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S-asymptotic ω-periodic solution for a coupled integro-differential equations with nonlocal conditions

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Abstract. The purpose of the present paper is to study the existence and attractivity of S-asymptotic ω-periodic mild solution to semilinear integro-differential systems with nonlocal conditions via resolvent operators in the sense given by Grimmer. The existence, as well as the uniqueness results are established by means of Perov and Schaefer fixed point theorems combined with a vector approach that uses matrices that converge to zero which was given in generalized Banach space. The obtained result is illustrated by an example at the end.

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1. Introduction

In 1964, Perov has extended the classical Banach contraction principle for contractive maps on vector-valued metrics [23]. In [15], R. Graef et al. gave the vector versions of some fixed point theorems, like Schaefer's fixed point theorem, in a vector Banach space. In recent years, many authors have investigated the existence of solutions for systems of ordinary differential, integral and semi-linear differential equations using the vector version fixed point theorems, see [6, 11, 15, 21, 22, 26, 2] and the references therein.

Physical problems inspired the nonlocal problems. Indeed, it is proved that nonlocal problems outperform standard Cauchy problems in applications. They are used to construct mathematical models for the evolution of phenomena

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such as nonlocal neutral networks, nonlocal pharmacokinetics, nonlocal pollution, and nonlocal combustion, among others. The nonlocal Cauchy problem has been studied first by Byszewski in 1991 (see, [7, 8, 9, 10]). Then, Balachandran and his collaborators have considered various classes of nonlinear integrodifferential systems [1]. Numerous authors have investigated qualitative aspects of many differential equations with nonlocal conditions, such as existence, uniqueness, and stability (see, [3, 4, 27]).

The existence of S-asymptotically almost periodic solutions is one of the most interesting subjects in qualitative theory of differential equations, due to its mathematical curiosity as well as its applications in physics and mathematical biology, among other areas. The interested reader is referred to [19, 18, 24, 29] and the sources given therein.

The goal of this paper is to study the existence and attractivity of S-Asymptotic ω-Periodic mild solutions for integrodifferential system with nonlocal conditions via resolvent operators of the form:

$$(1.1)\qquad \begin{cases} \vartheta'(\zeta) = A_1\vartheta(\zeta) + f_1\left(\zeta, \vartheta(\zeta), \varphi(\zeta), \Psi_1(\vartheta(\zeta), \varphi(\zeta))\right) \\ \qquad\qquad + \displaystyle\int_0^{\zeta} B_1(\zeta - s)\vartheta(s)ds, \text{ for } \zeta \in \Omega, \\ \varphi'(\zeta) = A_2\varphi(\zeta) + f_2\left(\zeta, \vartheta(\zeta), \varphi(\zeta), \Psi_2(\vartheta(\zeta), \varphi(\zeta))\right) \\ \qquad\qquad + \displaystyle\int_0^{\zeta} B_2(\zeta - s)\varphi(s)ds, \text{ for } \zeta \in \Omega, \\ \vartheta(0) = \vartheta_0 + \Upsilon_1(\vartheta, \varphi), \\ \varphi(0) = \varphi_0 + \Upsilon_2(\vartheta, \varphi), \end{cases}$$

where $\Omega = [0, +\infty)$, and for $i = 0, 1$, $A_i : D(A_i) \subset \Xi \to \Xi$ are the infinitesimal generators of a strongly continuous semigroup $\{T_i(\zeta)\}_{t \geq 0}$, $B_i(\zeta)$ are closed linear operators with domain $D(A_i) \subset D(B_i(\zeta))$, the operators Ψ_i are defined by

$$\Psi_i(\vartheta, \varphi)(\zeta) = \int_0^a g_i(\zeta, s, \vartheta(s), \varphi(s))ds, \quad a > 0.$$

The nonlinear terms $f_i : \Omega \times \Xi \times \Xi \times \Xi \to \Xi$, $\Upsilon_i \in BC(\Omega, \Xi) \times BC(\Omega, \Xi) \to \Xi$, are given functions, $BC(\Omega, \Xi)$ is a Banach space defined later, and $(\Xi, \|\cdot\|)$ a Banach space.

The following is how the paper is structured: Section 2 contains various important results and definitions required for this work. In Section 3, we present our primary existence results for problem (1.1). Lastly, an illustration is provided in Section 4 to show the application of our findings.

2. Preliminaries

Let $BC(\Omega,\Xi)$ denotes the Banach space of all bounded and continuous functions y mapping $\Omega := [0,+\infty)$ into Ξ, with

$$\|\xi\|_\infty = \sup_{\zeta\in\Omega} \|\xi(\zeta)\|.$$

A measurable function $\xi : [0;+\infty) \to \Xi$ is Bochner integrable if and only if $\|\xi\|$ is Lebesgue integrable. Let us denote by $L^1([0;+\infty),\Xi)$ the Banach space of measurable functions $\xi : [0;+\infty) \to \Xi$ which are Bochner integrable, with the norm

$$\|\xi\|_{L^1} = \int_0^{+\infty} \|\xi(\zeta)\| d\zeta.$$

2.1. Generalized Banach space

Definition 2.1. Let X be a vector metric space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A map $\|\cdot\| : X \to \mathbb{R}^n_+$ is called a norm on X if it satisfies the following properties:

- (i) $\|x\| = 0$ then $x = (0,\dots,0)$;
- (ii) $\|\lambda x\| = |\lambda|\|x\|$ for $x \in X, \lambda \in \mathbb{K}$;
- (iii) $\|x+y\| \leq \|x\| + \|y\|$ for every $x, y \in X$.

Remark 2.2. The pair $(X, \|\cdot\|_X)$ is called a generalized normed space. If the generalized metric generated by $\|\cdot\|_X$ (*i.e* $d(x,y) = \|x-y\|_X$) is complete then the espace $(X, \|\cdot\|_X)$ is called a generalized Banach space, where

$$\|x-y\|_X = \begin{pmatrix} \|x-y\|_1 \\ \vdots \\ \|x-y\|_n \end{pmatrix}.$$

Let $X = BC(\Omega,\Xi) \times BC(\Omega,\Xi)$ be endowed with the vector norm $\|\cdot\|_{X\times X}$ defined by $\|v\|_{X\times X} = (\|u_1\|_{BC}, \|u_2\|_{BC})$ for $v = (u_1,u_2)$. It is clear that $(BC(\Omega,\Xi)\times BC(\Omega,\Xi), \|\cdot\|_{X\times X})$ is a generalized Banach space. In the case of generalized Banach spaces in the sense of Perov, the notations of convergent sequence, Cauchy sequence, completeness, open and closed subset are similar to those for usual metric spaces.

Definition 2.3. A square matrix M of real numbers is said to be convergent to zero if and only if $M^n \to 0$ as $n \to +\infty$.

Theorem 2.4. *A square matrix M of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in$ C with det $(M - \lambda I) = 0$, where I denotes the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.*

Lemma 2.5 ([28]). *Let $M \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$. Then, the following assertions are equivalent:*

- *M is convergent towards zero,*
- *$M^k \to 0$ as $k \to +\infty$,*
- *The matrix $(I - M)$ is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots$$

- *The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Definition 2.6. Let $Q \in \mathcal{M}_{2\times 2}(\mathbb{R})$ is said to be order preserving (or positive) if $p_1 \leq p_0, q_1 \leq q_0$ imply

$$Q\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \geq Q\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

in the sense of components.

Lemma 2.7 ([25]). *Let*

$$Q = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

where $a, b, c, d \geq 0$ and $\det Q > 0$. Then, Q^{-1} is order preserving.

We consider the following Cauchy problem:

$$\begin{cases} w'(\zeta) = Aw(\zeta) + \int_0^\zeta B(\zeta - s)w(s)ds; \quad \text{for } \zeta \geq 0, \\ w(0) = w_0 \in \Xi. \end{cases} \tag{2.1}$$

The existence and properties of a resolvent operator have been discussed in [12, 16, 17]. In what follows, we suppose the following assumptions:

($R1$) A is the infinitesimal generator of a uniformly continuous semigroup $\{T(\zeta)\}_{\zeta > 0}$,

($R2$) For all $\zeta \geq 0, B(\zeta)$ is closed linear operator from $D(A)$ to Ξ and $B(\zeta) \in B(D(A), \Xi)$. For any $y \in D(A)$, the map $\zeta \to B(\zeta)y$ is bounded, differentiable and the derivative $\zeta \to B'(\zeta)y$ is bounded uniformly continuous on $\mathbb{R}^+$.

Theorem 2.8 ([16]). *Assume that ($R1$) – ($R2$) hold. Then, there exists a unique resolvent operator for the Cauchy problem (2.1).*

Lemma 2.9 ([16]). *Assume that ($R1$) – ($R2$) hold. The resolvent operator $(\Phi(\zeta))_{\zeta \geq 0}$ is compact for $\zeta > 0$ if and only if the semigroup $(T(\zeta))_{\zeta \geq 0}$ is compact for $\zeta > 0$.*

Lemma 2.10 ([20]). *Assume that* (R1) – ($R2$) *hold. If the resolvent operator $(\Phi(\zeta))_{\zeta \geq 0}$ is compact for $\zeta > 0$, then it is norm continuous (or continuous in the uniform operator topology) for $\zeta > 0$.*

Lemma 2.11 ([20])**.** *For any $\alpha > 0$ there exists a constant $\gamma = \gamma(\alpha)$ such that*

$$\|\Phi(\zeta + h) - \Phi(h)\Phi(\zeta)\|_{\mathcal{L}(\mathrm{X})} \leq \gamma h, \quad \textit{for} \quad 0 \leq h \leq \zeta \leq \alpha.$$

Now, we introduce some concepts and properties related to S-asymptotically ω-periodic functions.

