{\xi} \nabla \left(\Lambda - \frac{\beta S}{2} \right) \right) - 2\alpha Y(S) \end{aligned}$$

holds true provided $\xi(\Lambda) = 0$ and $\alpha \neq 0$.

Proof. By hypothesis, $\xi(\Lambda) = 0$ and since $\xi(S) = 0$, this implies that $\xi \left(\Lambda - \frac{\beta S}{2} \right) = 0$. Therefore, from (2.8) we have

$$(\phi X) \left(\Lambda - \frac{\beta S}{2} \right) = g \left(\xi, \nabla_X \nabla \left(\Lambda - \frac{\beta S}{2} \right) \right),$$

for any vector field X on M . Thus, Proposition 3.4 reduces to

$$\begin{aligned} \nabla_X \nabla \left(\Lambda - \frac{\beta S}{2} \right) &= 4\alpha QX - 2 \left(\Lambda - \frac{\beta S^*}{2} + 2\alpha(2n-1) \right) X \\ &\quad + 2 \left(\Lambda - \frac{\beta S^*}{2} - 2\alpha \right) \eta(X) \xi + g \left(\xi, \nabla_X \nabla \left(\Lambda - \frac{\beta S}{2} \right) \right) \xi \\ (3.30) \quad &\quad + \eta(X) \nabla_{\xi} \nabla \left(\Lambda - \frac{\beta S}{2} \right). \end{aligned}$$

Taking covariant derivative of (3.30) along Y , we get

$$\begin{aligned}
 \nabla_Y \nabla_X \nabla \left(\Lambda - \frac{\beta S}{2} \right) &= g \left(\xi, \nabla_Y \nabla_X \nabla \left(\Lambda - \frac{\beta S}{2} \right) \right) \xi \\
 &\quad - g \left(\phi Y, \nabla_X \nabla \left(\Lambda - \frac{\beta S}{2} \right) \right) \xi \\
 &\quad - g \left(\xi, \nabla_X \nabla \left(\Lambda - \frac{\beta S}{2} \right) \right) \phi Y \\
 &\quad + \eta(X) \nabla_Y \nabla_\xi \nabla \left(\Lambda - \frac{\beta S}{2} \right) - g(X, \phi Y) \nabla_\xi \nabla \left(\Lambda - \frac{\beta S}{2} \right) \\
 &\quad + 4\alpha (\nabla_Y Q) X - 2Y \left(\Lambda - \frac{\beta S^*}{2} \right) X \\
 &\quad + 2Y \left(\Lambda - \frac{\beta S^*}{2} \right) \eta(X) \xi - 2 \left(\Lambda - \frac{\beta S^*}{2} - 2\alpha \right) g(X, \phi Y) \xi \\
 &\quad - 2 \left(\Lambda - \frac{\beta S^*}{2} - 2\alpha \right) \eta(X) \phi Y.
 \end{aligned}$$

Utilizing the symmetry of *Hess*, anti-symmetric property of ϕ , (3.30) and the foregoing equation in (2.4), we get

$$\begin{aligned}
 R(X, Y) \nabla \left(\Lambda - \frac{\beta S}{2} \right) &= \nabla_X \nabla_Y \nabla \left(\Lambda - \frac{\beta S}{2} \right) - \nabla_Y \nabla_X \nabla \left(\Lambda - \frac{\beta S}{2} \right) \\
 &\quad - \nabla_{[X, Y]} \nabla \left(\Lambda - \frac{\beta S}{2} \right).
 \end{aligned}$$

Now, contracting the obtained results over X after a few steps of calculation of the foregoing expression and making use of the fact that $Tr\phi = 0 = \phi(\xi)$ and Lemma 2.4, the required result is obtained. \square

Now, we recall the commutation formula (see, [20])

$$(3.31) \quad \mathfrak{L}_Y \mathfrak{L}_X g - \mathfrak{L}_X \mathfrak{L}_Y g = \mathfrak{L}_{[Y, X]} g,$$

where $X, Y \in \chi(M)$. Since ξ is Killing, we have $\mathfrak{L}_\xi g = \mathfrak{L}_\xi Ric = 0$. Utilizing (3.1) in (3.31), we get

$$(3.32) \quad \mathfrak{L}_{[U, \xi]} g = -2\xi(\Lambda)g$$

where we used $\xi(S^*) = 0$ on M . Thus, the vector field $[U, \xi]$ is conformal and thus results in the following two cases:

Case I: $[U, \xi]$ is homothetic.