Definition 2.12. A function $f \in BC([0, +\infty), \Xi)$ is called S-asymptotically ω-periodic if there exists $\omega > 0$ such that

$$\lim_{\zeta \to +\infty} \|f(\zeta + \omega) - f(\zeta)\| = 0.$$

We denote by $Y = SAP_\omega(\Xi)$ the set of all S-asymptotically ω-periodic functions from $[0, +\infty)$ to Ξ. Note that $SAP_\omega(\Xi)$ is a Banach space with the sup-norm $\| \cdot \|_Y$.

Definition 2.13. A continuous function $f : [0, +\infty) \times \Xi \to \Xi$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if for each bounded subset K of Ξ, the set $\{f(\zeta, x) : (\zeta, x) \in [0, +\infty) \times K\}$ is bounded, and $\lim_{\zeta \to +\infty}(f(\zeta + \omega, x) - f(\zeta, x)) = 0$ uniformly in $\Xi \in K$.

Definition 2.14. A continuous function $f : [0, +\infty) \times \Xi \to \Xi$ is said to be asymptotically uniformly continuous on bounded sets if for every $\varepsilon > 0$ and for any bounded set $K \subseteq \Xi$, there exist constants $L_{\varepsilon,K} \geq 0$ and $\delta = \delta_{\varepsilon,K} > 0$ such that

$$\|f(\zeta, x) - f(\zeta, y)\| \leq \varepsilon,$$

for all $\zeta \geq L_{\varepsilon,K}$ and $x, y \in K$ with $\|x - y\| \leq \delta_{\varepsilon,K}$.

Lemma 2.15 ([18])**.** *Let Ξ, F be two Banach spaces, and $f : [0, +\infty) \times \Xi \to F$ be a function uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. If $x \in SAP_\omega(\Xi)$, then*

$$\lim_{\zeta \to +\infty} (f(\zeta + \omega, x(\zeta + \omega)) - f(\zeta, x(\zeta)) = 0.$$

Theorem 2.16 (Perov's fixed point theorem [23])**.** *Let (X, d) be a complete generalized metric space, with $d : X \times X \longrightarrow \mathbb{R}^n$ and let $N : X \longrightarrow X$, such that*

$$d(N(x), N(y)) \leq Md(x, y),$$

for all $x, y \in X$ and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is $M^k \longrightarrow 0$ as $k \longrightarrow +\infty$, then N has a unique fixed point $x_* \in X$, and we have

$$d\left(N^k\left(x_0\right), x_*\right) \leq M^k(I - M)^{-1} d\left(N\left(x_0\right), x_0\right)$$

for every $x_0 \in X$ and $k \geq 1$.

Theorem 2.17 (Schaefer's type theorem [15])**.** *Let $(X, \| \cdot \|_X)$ be a generalized Banach space and $N : X \to X$ is a continuous compact mapping. Moreover assume that the set*

$$\mathcal{A} = \{x \in X : x = \lambda N(x) \quad \textit{for some } \lambda \in (0, 1)\}$$

is bounded. Then N has a fixed point.

3. The main result

In this section we discuss the existence of a mild solution for the problem (1.1).

3.1. Existence of solutions

We obtain existence and uniqueness results by employing Perov's fixed point theorem.

Definition 3.1. A function $(\vartheta, \varphi) \in BC(\Omega, \Xi) \times BC(\Omega, \Xi) = X \times X$ is called a mild solution of problem (1.1) if it satisfies

$$\begin{aligned}\vartheta(\zeta) &= \Phi_1(\zeta)(\vartheta_0 + \Upsilon_1(\vartheta, \varphi)) \\ &\quad + \int_0^\zeta \Phi_1(\zeta - s) f_1\left(s, \vartheta(s), \varphi(s), \Psi_1(\vartheta(s), \varphi(s))\right) ds; \quad \zeta \in \Omega, \\ \varphi(\zeta) &= \Phi_2(\zeta)(\varphi_0 + \Upsilon_2(\vartheta, \varphi)) \\ &\quad + \int_0^\zeta \Phi_2(\zeta - s) f_2\left(s, \vartheta(s), \varphi(s), \Psi_2(\vartheta(s), \varphi(s))\right) ds; \quad \zeta \in \Omega.\end{aligned}$$

The hypotheses:

$(H1)$ For $i = 1, 2,\ f_i : \Omega \times \Xi \times \Xi \times \Xi \to \Xi$ are Carathéodory functions and there exist $p_i,\ q_i \in L^1(\Omega, \mathbb{R}^+)$ and a continuous nondecreasing functions $\psi_i, \phi_i : \Omega \to (0, +\infty)$ such that

$$\begin{aligned}\|f_i(\zeta, u, v, w(u, u)) - f_i(\zeta, \widehat{u}, \widehat{v}, w(\widehat{u}, \widehat{v}))\| \le p_i(\zeta)\psi_i(\|u - \widehat{u}\|) \\ + q_i(\zeta)\phi_i(\|v - \widehat{v}\|),\end{aligned}$$

$$\text{for } u,\ \widehat{u},\ v,\ \widehat{v},\ w \in \Xi,$$

with

$$\psi_i(\zeta) \le \zeta,\ \phi_i \le \zeta, \text{ and } f_i^0 = \|f_i(\cdot, 0, 0, 0)\| \in L^1(\Omega, \mathbb{R}^+).$$

$(H2)$ For $i = 1, 2$, $g_i : D_h \times \Xi \times \Xi \to \Xi$ are continuous and there exist continuous functions $g_{c_i},\ \widehat{g}_{c_i} : \Omega \to (0, +\infty)$ such that

$$\|g_i(\zeta, s, u, v) - g_i(\zeta, s, \widehat{u}, \widehat{v})\| \le g_{c_i}(\zeta)\|u - \widehat{u}\| + \widehat{g}_{c_i}(\zeta)\|v - \widehat{v}\|,$$

$$\text{for each } (\zeta, s) \in D_{g_i} \text{ and } u,\ \widehat{u},\ v,\ \widehat{v} \in \Xi,$$

with

$$\begin{aligned}\max\Big\{\sup_{\zeta\in\Omega}\{g_{c_i}(\zeta)\}, \sup_{\zeta\in\Omega}\{\widehat{g}_{c_i}(\zeta)\}, \sup_{(\zeta,s)\in D_{g_i}}\{\|g_i(\zeta, s, 0, 0)\|\}\Big\} \\ = \max\{g_{c_i}^*, \widehat{g}_{c_i}^*, g_i^*\} < +\infty.\end{aligned}$$

$(H3)$ For $i=1,2$, $\Upsilon_i : X\times X \to \Xi$ are Lipschitz functions, i.e there exist a positive constants $L_{\Upsilon_i}, \widehat{L}_{\Upsilon_i}$, such that

$$||\Upsilon_i(u,v)-\Upsilon_i(\widehat{u},\widehat{v})|| \leq L_{\Upsilon_i}\|u-\widehat{u}\|_X+\widehat{L}_{\Upsilon_i}\|v-\widehat{v}\|_X, \text{ for all } u,\ \widehat{u},\ v,\ \widehat{v}\in X.$$

$(H4)$ Assume that $(R1)-(R2)$ hold, and there exist $M_{\Phi_i}\geq 1$ and $\beta_i \geq 0$, such that

$$\|\Phi_i(\zeta)\|_{B(\Xi)} \leq M_{\Phi_i} e^{-\beta_i t}.$$

Theorem 3.2. *Assume that the conditions $(H1)-(H4)$ are satisfied, and the matrix*

$$\begin{pmatrix} M_{\Phi_1}\left(L_{\Upsilon_1}+\|p_1\|_{L^1}\right) & M_{\Phi_1}\left(\widehat{L}_{\Upsilon_1}+\|q_1\|_{L^1}\right) \\ \\ M_{\Phi_2}\left(L_{\Upsilon_2}+\|p_2\|_{L^1}\right) & M_{\Phi_2}\left(\widehat{L}_{\Upsilon_2}+\|q_2\|_{L^1}\right) \end{pmatrix}$$

converges to zero. Then, the system (1.1) has a unique mild solution.

Proof. Transform the problem (1.1) into a fixed point problem. Consider the operator
$\Theta : BC(\Omega,\Xi)\times BC(\Omega,\Xi) \to BC(\Omega,\Xi)\times BC(\Omega,\Xi)$ define by:

$$\Theta(\vartheta(\zeta),\varphi(\zeta)) = (\Theta_1(\vartheta(\zeta),\varphi(\zeta)),\Theta_2(\vartheta(\zeta),\varphi(\zeta))),$$

where

$$\begin{cases} \Theta_1(\vartheta(\zeta),\varphi(\zeta)) = \Phi_1(\zeta)(\vartheta_0+\Upsilon_1(\vartheta,\varphi)) \\ \qquad\qquad + \displaystyle\int_0^\zeta \Phi_1(\zeta-s) f_1\left(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s))\right) ds, \\ \Theta_2(\vartheta(\zeta),\varphi(\zeta)) = \Phi_2(\zeta)(\varphi_0+\Upsilon_2(\vartheta,\varphi)) \\ \qquad\qquad + \displaystyle\int_0^\zeta \Phi_2(\zeta-s) f_2\left(s,\vartheta(s),\varphi(s),\Psi_2(\vartheta(s),\varphi(s))\right) ds. \end{cases}$$

We show that Θ was well defined. Given $(\vartheta,\varphi)\in X\times X,\ \zeta\in\Omega$, we have

$$\begin{aligned} \|\Theta_1(\vartheta(\zeta),\varphi(\zeta))\| &\leq \|R_1(\zeta)\|(\|\vartheta_0\|+\|\Upsilon_1(\vartheta,\varphi)\|) \\ &\quad + \int_0^\zeta \|\Phi_1(\zeta-s)\|\|f_1\left(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s))\right)\|ds. \end{aligned}$$

From $(H1)$, we have

$$\begin{aligned} ||f_1(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s)))|| &\leq p_1(s)\psi_1(\|\vartheta(s)\|)+q_1(s)\phi_1(\|\varphi(s)\|) \\ &\quad + \|f_1(s,0,0,0)\|. \end{aligned}$$

Also, we have

$$||\Upsilon_1(\vartheta,\varphi)|| \leq L_{\Upsilon_1}\|\vartheta\|_X + \widehat{L}_{\Upsilon_1}\|\varphi\|_X + \Upsilon_1^0.$$

Then, we get

$$\begin{aligned}\|\Theta_1(\vartheta(\zeta),\varphi(\zeta))\| \leq{}& M_{\Phi_1}\left(\|\vartheta_0\| + L_{\Upsilon_1}\|\vartheta\|_X + \widehat{L}_{\Upsilon_1}\|\varphi\|_X + \Upsilon_1^0\right)\\ &+ M_{\Phi_1}\left(\|p_1\|_{L^1}\psi_1(\|\vartheta\|_X) + \|q_1\|_{L^1}\phi_1(\|\varphi\|_X)\right)\\ &+ M_{\Phi_1}\int_0^\zeta f_1^0(s)ds.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\|\Theta_2(\vartheta(\zeta),\varphi(\zeta))\| \leq{}& M_{\Phi_2}\left(\|\varphi_0\| + L_{\Upsilon_2}\|\vartheta\|_X + \widehat{L}_{\Upsilon_2}\|\varphi\|_X + \Upsilon_2^0\right)\\ &+ M_{\Phi_2}\left(\|p_2\|_{L^1}\psi_2(\|\vartheta\|_X) + \|q_2\|_{L^1}\phi_2(\|\varphi\|_X)\right)\\ &+ M_{\Phi_2}\int_0^\zeta f_2^0(s)ds.\end{aligned}$$

Thus,

$$\|\Theta(\vartheta,\varphi)\|_{X\times X} < +\infty.$$

Obviously, the fixed points of operator Θ are mild solutions of the problem (1.1).

We shall use the Perov's fixed point theorem to prove that Θ has a fixed point. Let $(\vartheta,\varphi),(\widehat{\vartheta},\widehat{\varphi}) \in X\times X$, then $(H1)$ and $(H2)$ imply

$$\begin{aligned}&\|\Theta_1(\vartheta(\zeta),\varphi(\zeta)) - \Theta_1(\widehat{\vartheta}(\zeta),\widehat{\varphi}(\zeta))\|\\ &\quad\leq M_{\Phi_1}(L_{\Upsilon_1}\|\vartheta-\widehat{\vartheta}\|_X + \widehat{L}_{\Upsilon_1}\|\varphi-\widehat{\varphi}\|_X)\\ &\qquad + M_{\Phi_1}\int_0^\zeta p_1(s)\psi_1(\|\vartheta(s)-\widehat{u}(s)\|) + q_1(s)\phi_1(\|v(s)-\widehat{v}(s)\|)ds\\ &\quad\leq M_{\Phi_1}\left(L_{\Upsilon_1} + \|p_1\|_{L^1}\right)\|\vartheta-\widehat{\vartheta}\|_X + M_{\Phi_1}\left(\widehat{L}_{\Upsilon_1} + \|q_1\|_{L^1}\right)\|\varphi-\widehat{\varphi}\|_X.\end{aligned}$$

Similarly, we get

$$\begin{aligned}&\|\Theta_2(\vartheta(\zeta),\varphi(\zeta)) - \Theta_2(\widehat{\vartheta}(\zeta),\widehat{\varphi}(\zeta))\|\\ &\quad\leq M_{\Phi_2}\left(L_{\Upsilon_2} + \|p_2\|_{L^1}\right)\|\vartheta-\widehat{\vartheta}\|_X + M_{\Phi_2}\left(\widehat{L}_{\Upsilon_2} + \|q_2\|_{L^1}\right)\|\varphi-\widehat{\varphi}\|_X.\end{aligned}$$

Then, we have

$$\begin{aligned}&\|\Theta(\vartheta,\varphi) - \Theta(\widehat{\vartheta},\widehat{\varphi})\|_{X\times X}\\ &\quad\leq \begin{pmatrix} M_{\Phi_1}\left(L_{\Upsilon_1} + \|p_1\|_{L^1}\right) & M_{\Phi_1}\left(\widehat{L}_{\Upsilon_1} + \|q_1\|_{L^1}\right)\\ \\ M_{\Phi_2}\left(L_{\Upsilon_2} + \|p_2\|_{L^1}\right) & M_{\Phi_2}\left(\widehat{L}_{\Upsilon_2} + \|q_2\|_{L^1}\right)\end{pmatrix}\begin{pmatrix}\|\vartheta-\widehat{\vartheta}\|_X\\ \|\varphi-\widehat{\varphi}\|_X\end{pmatrix}.\end{aligned}$$

Applying now Theorem 2.16, we conclude that Θ has a unique fixed point, which is a mild solution of problem (1.1). ☐

Now, we study the existence of a mild solution by using Schaefer's fixed point theorem.
In the assumption (H1), we suppose that for every $M_1, M_2 \geq 0$, we have

$$\lim_{\zeta \to +\infty} \sup_{\zeta \in \Omega} \int_0^{\zeta} e^{-\beta(\zeta - s)} \big[M_1 p_i(s) + M_2 q_i(s)\big] ds = 0.$$

Let

$$\widehat{M} = \begin{pmatrix} 1 - M_{\Phi_1}\left(L_{\Upsilon_1} + \|p_1\|_{L^1}\right) & -M_{\Phi_1}\left(\widehat{L}_{\Upsilon_1} + \|q_1\|_{L^1}\right) \\ \\ -M_{\Phi_2}\left(L_{\Upsilon_2} + \|p_2\|_{L^1}\right) & 1 - M_{\Phi_2}\left(\widehat{L}_{\Upsilon_2} + \|q_2\|_{L^1}\right) \end{pmatrix}.$$

Theorem 3.3. *Assume that the conditions* $(H1)$*-*$(H4)$ *are satisfied and for* $i = 1, 2$,

$$M_{\Phi_i} \max\{L_{\Upsilon_i} + \|p_i\|_{L^1}, \widehat{L}_{\Upsilon_i} + \|q_i\|_{L^1}\} < 1, \ \textit{and} \ \det\left(\widehat{M}\right) > 0.$$

Then, the system (1.1) *has at least one mild solution.*

Proof. We use Theorem 2.17 to prove that Θ has a fixed point. We have divided the proof into four steps.

Step 1 : Θ is continuous.
Let $(\vartheta_n, \varphi_n)_{n \in \mathbb{N}}$ be a couple of sequences such that $(\vartheta_n, \varphi_n) \to (\vartheta^*, \varphi^*)$, then for $\zeta \in \Omega$, we have

$$\begin{aligned} &\|(\Theta_1(\vartheta_n, \varphi_n))(\zeta) - (\Theta_1(\vartheta^*, \varphi^*))(\zeta)\| \\ &\quad \leq M_{\Phi_1} \int_0^{\zeta} \|f_1\left(s, \vartheta_n(s), \varphi_n(s), \Psi_1(\vartheta_n(s), \varphi_n(s))\right) \\ &\qquad - f_1(s, \vartheta^*(s), \varphi^*(s), \Psi_1(\vartheta^*(s), \varphi^*(s)))\| ds \\ &\qquad + M_{\Phi_1}\|\Upsilon_1(\vartheta_n, \varphi_n) - \Upsilon_1(\vartheta^*, \varphi^*)\|. \end{aligned}$$

By the continuity of g_1, we get

$$g_1(\zeta, s, \vartheta_n(s), \varphi_n(s)) \to g_1(\zeta, s, \vartheta^*(s), \varphi^*(s)) \quad \textit{as} \ \ n \to +\infty.$$

And, we have

$$\begin{aligned} \|g_1(\zeta, s, \vartheta_n(s), \varphi_n(s)) - g_1(\zeta, s, \vartheta^*(s), \varphi^*(s))\| \leq g^*_{c_i}\|\vartheta_n(s) - \vartheta^*(s)\| \\ + \widehat{g}^*_{c_i}\|\varphi_n(s) - \varphi^*(s)\|. \end{aligned}$$

By Lebesgue dominated convergence theorem, we obtain

$$\int_0^{\zeta} g_1(\zeta, s, \vartheta_n(s), \varphi_n(s)) ds \to \int_0^{\zeta} g_1(\zeta, s, \vartheta^*(s), \varphi^*(s)) ds, \quad \textit{as} \ \ n \to +\infty.$$

Hence

$$\|\Theta_1(\vartheta_n, \varphi_n) - \Theta_1(\vartheta^*, \varphi^*)\|_X \to 0, \quad \textit{as} \ \ n \to +\infty.$$

Similarly, we get

$$\|\Theta_2(\vartheta_n,\varphi_n)-\Theta_2(\vartheta^*,\varphi^*)\|_X \to 0, \quad as \;\; n\to+\infty.$$

Thus, Θ is continuous.

Let B_δ be defined by

$$B_\delta=\{(\vartheta,\varphi)\in X\times X : (\|\vartheta\|_X,\|\varphi\|_X)\leq(\delta_1,\delta_2)\},$$

with $\delta_i>0$. The set B_δ is bounded, closed and convex.