Case II: $[U, \xi]$ is non-homothetic.

Proceeding with the calculation as in (Theorem 1.4, [7]), we can conclude that Case I cannot happen as it results in a contradiction of Λ being a constant regardless of the fact that it is non-constant. Thus, invoking Okumura's theorem in Case II yields that the manifold is isometric to the unit sphere \mathbb{S}^{2n+1} . Hence, we can state the following theorem.

Theorem 3.6. *A complete Sasakian manifold admitting almost \ast -RYS of $\dim > 3$ with $\Lambda \neq \text{constant}$ is isometric to the unit sphere \mathbb{S}^{2n+1} provided $\alpha \neq 0$.*

Remark 3.7. For a particular value of $\alpha = 1, \beta = -2\rho$, the above theorem reduces to Theorem 1.4 of [7].

Suppose that the vector field U is parallel to the Reeb vector field ξ . This means that $U = \sigma\xi$, where σ is some smooth function on M . Then from (2.8), we get

$$(3.33) \quad (\mathfrak{L}_U g)(X, Y) = X(\sigma)\eta(Y) + Y(\sigma)\eta(X).$$

Using the antisymmetry of ϕ , (3.1) implies

$$(3.34) \quad \begin{aligned} X(\sigma)\eta(Y) + Y(\sigma)\eta(X) + 2\alpha Ric(X, Y) &= [2\Lambda - \beta S^\ast + 2\alpha(2n - 1)]g(X, Y) \\ &+ 2\alpha\eta(X)\eta(Y). \end{aligned}$$

Putting $X = Y = \xi$ in (3.34) and using (2.8), we get $\xi(\sigma) = \Lambda - \frac{\beta S^\ast}{2}$. Similarly, setting $Y = \xi$ in (3.34) yields

$$(3.35) \quad X(\sigma) = \xi(\sigma)\eta(X)$$

Taking the covariant differentiation of (3.35) along $Y \in \chi(M)$ and using (2.8), we get

$$g(\nabla_Y \nabla \sigma, X) = Y(\xi(\sigma))\eta(X) + \xi(\sigma)g(\phi X, Y).$$

Using the symmetric property of $Hess_\sigma$, it follows that

$$X(\xi(\sigma))\eta(Y) - Y(\xi(\sigma))\eta(X) = 2\xi(\sigma)g(\phi X, Y),$$

which further gives

$$\xi(\sigma)d\eta(X, Y) = 0, \forall X, Y \perp \xi$$

Now, since $d\eta$ is non-zero, we get $\xi(\sigma) = 0$ and hence $\nabla\sigma = 0$. This implies σ is constant and thus U is Killing. Further, $2\Lambda - \beta S^\ast = 0$ which leads to the fact that

$$Ric(X, Y) = (2n - 1)g(X, Y) + \eta(X)\eta(Y).$$

Hence, M is \ast -Ricci flat and \ast -scalar curvature $S^\ast = 0$. Moreover, $S = 4n^2$ and $\Lambda = 0$. Therefore, we can conclude the following theorem.