Step 2 : Θ is a completely continuous operator.
Claim 1 : $\Theta(B_\delta)$ is bounded.
Let $(\vartheta,\varphi)\in B_\delta$ and $\zeta\in\Omega$, from $(H1)$-$(H3)$, it follows that

$$||f_1(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s)))||\leq p_1(s)\psi_1(\delta_1)+q_1(s)\phi_1(\delta_2)+\|f_1(s,0,0,0)\|,$$

and

$$||\Upsilon_1(\vartheta,\varphi)||\leq L_{\Upsilon_1}\delta_1+\widehat{L}_{\Upsilon_1}\delta_2+\Upsilon_1^0.$$

Then, we get

$$\begin{aligned}\|\Theta_1(\vartheta(\zeta),\varphi(\zeta))\| &\leq M_{\Phi_1}\Big(\|\vartheta_0\|+L_{\Upsilon_1}\delta_1+\widehat{L}_{\Upsilon_1}\delta_2+\Upsilon_1^0+\|p_1\|_{L^1}\psi_1(\delta_1)\\ &\quad+\|q_1\|_{L^1}\phi_1(\delta_2)\Big)+M_{\Phi_1}\int_0^\zeta f_1^0(s)ds\\ &<+\infty.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\|\Theta_2(\vartheta(\zeta),\varphi(\zeta))\| &\leq M_{\Phi_2}\Big(\|\varphi_0\|+L_{\Upsilon_2}\delta_1+\widehat{L}_{\Upsilon_2}\delta_2+\Upsilon_2^0+\|p_2\|_{L^1}\psi_2(\delta_1)\\ &\quad+\|q_2\|_{L^1}\phi_2(\delta_2)\Big)+M_{\Phi_1}\int_0^\zeta f_2^0(s)ds\\ &<+\infty.\end{aligned}$$

Thus,

$$\|\Theta(\vartheta,\varphi)\|_{X\times X}<+\infty.$$

Claim 2: The set $\Theta(B_\delta)$ is equicontinuous.
For $(\vartheta,\varphi)\in B_\delta$ and $\kappa_1,\kappa_2\in\Omega$, we have

$$\begin{aligned}&\|\Theta_1(\vartheta,\varphi)(\kappa_1)-\Theta_1(\vartheta,\varphi)(\kappa_2)\|\\ &\quad\leq\|R(\kappa_1)-R(\kappa_2)\|_{B(\Xi)}(\|\vartheta_0\|+L_{\Upsilon_1}\delta_1+\widehat{L}_{\Upsilon_1}\delta_2+\Upsilon_1^0)\\ &\qquad+\int_0^{\kappa_1}\|R(\kappa_1-s)-R(\kappa_2-s)\|_{B(\Xi)}\big(p_1(s)\psi_1(\delta_1)+q_1(s)\phi_1(\delta_2)\big)ds\\ &\qquad+M_{\Phi_i}\int_{\kappa_1}^{\kappa_2}\big(p_1(s)\psi_1(\delta_1)+q_1(s)\phi_1(\delta_2)\big)ds.\end{aligned}$$

By the strong continuity of $\Phi_1(\cdot)$ and $(H1)$, we have

$$\|\Theta_1(\vartheta,\varphi)(\kappa_1)-\Theta_1(\vartheta,\varphi)(\kappa_2)\|\to 0 \ as\ \kappa_1\to\kappa_2.$$

Similarly, we get

$$\|\Theta_2(\vartheta,\varphi)(\kappa_1)-\Theta_2(\vartheta,\varphi)(\kappa_2)\|\to 0 \ as\ \kappa_1\to\kappa_2.$$

Hence, the set $\Theta(B_\delta)$ is equicontinuous.
Claim 3: The set $\Theta(B_\delta)$ is equiconvergent.
Let $(\vartheta,\varphi)\in B_\delta$ and $\zeta\in\Omega$. By $(H1),(H3)$, we have

$$\begin{aligned}\|\Theta_1(\vartheta(\zeta),\varphi(\zeta))\| &\le M_{\Phi_1}e^{-\beta_1 t}\left(\|\vartheta_0\|+L_{\Upsilon_1}\delta_1+\widehat{L}_{\Upsilon_1}\delta_2+\Upsilon_1^0\right)\\ &\quad+M_{\Phi_1}\int_0^\zeta e^{-\beta_1(\zeta-s)}\left(p_1(s)\psi_1(\delta_1)+q_1(s)\phi_1(\delta_2)\right)\\ &\quad+M_{\Phi_1}\int_0^\zeta f_1^0(s)ds\\ &\xrightarrow[\zeta\to+\infty]{} M_{\Phi_1}\int_0^\zeta f_1^0(s)ds.\end{aligned}$$

Then

$$\|\Theta_1(\vartheta(\zeta),\varphi(\zeta))-\Theta_1(\vartheta(+\infty),\varphi(+\infty))\|\xrightarrow[\zeta\to+\infty]{}0.$$

Similarly, we get

$$\|\Theta_2(\vartheta(\zeta),\varphi(\zeta))-\Theta_2(\vartheta(+\infty),\varphi(+\infty))\|\xrightarrow[\zeta\to+\infty]{}0.$$

Claim 4: The set $\Theta(B_\delta(\zeta))$ is relatively compact.
If $\zeta=0$, $\{\Theta(\vartheta(0),\varphi(0))\ :\ (\vartheta,\varphi)\in B_\delta\}$ is compact. Let $\zeta>0$ be fixed, $\eta\in(0,\zeta)$ and $(\vartheta,\varphi)\in B_\delta$, we define the operator

$$\Theta_\eta(\vartheta(\zeta),\varphi(\zeta))=(\Theta_1^\eta(\vartheta(\zeta),\varphi(\zeta)),\Theta_2^\eta(\vartheta(\zeta),\varphi(\zeta))),$$

where

$$\begin{cases}\Theta_1^\eta(\vartheta(\zeta),\varphi(\zeta))=\Phi_1(\eta+\zeta)(\vartheta_0+\Upsilon_1(\vartheta,\varphi))\\ \qquad+\Phi_1(\eta)\displaystyle\int_0^{\zeta-\eta}\Phi_1(\zeta-\eta-s)f_1\left(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s))\right)ds,\\ \Theta_2^\eta(\vartheta(\zeta),\varphi(\zeta))=\Phi_2(\eta+\zeta)(\varphi_0+\Upsilon_2(\vartheta,\varphi))\\ \qquad+\Phi_2(\eta)\displaystyle\int_0^{\zeta-\eta}\Phi_2(\zeta-\eta-s)f_2\left(s,\vartheta(s),\varphi(s),\Psi_2(\vartheta(s),\varphi(s))\right)ds,\end{cases}$$

and the operator

$$\widehat{\Theta}_\eta(\vartheta(\zeta),\varphi(\zeta))=(\widehat{\Theta}_1^\eta(\vartheta(\zeta),\varphi(\zeta)),\widehat{\Theta}_2^\eta(\vartheta(\zeta),\varphi(\zeta))),$$

where

$$\begin{cases} \widehat{\Theta}_1^\eta(\vartheta(\zeta),\varphi(\zeta)) = \Phi_1(\eta+\zeta)(\vartheta_0+\Upsilon_1(\vartheta,\varphi)) \\ \qquad\qquad + \displaystyle\int_0^{\zeta-\eta} \Phi_1(\zeta-s) f_1\left(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s))\right) ds, \\ \widehat{\Theta}_2^\eta(\vartheta(\zeta),\varphi(\zeta)) = \Phi_2(\eta+\zeta)(\varphi_0+\Upsilon_2(\vartheta,\varphi)) \\ \qquad\qquad + \displaystyle\int_0^{\zeta-\eta} \Phi_2(\zeta-s) f_2\left(s,\vartheta(s),\varphi(s),\Psi_2(\vartheta(s),\varphi(s))\right) ds. \end{cases}$$

Since $\Phi_i(\cdot)$ are compact and by Lemma 2.11, the sets

$$\{\Theta_i^\eta(\vartheta(\zeta),\varphi(\zeta)) : (\vartheta,\varphi)\in B_\delta\}_{i=1,2}$$

are relatively compact. Moreover for $(\vartheta,\varphi)\in B_\delta$, we obtain

$$\begin{aligned} \|\Theta_1^\eta(\vartheta(\zeta),\varphi(\zeta)) - \widehat{\Theta}_1^\eta(\vartheta(\zeta),\varphi(\zeta))\| \le & \int_0^{\zeta-\eta} \|\Phi_1(\eta)\Phi_1(\zeta-\eta-s)-\Phi_1(\zeta-s)\| \\ & \times \|f_1\left(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s))\right)\| ds. \end{aligned}$$

From Lemma 2.11 and (H1), we get

$$\|\Theta_1^\eta(\vartheta(\zeta),\varphi(\zeta)) - \widehat{\Theta}_1^\eta(\vartheta(\zeta),\varphi(\zeta))\| \le \gamma\eta(\|p_1\|_{L^1}\psi_1(\delta_1)+\|q_1\|_{L^1}\phi_1(\delta_2)) \xrightarrow[\eta\to0]{} 0.$$

Similarly, we get

$$\|\Theta_2^\eta(\vartheta(\zeta),\varphi(\zeta)) - \widehat{\Theta}_2^\eta(\vartheta(\zeta),\varphi(\zeta))\| \le \gamma\eta(\|p_2\|_{L^1}\psi_2(\delta_1)+\|q_2\|_{L^1}\phi_2(\delta_2)) \xrightarrow[\eta\to0]{} 0.$$

Therefore, the set $\left\{\widehat{\Theta}^\eta(\vartheta(\zeta),\varphi(\zeta)) \ : \ (\vartheta,\varphi)\in B_\delta\right\}$ is precompact.

Applying this idea again, we obtain

$$\begin{aligned} &\|\Theta_1(\vartheta(\zeta),\varphi(\zeta)) - \widehat{\Theta}_1^\eta(\vartheta(\zeta),\varphi(\zeta))\| \\ &\quad \le \|\Phi_1(\zeta)-\Phi_1(\zeta+\eta)\| \left(\|\vartheta_0\| + L_{\Upsilon_1}\delta_1 + \widehat{L}_{\Upsilon_1}\delta_2 + \Upsilon_1^0\right) \\ &\qquad + M_{\Phi_1}\left(\psi_1(\delta_1)\int_{\zeta-\eta}^{\zeta} p_1(s) + \phi_1(\delta_2)\int_{\zeta-\eta}^{\zeta} q_1(s)ds\right) \\ &\quad \xrightarrow[\eta\to0]{} 0, \end{aligned}$$

and

$$\begin{aligned} &\|\Theta_2(\vartheta(\zeta),\varphi(\zeta)) - \widehat{\Theta}_2^\eta(\vartheta(\zeta),\varphi(\zeta))\| \\ &\quad \le \|\Phi_2(\zeta)-\Phi_2(\zeta+\eta)\| \left(\|\varphi_0\| + L_{\Upsilon_2}\delta_1 + \widehat{L}_{\Upsilon_2}\delta_2 + \Upsilon_2^0\right) \\ &\qquad + M_{\Phi_2}\left(\psi_2(\delta_1)\int_{\zeta-\eta}^{\zeta} p_2(s) + \phi_2(\delta_2)\int_{\zeta-\eta}^{\zeta} q_2(s)ds\right) \\ &\quad \xrightarrow[\eta\to0]{} 0. \end{aligned}$$

Thus, $\Theta(B_\delta(\zeta))$ is precompact. Consequently $\Theta(B_\delta)$ is relatively compact.

Step 3 : The following set

$$\Delta = \{(\vartheta,\varphi) \in X \times X \;:\; (\vartheta,\varphi) = \lambda\Theta(\vartheta,\varphi),\ \text{for some } \lambda \in (0,1)\},$$

is bounded.
Let $(\vartheta,\varphi) \in \Delta$ and $\lambda \in (0,1)$ be such that

$$(\vartheta(\zeta),\varphi(\zeta)) = \lambda\Theta(\vartheta(\zeta),\varphi(\zeta)).$$

Then, we have

$$\begin{cases} \vartheta(\zeta) &=& \lambda\Theta_1(\vartheta(\zeta),\varphi(\zeta)), \\ \varphi(\zeta) &=& \lambda\Theta_2(\vartheta(\zeta),\varphi(\zeta)). \end{cases}$$

Thus

$$\begin{aligned} \|\vartheta(\zeta)\| \le M_{\Phi_1}\left(\|\vartheta_0\| + L_{\Upsilon_1}\|\vartheta\|_X + \widehat{L}_{\Upsilon_1}\|\varphi\|_X + \Upsilon_1^0\right) \\ + M_{\Phi_1}\left(\|p_1\|_{L^1}\|\vartheta\|_X + \|q_1\|_{L^1}\|\varphi\|_X\right) + M_{\Phi_1}\int_0^\zeta f_1^0(s)ds. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \|\varphi(\zeta)\| \le M_{\Phi_2}\left(\|\varphi_0\| + L_{\Upsilon_2}\|\vartheta\|_X + \widehat{L}_{\Upsilon_2}\|\varphi\|_X + \Upsilon_2^0\right) \\ + M_{\Phi_2}\left(\|p_2\|_{L^1}\|\vartheta\|_X + \|q_2\|_{L^1}\|\varphi\|_X\right) + M_{\Phi_2}\int_0^\zeta f_2^0(s)ds. \end{aligned}$$

Therefore

$$\begin{aligned} &\begin{pmatrix} 1 - M_{\Phi_1}\left(L_{\Upsilon_1} + \|p_1\|_{L^1}\right) & -M_{\Phi_1}\left(\widehat{L}_{\Upsilon_1} + \|q_1\|_{L^1}\right) \\ -M_{\Phi_2}\left(L_{\Upsilon_2} + \|p_2\|_{L^1}\right) & 1 - M_{\Phi_2}\left(\widehat{L}_{\Upsilon_2} + \|q_2\|_{L^1}\right) \end{pmatrix} \begin{pmatrix} \|\vartheta\|_X \\ \|\varphi\|_X \end{pmatrix} \\ &\le \begin{pmatrix} M_{\Phi_1}\left(\|\vartheta_0\| + \displaystyle\int_0^\zeta f_1^0(s)ds\right) \\ M_{\Phi_2}\left(\|\varphi_0\| + \displaystyle\int_0^\zeta f_2^0(s)ds\right) \end{pmatrix}. \end{aligned}$$

From Lemma 2.7, $\widehat{M}^{-1}$ is order preserving, then we get

$$\begin{pmatrix} \|\vartheta\|_X \\ \|\varphi\|_X \end{pmatrix} \le \widehat{M}^{-1} \begin{pmatrix} M_{\Phi_1}\left(\|\vartheta_0\| + \displaystyle\int_0^\zeta f_1^0(s)ds\right) \\ M_{\Phi_2}\left(\|\varphi_0\| + \displaystyle\int_0^\zeta f_2^0(s)ds\right) \end{pmatrix}.$$

Thus, the set Δ is bounded, hence we deduce from Theorem 2.17, that Θ has a fixed point. Consequently, system (1.1) has at least one mild solution on Ω. □

3.2. Asymptotic periodicity of solutions

We will need the following hypothesis:

($H5$) The function f is uniformly S-asymptotically ω-periodic and asymptotically uniformly continuous on bounded sets.

Theorem 3.4. *Assume that the conditions* ($H1$)-($H5$) *are satisfied. Then the system (1.1) has an S-asymptotically ω-periodic mild solution.*

Proof. Firstly, we will prove that

$$\Theta(SAB_\omega(\Omega,\Xi)\times SAB_\omega(\Omega,\Xi))\subset SAB_\omega(\Omega,\Xi)\times SAB_\omega(\Omega,\Xi).$$

Let $(\vartheta,\varphi)\in SAB_\omega(\Omega,\Xi)$ and $\zeta\in\Omega$, then we have

$$\begin{aligned}
&\|\Theta_1(\vartheta(\zeta+\omega),\varphi(\zeta+\omega))-\Theta_1(\vartheta(\zeta),\varphi(\zeta))\|\\
&\quad\le M_{\Phi_1}(\|\vartheta_0\|+\|\Upsilon_1(\vartheta,\varphi)\|)(e^{-\beta_1(\zeta+\omega)}+e^{-\beta_1\zeta})\\
&\qquad+\left\|\int_0^{\zeta+\omega}\Phi_1(\zeta+\omega-s)f_1\left(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s))\right)ds\right.\\
&\qquad\left.-\int_0^{\zeta}\Phi_1(\zeta-s)f_1\left(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s))\right)ds\right\|\\
&\quad\le M_{\Phi_1}\left(\|\vartheta_0\|+L_{\Upsilon_1}\|\vartheta\|_Y+\widehat{L}_{\Upsilon_1}\|\varphi\|_Y+\Upsilon_1^0\right)\left(e^{-\beta_1(\zeta+\omega)}+e^{-\beta_1\zeta}\right)\\
&\qquad+\int_0^{\omega}\|\Phi_1(\zeta+\omega-s)\|\|f_1\left(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s))\right)\|ds\\
&\qquad+\int_0^{\zeta}\Phi_1(\zeta-s)f_1\left(s+\omega,\vartheta(s+\omega),\varphi(s+\omega),\Psi_1(\vartheta,\varphi)(s+\omega)\right)\\
&\qquad-f_1\left(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s))\right)ds\\
&\quad\le M_{\Phi_1}\left(\|\vartheta_0\|+L_{\Upsilon_1}\|\vartheta\|_Y+\widehat{L}_{\Upsilon_1}\|\varphi\|_Y+\Upsilon_1^0\right)\left(e^{-\beta_1(\zeta+\omega)}+e^{-\beta_1\zeta}\right)\\
&\qquad+M_{\Phi_1}\int_0^{\zeta}e^{-\beta_1(\zeta+\omega-s)}(\psi_1(\|\vartheta(s)\|)p_1(s)+\phi_1(\|\varphi(s)\|)q_1(s)+\|f_1^0(s)\|)ds\\
&\qquad+\int_0^{L_\varepsilon}\|\Phi_1(\zeta-s)\|\|f_1\left(s+\omega,\vartheta(s+\omega),\varphi(s+\omega),\Psi_1(\vartheta,\varphi)(s+\omega)\right)\\
&\qquad-f_1\left(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s))\right)\|ds\\
&\qquad+\int_{L_\varepsilon}^{\zeta}\|\Phi_1(\zeta-s)\|\|f_1\left(s+\omega,\vartheta(s+\omega),\varphi(s+\omega),\Psi_1(\vartheta,\varphi)(s+\omega)\right)\\
&\qquad-f_1\left(s,\vartheta(s),\varphi(s),\Psi_1(\vartheta(s),\varphi(s))\right)\|ds\\
&\quad\le M_{\Phi_1}\left(\|\vartheta_0\|+L_{\Upsilon_1}\|\vartheta\|_Y+\widehat{L}_{\Upsilon_1}\|\varphi\|_Y+\Upsilon_1^0\right)\left(e^{-\beta_1(\zeta+\omega)}+e^{-\beta_1\zeta}\right)\\
&\qquad+M_{\Phi_1}\int_0^{\zeta}e^{-\beta_1(\zeta+\omega-s)}(2\psi_1(\|\vartheta\|_Y)p_1(s)+2\phi_1(\|\varphi\|_Y)q_1(s)+\|f_1^0(s)\|)ds\\
&\qquad+\varepsilon M_{\Phi_1}\left(\frac{1-e^{-\beta_1(\zeta-L_\varepsilon)}}{\beta_1}\right).
\end{aligned}$$