Theorem 3.8. *If U is parallel to the characteristic vector field ξ on a Sasakian manifold M admitting almost \ast -RYS with $\alpha \neq 0$, then U is Killing and \ast -Ricci flat with constant scalar curvature $4n^2$. Moreover, the soliton is steady for any σ .*

Now, let us consider that U is an infinitesimal contact transformation on M . Thus, setting $Y = \xi$ in (3.1), we get

$$(3.36) \quad (\mathfrak{L}_U g)(X, \xi) = (2\Lambda - \beta S^\ast)\eta(X).$$

Then, from $\eta(\xi) = 1$, we obtain by taking Lie derivative along U

$$(\mathfrak{L}_U \eta)(\xi) = -\eta(\mathfrak{L}_U \xi) = \Lambda - \frac{\beta S^*}{2},$$

and from Definition 2.2, $\psi = \Lambda - \frac{\beta S^*}{2}$. In view of this and the Lie derivative of $\eta(X) = g(X, \xi)$ along U gives

$$(3.37) \quad g(\mathfrak{L}_U \xi, X) = - \left(\Lambda - \frac{\beta S^*}{2} \right) \eta(X).$$

Taking exterior derivative of $\mathfrak{L}_U \eta = \psi \eta$ yields

$$(3.38) \quad \begin{aligned} (\mathfrak{L}_U d\eta)(X, Y) &= d(\mathfrak{L}_U \eta)(X, Y) \\ &= \frac{1}{2} [X(\psi)\eta(Y) - Y(\psi)\eta(X)] + \psi d\eta(X, Y). \end{aligned}$$

Now, Lie derivative of (2.2) along U and using Definition 2.2, (3.1) and (3.38) yields

$$(3.39) \quad \begin{aligned} 2(\mathfrak{L}_U \phi)(X) + 2[2\Lambda - \beta S^* + 2\alpha(2n - 1)]\phi X &= 4\phi QX - X(\psi)\xi \\ &+ \eta(X)\nabla\psi. \end{aligned}$$

Utilizing $\phi\xi = 0$, we have $(\mathfrak{L}_U \phi)\xi = 0$ and setting $X = \xi$ in (3.39) leads to

$$\nabla\psi = \xi(\psi)\xi$$

Hence, ψ is constant on M . Thus, utilizing $\psi = \Lambda - \frac{\beta S^*}{2}$ and (2.1) in (3.39), we obtain

$$(3.40) \quad \begin{aligned} (\mathfrak{L}_U \phi)(\phi X) &= \phi(\mathfrak{L}_U \phi)(X) \\ &= -2QX + \left(\Lambda - \frac{\beta S^*}{2} + 2\alpha(2n - 1) \right) X \\ &- \left(\Lambda - \frac{\beta S^*}{2} - 2\alpha \right) \eta(X)\xi, \end{aligned}$$

where we have used $Q\phi = \phi Q$. Furthermore, taking Lie derivative of (2.1) yields

$$(3.41) \quad (\mathfrak{L}_U \phi)(\phi X) + \phi(\mathfrak{L}_U \phi)(X) = (\mathfrak{L}_U \eta)(X)\xi + \eta(X)\mathfrak{L}_U \xi.$$

Combining Definition 2.2 and equations (2.8), (3.40) and (3.41), we get

$$(3.42) \quad 2\alpha Ric = \left[\Lambda - \frac{\beta S^*}{2} + 2\alpha(2n - 1) \right] g - \left(\Lambda - \frac{\beta S^*}{2} - 2\alpha \right) \eta \otimes \eta.$$

Utilizing (3.42) in (3.39) yields $\mathfrak{L}_U \phi = 0$ which implies that U leaves ϕ invariant. Taking Lie derivative of the possible volume form $\omega = \eta \wedge (d\eta)^n \neq 0$ along U yields $\mathfrak{L}_U \omega = (n + 1)\psi\omega$. Invoking the result $\mathfrak{L}_U \omega = (div U)\omega$ implies

$\operatorname{div}U = (n + 1)\psi$ and then integrating it over a compact M where we applied divergence theorem to get $\psi = 0$ and thus $\Lambda = \frac{\beta S^*}{2}$. Hence, (3.42) becomes

$$\operatorname{Ric} = (2n - 1)g + \eta \otimes \eta,$$

which gives $S = 4n^2$. Thus, M is $*$ -Ricci flat and $S^* = 0$. Further, V is Killing and $\Lambda = 0$. Moreover, from (3.37) and (2.8), $U(\eta) = U(\xi) = 0$ which leads to the following theorem.