Then

$$\|\Theta_1(\vartheta(\zeta+\omega),\varphi(\zeta+\omega))-\Theta_1(\vartheta(\zeta),\varphi(\zeta))\| \xrightarrow[\zeta\to+\infty]{} 0.$$

Similarly, we get

$$\|\Theta_2(\vartheta(\zeta+\omega),\varphi(\zeta+\omega))-\Theta_2(\vartheta(\zeta),\varphi(\zeta))\| \xrightarrow[\zeta\to+\infty]{} 0.$$

Now, let $(\vartheta^*,\varphi^*)\in BC(\Omega,\Xi)\times BC(\Omega,\Xi)$ be a solution of (1.1), then from Theorem 3.3 we have

$$(\vartheta^*(\zeta),\varphi^*(\zeta))=\Theta(\vartheta^*(\zeta),\varphi^*(\zeta)).$$

Then,

$$\begin{aligned}\|(\vartheta^*(\zeta+\omega),\varphi^*(\zeta+\omega))-(\vartheta^*(\zeta),\varphi^*(\zeta))\| = \|&\Theta(\vartheta^*(\zeta+\omega),\varphi^*(\zeta+\omega))\\ &-\Theta(\vartheta^*(\zeta),\varphi^*(\zeta))\|.\end{aligned}$$

Thus,

$$\|(\vartheta^*(\zeta+\omega),\varphi^*(\zeta+\omega))-(\vartheta^*(\zeta),\varphi^*(\zeta))\| \xrightarrow[\zeta\to+\infty]{} 0.$$

Consequently, $(\vartheta^*,\varphi^*)\in SAB_\omega(\Omega,\Xi)\times SAB_\omega(\Omega,\Xi)$. □

3.3. Attractivity of solutions

In this section we study the local attractivity of solutions for the problem (1.1).
Firstly, we introduce the following concept of attractivity of solutions as in [13].

Definition 3.5. We say that solutions of (1.1) are locally attractive if there exists a closed ball $B(z^*,\gamma)$ in the generalized Banach space $Y\times Y$ for some $z^*=(z_1^*,z_2^*)\in Y\times Y$ such that for arbitrary solutions $z=(z_1,z_2)$ and $\widetilde{z}=(\widetilde{z_1},\widetilde{z_2})$ of (1.1) belonging to $B(z^*,\gamma)$, we have that

$$\lim_{t\to+\infty}((z_1(\zeta),z_2(\zeta))-(\widetilde{z_1}(\zeta),\widetilde{z_2}(\zeta)))=0_{\mathbb{R}^2}.$$

When the last limit is uniform with respect to $B(z^*,\gamma)$, solutions of problem (1.1) are said to be uniformly locally attractive (or equivalently that solutions of (1.1) are locally asymptotically stable).

Let z^* be a solution of (1.1) and $B_\gamma=B(z^*,\gamma)$ the closed ball in $Y\times Y$, with $\gamma=(\gamma_1,\gamma_2)>0$, and $\det(M^*)>0$, where

$$M^*=\begin{pmatrix}1-2M_{\Phi_1}\left(L_{\Upsilon_1}+\|p_1\|_{L^1}\right) & -2M_{\Phi_1}\left(\widehat{L}_{\Upsilon_1}+\|q_1\|_{L^1}\right)\\ \\ -2M_{\Phi_2}\left(L_{\Upsilon_2}+\|p_2\|_{L^1}\right) & 1-2M_{\Phi_2}\left(\widehat{L}_{\Upsilon_2}+\|q_2\|_{L^1}\right).\end{pmatrix}$$

Theorem 3.6. *Suppose that hypotheses* $(H1)$-$(H4)$ *hold. Then, the solutions of problem (1.1) are uniformly locally attractive.*

Proof. For $(z_1, z_2) \in B(z^*, \gamma)$, $\zeta \in \Omega$ and by $(H1)$ and $(H3)$, we have

$$\|\Theta(z_1, z_2)(\zeta) - z^*(\zeta)\| = \begin{pmatrix} \|\Theta_1(z_1(\zeta), z_1(\zeta)) - \Theta_1(z_1^*(\zeta), z_2^*(\zeta))\| \\ \|\Theta_2(z_1(\zeta), z_1(\zeta)) - \Theta_2(z_1^*(\zeta), z_2^*(\zeta))\| \end{pmatrix}.$$

Then, we have

$$\begin{aligned} &\|\Theta_1(z_1(\zeta), z_1(\zeta)) - \Theta_1(z_1^*(\zeta), z_2^*(\zeta))\| \\ &\quad \le M_{\Phi_1}(L_{\Upsilon_1}\|z_1 - z_1^*\|_X + \widehat{L}_{\Upsilon_1}\|z_2 - z_2^*\|_X) \\ &\qquad + M_{\Phi_1}\int_0^\zeta p_1(s)\psi_1(\|z_1(s) - z_1^*(s)\|) + q_1(s)\phi_1(\|z_2(s) - z_2^*(s)\|)ds \\ &\quad \le 2M_{\Phi_1}\left(L_{\Upsilon_1} + \|p_1\|_{L^1}\right)\gamma_1 + 2M_{\Phi_1}\left(\widehat{L}_{\Upsilon_1} + \|q_1\|_{L^1}\right)\gamma_2. \end{aligned}$$

Similarly, we get

$$\begin{aligned} &\|\Theta_2(z_1(\zeta), z_1(\zeta)) - \Theta_2(z_1^*(\zeta), z_2^*(\zeta))\| \\ &\quad \le 2M_{\Phi_2}\left(L_{\Upsilon_2} + \|p_2\|_{L^1}\right)\gamma_1 + 2M_{\Phi_2}\left(\widehat{L}_{\Upsilon_1} + \|q_2\|_{L^1}\right)\gamma_2. \end{aligned}$$

Thus,

$$\begin{aligned} &\|\Theta(z_1, z_2)(\zeta) - z^*(\zeta)\| \\ &\quad \le \begin{pmatrix} 2M_{\Phi_1}\left(L_{\Upsilon_1} + \|p_1\|_{L^1}\right) & 2M_{\Phi_1}\left(\widehat{L}_{\Upsilon_1} + \|q_1\|_{L^1}\right) \\ 2M_{\Phi_2}\left(L_{\Upsilon_2} + \|p_2\|_{L^1}\right) & 2M_{\Phi_2}\left(\widehat{L}_{\Upsilon_2} + \|q_2\|_{L^1}\right) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}. \end{aligned}$$

From Lemma 2.7, M_1 is order preserving, then we obtain

$$0 < \begin{pmatrix} 1 - 2M_{\Phi_1}\left(L_{\Upsilon_1} + \|p_1\|_{L^1}\right) & -2M_{\Phi_1}\left(\widehat{L}_{\Upsilon_1} + \|q_1\|_{L^1}\right) \\ -2M_{\Phi_2}\left(L_{\Upsilon_2} + \|p_2\|_{L^1}\right) & 1 - 2M_{\Phi_2}\left(\widehat{L}_{\Upsilon_2} + \|q_2\|_{L^1}\right) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}.$$

This proves that $\Theta(B_\gamma) \subset B_\gamma$.

So, for each $(z_1, z_2), (\widetilde{z}_1, \widetilde{z}_2) \in B(z^*, \gamma)$ solutions of problem (1.1) and $\zeta \in \Omega$, we have

$$\|(z_1, z_2)(\zeta) - (\widetilde{z}_1, \widetilde{z}_2)(\zeta)\| = \|\Theta((z_1, z_2)(\zeta)) - \Theta((\widetilde{z}_1, \widetilde{z}_2)(\zeta))\|.$$

Then

$$\begin{aligned} &\|\Theta_1((z_1, z_2)(\zeta)) - \Theta_1((\widetilde{z}_1, \widetilde{z}_2)(\zeta))\| \\ &\quad \le M_{\Phi_1}(L_{\Upsilon_1}\|z_1 - \widetilde{z}_1\|_X + \widehat{L}_{\Upsilon_1}\|z_2 - \widetilde{z}_2\|_X) \\ &\qquad + M_{\Phi_1}\int_0^\zeta p_1(s)\psi_1(\|z_1(s) - \widetilde{z}_1(s)\|) + q_i(s)\phi_1(\|z_2(s) - \widetilde{z}_2(s)\|)ds \\ &\quad \le 2M_{\Phi_1}(L_{\Upsilon_1}\gamma_1 + \widehat{L}_{\Upsilon_1}\gamma_2)e^{-\beta_1\zeta} \\ &\qquad + M_{\Phi_1}\int_0^\zeta e^{-\beta_1(\zeta - s)}\left(\psi_1(\gamma_1)p_1(s) + \phi_1(\gamma_2)q_1(s)\right)ds. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\|\Theta_2((z_1,z_2)(\zeta))-\Theta_2((\widetilde{z}_1,\widetilde{z}_2)(\zeta))\| \leq{}& 2M_{\Phi_2}(L_{\Upsilon_2}\gamma_1+\widehat{L}_{\Upsilon_2}\gamma_2)e^{-\beta_2\zeta}\\ &+M_{\Phi_2}\int_0^{\zeta}e^{-\beta_2(\zeta-s)}\Big(\psi_2(\gamma_1)p_2(s)\\ &+\phi_2(\gamma_2)q_2(s)\Big)ds.\end{aligned}$$

We conclude that $\|(z_1,z_2)(\zeta)-(\widetilde{z}_1,\widetilde{z}_2)(\zeta)\|\to 0, \quad as\ \zeta\to+\infty.$ □

4. An Example

Consider the following class of partial integro-differential system:

(4.1)

$$\begin{cases}\frac{\partial}{\partial\zeta}\varpi_1(\zeta,x)-\Delta(\theta,\varpi_1(\zeta,x))-\displaystyle\int_0^{\zeta}\Gamma_1(\zeta-s)\Delta(\theta,\varpi_1(s,x))ds\\ \quad=\rho_1(\zeta)\frac{\sin\left(e^{-\sigma\zeta}\right)}{(\zeta^2+1)}\displaystyle\int_0^a\frac{\ln(1+e^{-\zeta^2})\,(1+\|\varpi_1(s)\|+\|\varpi_2(s)\|)\,e^{-\sigma(\zeta-s)}}{1+2\zeta^2+s^2}ds\\ \qquad+\frac{(\varpi_1(\zeta,x)+\varpi_2(\zeta,x))\pi\sin\left(e^{-\sigma\zeta}\right)}{77(\zeta^2+1)}\rho_1(\zeta)\ \text{ if } \zeta\in\Omega \ \text{ and } \ x\in\widetilde{\Omega}=(0,1),\\ \frac{\partial}{\partial\zeta}\varpi_2(\zeta,x)-\Delta(\widehat{\theta},\varpi_2(\zeta,x))-\displaystyle\int_0^{\zeta}\Gamma_2(\zeta-s)\Delta(\widehat{\theta},\varpi_2(s,x))ds\\ \quad=\frac{(e^a-1)\rho_2(\zeta)\cos\left(e^{-\sigma\pi}\right)(\varpi_1(\zeta,x)+\varpi_2(\zeta,x))}{198(t^2+1)e^{t+(\pi+1)a}}\\ \qquad+e^{-\sigma\zeta}\rho_2(\zeta)\displaystyle\int_0^a e^{s-\zeta-(\pi+1)a}\sqrt{\|\varpi_1(s)\|+\|\varpi_2(s)\|}ds,,\\ \varpi_i(\zeta,\tau)=0, \qquad \text{for } i=1,2,\ t\geq 0,\ \text{and } \zeta\in\partial\widetilde{\Omega},\\ ,\\ \varpi_1(0,x)=\varpi_1^0(x)+\frac{1}{107+e^{\zeta}}\sum_{j=1}^3\ln\left(1+\|\varpi_1(j)\|\right)\left(1+\|\varpi_2(j)\|\right),\\ \varpi_2(0,x)=\varpi_2^0(x)+\frac{e^{-\zeta+\frac{1}{4}}}{333}\sum_{j=1}^4(\|\varpi_1(2^j)\|+\|\varpi_2(2^j)\|),\end{cases}$$

where $\zeta\in\Omega,\ and\ x\in(0,1)$, $\Omega=\mathbb{R}^+$, $\widetilde{\Omega}=(0;1)$, $\rho_i:\mathbb{R}^+\mapsto\mathbb{R}$ are continuous, $\theta_1,\ \theta_2,\ \widehat{\theta}_1\ \widehat{\theta}_2\in\mathbb{R}$, $\sigma\geq\beta_i$.

The operator Δ defined by

$$\Delta(\theta,\xi)=\frac{\partial}{\partial x}\left(\frac{\partial\xi(\zeta,x)}{\partial x}+\theta_1\xi(\zeta,x)\right)+\theta_2\xi(\zeta,x).$$

Let

$$\Xi=H:=L^2(0,1)=\left\{\xi:(0,1)\longrightarrow\mathbb{R}:\int_0^1|\xi(x)|^2dx<+\infty\right\},$$

be the Hilbert space with the scalar product $\langle\xi,v\rangle=\displaystyle\int_0^1\xi(x)v(x)dx.$

We define the operators A_i induced on H as follows:

$$\mathrm{A}_1z=z''+\theta_1z'+\theta_2z,\quad \theta_1,\theta_2\in\mathbb{R}\text{ and }D(\mathrm{A}_1)=H^2(0,1)\cap H_0^1(0,1),$$

$$\mathrm{A}_2 z = z'' + \widehat{\theta}_1 z' + \widehat{\theta}_2 z, \quad \widehat{\theta}_1, \widehat{\theta}_2 \in \mathbb{R} \text{ and } D(\mathrm{A}_2) = H^2(0,1) \cap H_0^1(0,1),$$

which are the infinitesimal generators of an analytic semigroup $(G_1(\zeta))_{\zeta \geq 0}$, $(G_2(\zeta))_{\zeta \geq 0}$ on H. Since the semigroup generated by A_1 is analytic (respectively, A_2), then it is norm continuous for $\zeta > 0$ which implies that resolvent operator is operator-norm continuous for $\zeta > 0$ (see [14]). We define also the operators $B_i(\zeta) : H \mapsto H$ by:

$$B_i(\zeta) z = \Gamma_i(\zeta) A_i z, \ \textit{for } t \geq 0, \ z \in D(A_i).$$

As in [5, 16], for $i = 1, 2$ and some $\widehat{r_i} > r_i > 1$, we assume that $\|\Gamma_i(\zeta)\| \leq \frac{e^{-\widehat{r_i}\zeta}}{r_i}$, and $\|\Gamma_i'(\zeta)\| \leq \frac{e^{-\widehat{r_i}\zeta}}{r_i^2}$, we get that $\|\Phi_i(\zeta)\| \leq e^{-\widehat{\sigma_i}\zeta}$, where $\widehat{\sigma_i} = 1 - r_i^{-1}$. Thus $(H4)$ holds with $M_{\Phi_i} = 1$ and $\beta_i = 1 - r_i^{-1}$.

Now, if $\rho(\zeta) = 1$. We put $\varpi(\zeta)(x) = \varpi(\zeta, x)$, for $\zeta \in [0, +\infty)$, and define

$$\begin{aligned} f_1(\zeta, \phi_1, \phi_2, \Psi_1(\phi_1, \phi_2)) = {} & \frac{\rho_1(\zeta) \sin\left(e^{-\sigma\zeta}\right) \Psi_1(\phi_1, \phi_2)}{77\left(\zeta^2 + 1\right)} \\ & + \frac{\pi \rho_1(\zeta)(\phi_1(\zeta) + \phi_2(\zeta)) \sin\left(e^{-\sigma\zeta}\right)}{77(\zeta^2 + 1)} \end{aligned}$$

$$\Psi_1(\phi_1, \phi_2) = \int_0^a \frac{\ln(1 + e^{-\zeta^2})(1 + \|\phi_1(s)\| + \|\phi_2(s)\|) e^{-\sigma(\zeta - s)}}{1 + 2\zeta^2 + s^2} ds,$$

$$\begin{aligned} f_2(\zeta, \phi_1, \phi_2, \Psi_2(\phi_1, \phi_2)) = {} & \frac{\rho_2(\zeta) \cos\left(e^{-\sigma\pi}\right)(\phi_1(\zeta) + \phi_2(\zeta))}{198\left(\zeta^2 + 1\right) e^{\zeta + (\pi+1)a}} \left(e^a - 1\right) \\ & + \frac{e^{-\sigma\zeta} \rho_2(\zeta) \Psi_2(\phi_1, \phi_2)}{198\left(\zeta^2 + 1\right)}, \end{aligned}$$

$$\Psi_2(\phi_1, \phi_2) = \int_0^a e^{s - \zeta - (\pi+1)a} \sqrt{\|\phi_1(s)\| + \|\phi_2(s)\|} ds,$$

$$\Upsilon_1(\zeta, \phi_1, \phi_2) = \frac{1}{707 + e^{\zeta}} \sum_{j=1}^{4} \ln(1 + \|\phi_1(j)\|)(1 + \|\phi_2(j)\|),$$

$$\Upsilon_2(\zeta, \phi_1, \phi_2) = \frac{e^{-\zeta + \frac{1}{4}}}{333} \sum_{j=1}^{3} (\|\phi_1(2^j)\| + \|\phi_2(2^j)\|).$$

We can write (4.1) in the form

$$
\begin{cases}
\vartheta'(\zeta) = A_1\vartheta(\zeta) + f_1\left(\zeta, \vartheta(\zeta), y(\zeta), \Psi_1(\vartheta(\zeta), \varphi(\zeta))\right) \\
\qquad\qquad + \int_0^{\zeta} B_1(\zeta - s)\vartheta(s)ds, \text{ for } \zeta \in I, \\
\varphi'(\zeta) = A_2\varphi(\zeta) + f_2\left(\zeta, \vartheta(\zeta), \varphi(\zeta), \Psi_1(\vartheta(\zeta), \varphi(\zeta))\right) \\
\qquad\qquad + \int_0^{\zeta} B_2(\zeta - s)\vartheta(s)ds, \text{ for } \zeta \in I, \\
\vartheta(0) = \vartheta_0 + \Upsilon_1(\vartheta, \varphi), \\
\varphi(0) = \varphi_0 + \Upsilon_2(\vartheta, \varphi).
\end{cases} \tag{4.2}
$$

For $\zeta \in \Omega$, we have

$$
\begin{aligned}
&\|f_1(\zeta, \phi_1(\zeta), \phi_2(\zeta), \Psi_1(\phi_1(\zeta), \phi_2(\zeta))) - f_1(\zeta, \widehat{\phi}_1(\zeta), \widehat{\phi}_2(\zeta), \Psi_1(\widehat{\phi}_1(\zeta), \widehat{\phi}_2(\zeta)))\| \\
&\quad \leq \frac{e^{-\sigma\zeta}}{77}\left(e^{-\sigma\zeta-\zeta^2}\frac{\pi}{2} + \frac{\pi}{\zeta^2+1}\right)\left(\|\phi_1(\zeta) - \widehat{\phi}_1(\zeta)\| + \|\phi_2(\zeta) - \widehat{\phi}_2(\zeta)\|\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\|f_2(\zeta, \phi_1(\zeta), \phi_2(\zeta), \Psi_2(\phi_1(\zeta), \phi_2(\zeta))) - f_2(\zeta, \widehat{\phi}_1(\zeta), \widehat{\phi}_2(\zeta), \Psi_2(\widehat{\phi}_1(\zeta), \widehat{\phi}_2(\zeta)))\| \\
&\quad \leq \frac{e^{-\sigma\zeta}}{99(\zeta^2+1)}\left(\int_0^a e^{s-\zeta-(\pi+1)a}ds\right)\left(\|\phi_1(\zeta) - \widehat{\phi}_1(\zeta)\| + \|\phi_2(\zeta) - \widehat{\phi}_2(\zeta)\|\right).
\end{aligned}
$$