Theorem 3.9. *If U is an infinitesimal contact transformation on a Sasakian manifold M admitting almost $*$ -RYS with $\alpha \neq 0$, then M is $*$ -Ricci flat and $S = 4n^2$. Moreover, U becomes an infinitesimal automorphism and the soliton is steady for any values of β .*

Again, recall the formula

$$(3.43) \quad \nabla_Y \nabla_X U - \nabla_{\nabla_Y X} U + R(U, Y)X = (\mathfrak{L}_U \nabla)(Y, X).$$

Setting $X = Y = \xi$ in the foregoing equation and utilizing Proposition 3.3, we obtain

$$(3.44) \quad \nabla_\xi \nabla_\xi U + R(U, \xi)\xi = \xi(2\Lambda - \beta S^*)\xi - \nabla \left(\Lambda - \frac{\beta S^*}{2} \right).$$

Suppose that U is a Jacobi field, that is,

$$\nabla_\xi \nabla_\xi U + R(U, \xi)\xi = 0.$$

Let $\gamma = \Lambda - \frac{\beta S^*}{2}$. Utilizing the above equation into (3.44) yields $2\xi(\gamma)\xi = \nabla\gamma$. Also,

$$X(\xi(\gamma))\eta(Y) - \xi(\gamma)g(\phi X, Y) = \frac{1}{2}g(\nabla_X \nabla \gamma, Y).$$

Making use of the symmetric and anti-symmetric properties of $Hess_\gamma$ and ϕ respectively, it follows that

$$\xi(\gamma)d\eta(X, Y) = 0, \forall X, Y \perp \xi.$$

This implies that $\xi(\gamma) = 0$ and consequently, $\nabla\gamma = 0$ and hence $\gamma = \Lambda - \frac{\beta S^*}{2}$ is constant. Thus, we can state the following result.

Theorem 3.10. *If U is a Jacobi field along trajectories of ξ on a Sasakian manifold M admitting almost $*$ -RYS, then the soliton reduces to $*$ -RYS provided $\alpha \neq 0$.*

Let us now see an example of a Sasakian manifold satisfying gradient almost $*$ -RYS.

Example 3.11. From Example 3.1 of Ghosh and Patra [9], we see that $*$ -Ricci tensor on a Sasakian manifold satisfies the equation

$$(3.45) \quad \operatorname{Ric}^*(X, Y) = [(n + 1)c - (n - 1)]g(X, Y), \forall X, Y \perp \xi$$

where c is a constant ϕ sectional curvature. Again, using Example 4.1 of [9], we define a vector field U on the unit sphere \mathbb{S}^{2n+1} such that $U = -D\gamma + \omega\xi$, where D is the gradient operator on the sphere and ω is constant. It follows that U is conformal from Obata's theorem and (2.8). On applying a D -homothetic deformation to the unit sphere, we obtain a Sasakian structure on \mathbb{S}^{2n+1} with constant $c = \frac{4}{a} - 3$. Now, choosing $a = \frac{2(n+1)}{2n+1}$, we also observe that from (3.45) the \ast -Ricci tensor vanishes and thus the Ricci tensor satisfies (3.18). Hence, this example satisfies Theorem 3.2.

4. Conclusion

Throughout the paper, we studied almost \ast -Ricci-Yamabe solitons with $\alpha \neq 0$ on a Sasakian manifold M . Following the method used in [7] and extending their results, we give analytic answer to the question raised at the beginning of this paper and hence we proved that if a complete Sasakian manifold admits almost \ast -Ricci-Yamabe soliton and gradient almost \ast -Ricci-Yamabe soliton as its metric, then it is isometric to the unit sphere \mathbb{S}^{2n+1} under the condition that α is non-zero. Furthermore, we obtained certain conditions for the soliton to become steady. Also, we found that if the potential vector field U is assumed to be an infinitesimal contact transformation, it becomes an infinitesimal automorphism. Lastly, we give an example constructed in [9] to verify our results. However, we studied the solitons only on a Sasakian manifold and found the results, further work of the almost \ast -Ricci-Yamabe solitons on Riemannian manifold can be carried out and is highly suggested.

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