So, $\psi_i(\zeta) = \phi_i(\zeta) = \zeta;\ i = 1, 2$, are continuous nondecreasing functions from $(0, +\infty)$ to $(0, +\infty)$. Also, we have

$$
p_1(\zeta) = q_1(\zeta) = \frac{e^{-\sigma\zeta}}{77}(e^{-\sigma\zeta-\zeta^2}\frac{\pi}{2} + \frac{\pi}{\zeta^2+1}) \in L^1(\Omega, \mathbb{R}^+),
$$

and

$$
p_2(\zeta) = q_2(\zeta) = \frac{e^{-\sigma\zeta}}{99(\zeta^2+1)}\left(\int_0^a e^{s-\zeta-(\pi+1)a}ds\right) \in L^1(\Omega, \mathbb{R}^+).
$$

Now, about g_1, g_2, Υ_1 and Υ_2, we have

$$
\begin{aligned}
\|g_1(\zeta, s, \phi_1, \phi_2) - g_1(\zeta, s, \widehat{\phi}_1, \widehat{\phi}_2)\| &\leq e^{s-\zeta-(\pi+1)a} \\
&\quad \times \left(\|\phi_1(\zeta) - \widehat{\phi}_1(\zeta)\| + \|\phi_2(\zeta) - \widehat{\phi}_2(\zeta)\|\right),
\end{aligned}
$$

$$
\begin{aligned}
\|g_2(\zeta, s, \phi_1, \phi_2) - g_2(\zeta, s, \widehat{\phi}_1, \widehat{\phi}_2)\| &\leq \frac{\ln(1+e^{-\zeta^2})e^{-\sigma(\zeta-s)}}{1+2\zeta^2+s^2} \\
&\quad \times \left(\|\phi_1(\zeta) - \widehat{\phi}_1(\zeta)\| + \|\phi_2(\zeta) - \widehat{\phi}_2(\zeta)\|\right),
\end{aligned}
$$

$$\|\Upsilon_1(\zeta,\phi_1,\phi_2)-\Upsilon_1(\zeta,\widehat{\phi}_1,\widehat{\phi}_2)\| \leq \frac{1}{177}\left(\|\phi_1(\zeta)-\widehat{\phi}_1(\zeta)\|+\|\phi_2(\zeta)-\widehat{\phi}_2(\zeta)\|\right),$$

$$\|\Upsilon_2(\zeta,\phi_1,\phi_2)-\Upsilon_2(\zeta,\widehat{\phi}_1,\widehat{\phi}_2)\| \leq \frac{e^{\frac{1}{4}}}{111}\left(\|\phi_1(\zeta)-\widehat{\phi}_1(\zeta)\|+\|\phi_2(\zeta)-\widehat{\phi}_2(\zeta)\|\right).$$

And

$$g^*_{c_1}=\widehat{g}^*_{c_1}=e^{-(\pi+1)a},\ g^*_{c_2}=\widehat{g}^*_{c_2}=\ln(2),\ L_{\Upsilon_1}=L_{\widehat{\Upsilon}_1}=\frac{1}{177},\ L_{\Upsilon_2}=L_{\widehat{\Upsilon}_2}=\frac{e^{\frac{1}{4}}}{111}.$$

Also, we have

$$\frac{1}{177}+\frac{\pi}{308}+\frac{\pi^2}{154}\simeq 0,076$$
$$\frac{e^{\frac{1}{4}}}{111}+\frac{\pi}{198}\simeq 0,027.$$

Then, $\rho(\widetilde{M})\in(0,1),\ \det(\widehat{M})=0,897,\ \det(M^*)=0,794$ where

$$\widetilde{M}=\begin{pmatrix} M_{\Phi_1}\left(L_{\Upsilon_1}+\|p_1\|_{L^1}\right) & M_{\Phi_1}\left(\widehat{L}_{\Upsilon_1}+\|q_1\|_{L^1}\right) \\ & \\ M_{\Phi_2}\left(L_{\Upsilon_2}+\|p_2\|_{L^1}\right) & M_{\Phi_2}\left(\widehat{L}_{\Upsilon_2}+\|q_2\|_{L^1}\right)\end{pmatrix}.$$

And $\widehat{M}=I-\widetilde{M},\ M^*=I-2\widetilde{M}$. Therefore, $\widetilde{M}$ converges to zero and $\widehat{M},\ M^*$ are order preserving. By Theorem 3.2 and Theorem 3.6, we deduce that (4.1) has a unique mild solution, which is locally attractive.

Now, we assume that $\rho(\zeta)=\sin(\ln(\zeta+1))$. Then, we have

$$\begin{aligned}&\|f_1(\zeta,\phi_1(\zeta),\phi_2(\zeta),\Psi_1(\phi_1(\zeta),\phi_2(\zeta)))-f_1(\zeta,\widehat{\phi}_1(\zeta),\widehat{\phi}_2(\zeta),\Psi_1(\widehat{\phi}_1(\zeta),\widehat{\phi}_2(\zeta)))\|\\ &\quad\leq\frac{3\pi}{154}\left(\|\phi_1(\zeta)-\widehat{\phi}_1(\zeta)\|+\|\phi_2(\zeta)-\widehat{\phi}_2(\zeta)\|\right),\end{aligned}$$

and

$$\begin{aligned}&\|f_2(\zeta,\phi_1(\zeta),\phi_2(\zeta),\Psi_2(\phi_1(\zeta),\phi_2(\zeta)))-f_2(\zeta,\widehat{\phi}_1(\zeta),\widehat{\phi}_2(\zeta),\Psi_2(\widehat{\phi}_1(\zeta),\widehat{\phi}_2(\zeta)))\|\\ &\quad\leq\frac{e^{-\pi a}}{99}\left(\|\phi_1(\zeta)-\widehat{\phi}_1(\zeta)\|+\|\phi_2(\zeta)-\widehat{\phi}_2(\zeta)\|\right).\end{aligned}$$

Now, for any $\omega>0$, we have

$$|\sin(\ln(\zeta+\omega+1))-\sin(\ln(\zeta+1))|\xrightarrow[\zeta\to+\infty]{}0.$$

In other hand, if we put

$$\Lambda_1(\zeta)=\frac{\sin\left(e^{-\sigma\zeta}\right)\Psi_1(\phi_1,\phi_2)}{77\left(\zeta^2+1\right)}+\frac{\pi(\phi_1(\zeta)+\phi_2(\zeta))\sin\left(e^{-\sigma\zeta}\right)}{77(\zeta^2+1)}.$$

$$\Lambda_2(\zeta)=\frac{\cos\left(e^{-\sigma\pi}\right)(\phi_1(\zeta)+\phi_2(\zeta))}{198\left(\zeta^2+1\right)e^{\zeta+(\pi+1)a}}\left(e^a-1\right)+\frac{e^{-\sigma\zeta}\Psi_2(\phi_1,\phi_2)}{198\left(\zeta^2+1\right)}.$$

For $\phi_1,\ \phi_2\ \in \widetilde{H} = H \cap SAB_\omega$, $\Lambda_1,\ \Lambda_2$, are bounded and we have

$$\|\Lambda_1(\zeta)\| \xrightarrow[\zeta\to+\infty]{} 0, \quad \text{and } \|\Lambda_2(\zeta)\| \xrightarrow[\zeta\to+\infty]{} 0.$$

Then

$$\begin{aligned}
&\|f_1(\zeta+\omega, \phi_1(\zeta), \phi_2(\zeta), \Psi_1(\phi_1(\zeta), \phi_2(\zeta))) - f_1(\zeta, \phi_1(\zeta), \phi_2(\zeta), \Psi_1(\phi_1(\zeta), \phi_2(\zeta)))\| \\
&\quad \le \|\rho_1(\zeta+\omega) - \rho_1(\zeta)\|\|\Lambda_1(\zeta+\omega)\| + \|\rho_1(\zeta)\|(\|\Lambda_1(\zeta)\| + \|\Lambda_1(\zeta+\omega)\|) \\
&\quad \xrightarrow[\zeta\to+\infty]{} 0.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\|f_2(\zeta+\omega, \phi_1(\zeta), \phi_2(\zeta), \Psi_1(\phi_1(\zeta), \phi_2(\zeta))) - f_1(\zeta, \phi_1(\zeta), \phi_2(\zeta), \Psi_1(\phi_1(\zeta), \phi_2(\zeta)))\| \\
&\quad \le \|\rho_2(\zeta+\omega) - \rho_2(\zeta)\|\|\Lambda_2(\zeta+\omega)\| + \|\rho_2(\zeta)\|(\|\Lambda_2(\zeta+\omega)\| + \|\Lambda_2(\zeta)\|) \\
&\quad \xrightarrow[\zeta\to+\infty]{} 0.
\end{aligned}$$

Therefore, all conditions of Theorem 3.3, Theorem 3.4 and Theorem 3.6 are verified. Consequently, the problem (4.1) has at least one S-asymptotically ω-periodic mild solution, which is locally attractive.

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