

Quarter-symmetric metric connection on Sasaki-Kenmotsu manifold

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Abstract. In this paper, we delve into the study of quarter-symmetric metric connection (*QSMC*) on Sasaki-Kenmotsu manifolds. Our investigation encompasses the analysis of the Riemann curvature tensor, Ricci tensor and associated properties within the context of the *QSMC*. Additionally, we establish the non-existence of recurrent, A-recurrent and pseudo-symmetric Sasaki-Kenmotsu manifolds under the *QSMC*. Moreover, we explore conformally flat and ρ_1 -conformally flat Sasaki-Kenmotsu manifolds with respect to the *QSMC*, presenting several intriguing results. The primary focus of this paper lies in elucidating the implementation of the *QSMC* on Sasaki-Kenmotsu manifolds.

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1. Introduction

This research article primarily delves into the implementation of *QSMC* denoted by D^1 on Sasaki-Kenmotsu manifolds. Previously, Golab [6] introduced these connections on differentiable manifolds while Rastogi [9, 10] pursued structured contact metric manifolds and their subclasses such as K-contact, Sasakian, Kenmotsu, and cosymplectic manifolds.

Yoldas and Yasar [15] investigated the properties of torse-forming vector fields and certain curvature conditions on Kenmotsu manifolds. In [14], Yoldas explored the characteristics of α -cosymplectic manifolds, focusing on specific curvature conditions and vector fields. Additionally, in [7], the authors examined submanifolds of real space forms equipped with a Ricci quarter-symmetric metric connection.

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Mandal and De [4], in a prior work, examined the *QSMC* D^1 on Sasaki manifolds. Initially, they established the existence of this connection on Riemannian manifolds and derived the relationship between the Riemannian connection and the *QSMC* on Sasaki manifolds. Additionally, they characterized the manifold using the projective curvature tensor with respect to the D^1 connection.

Subsequently, other researchers investigated the existence of almost-pseudo-symmetric and Ricci-symmetric Sasakian manifolds satisfying the D^1 connection. Furthermore, Prasad and Haseeb [8] explored curvature conditions on the conformal curvature tensor in K -contact manifolds with respect to the D^1 connection. Similarly, Sular et al. [11] investigated the *QSMC* D^1 in Kenmotsu manifolds. Initially, they determined the Riemann curvature tensor and Ricci tensor of Kenmotsu manifolds with respect to the D^1 connection. Moreover, they established the non-existence of generalized recurrent and pseudo-symmetric Kenmotsu manifolds with respect to certain metric connections.

On an m -dimensional Riemannian manifold, the *QSMC* is defined by its torsion tensor of the form:

$$(1.1) \quad T^1(X, Y) = \eta(Y)AX - \eta(X)AY,$$

since D^1 is the *QSMC*, hence we have

$$(1.2) \quad T^1(X, Y) = D^1_X Y - D^1_Y X - [X, Y],$$

where η is a 1-form and A is a $(1, 1)$ tensor field. Providing the torsion tensor $T^1(X, Y) = 0$ then the connection D^1 becomes symmetric, otherwise it is non-symmetric. If the Riemannian metric g is compatible with respect to the *QSMC*, i.e., it satisfies the condition $D^1 g = 0$, then we say that the linear connection D^1 is a metric connection. Specifically, the linear connection is both symmetric and metric connection if and only if it is the Levi-Civita connection.

The semi-symmetry of the *QSMC* is characterized by the reduction of equation (1.1) to

$$(1.3) \quad T^1(X, Y) = \eta(Y)X - \eta(X)Y,$$

where $X, Y \in \chi(M)$, representing vector fields on the manifold M . In their previous work [4], the authors demonstrated the existence of the *QSMC* D^1 on a Riemannian manifold. The relation between the *QSMC* D^1 and the Levi-Civita connection D^g in a Riemannian manifold M is expressed as:

$$(1.4) \quad D^1_X Y = D^g_X Y - \eta(X)AY.$$

Drawing inspiration from this prior research, we initially explore the *QSMC* on a Sasaki-Kenmotsu manifold, examining Riemannian, Ricci curvature tensors, and scalar curvatures concerning the *QSMC*. Subsequently, we establish the non-existence of generalized recurrent, A -recurrent, and pseudo-symmetric Sasaki-Kenmotsu manifolds concerning the *QSMC*.

2. Preliminaries

Let M denote a $(2n + 2)$ -dimensional manifold of class C^∞ equipped with an almost bi-contact structure $(A, \rho_1, \rho_2, \eta, \omega)$, where A is a $(1, 1)$ -tensor field, η and ω are the characteristic vector fields corresponding to the 1-forms ρ_1 and ρ_2 , respectively. These structures satisfy the following conditions:

$$(2.1) \quad \eta(\rho_1) = \omega(\rho_2) = 1, \quad A^2 = -I + \eta \otimes \rho_1 + \omega \otimes \rho_2, \quad tr A = 0,$$

$$(2.2) \quad \eta(\rho_2) = \omega(\rho_1) = A\rho_1 = A\rho_2 = 0, \quad \text{and} \quad \eta \circ A = \omega \circ A = 0.$$

Here, I represents the identity transformation of $\Gamma(M)$ (algebra of all vector fields on M). If the Riemannian metric g admits the almost bi-contact structure, i.e. if it satisfies

$$(2.3) \quad g(AX, AY) = g(X, Y) - \eta(X)\eta(Y) - \omega(X)\omega(Y),$$

for all $X, Y \in \Gamma(M)$, then $(A, \rho_1, \rho_2, \eta, \omega, g)$ is called an almost bi-contact metric structure, and $(M, A, \rho_1, \rho_2, \eta, \omega, g)$ is called an almost bi-contact metric manifold. Here, $\eta(X) = g(X, \rho_1)$ and $\omega(X) = g(X, \rho_2)$. The fundamental 2-form \hat{A} of the almost bi-contact metric structure is defined by $\hat{A}(X, Y) = g(X, AY)$.

If D is the torsion-free metric-compatible connection of g on $(M, A, \rho_1, \rho_2, \eta, \omega)$, then it can be proved that

$$(2.4) \quad (D_X \eta)Y = g(D_X \rho_1, Y) \quad \text{and} \quad (D_X \omega)Y = g(D_X \rho_2, Y),$$

$$(2.5) \quad (D_X A)(Y, Z) = g(Y, (D_X A)Z) = -g((D_X A)Y, Z)$$

for all X, Y and $Z \in \Gamma(M)$.

Definition 2.1. [2] An almost bi-contact metric structure $(A, \rho_1, \rho_2, \eta, \omega, g)$ on M is called a Sasaki-Kenmotsu structure if $(A, \rho_1, \rho_2, \eta, \omega, g)$ is normal, ω is closed, and $d\eta = A - 2\eta \wedge \omega$.

Remark 2.2. [2] Differentiating the formula $d\eta = A - 2\eta \wedge \omega$ and using $d\omega = 0$ on Sasaki-Kenmotsu manifold, we get $dA = 2d\eta \wedge \omega$, which easily gives $dA = 2\omega \wedge A$.

Let $D_\eta := \{X \in \Gamma(M) | \eta(X) = 0\}$ and $D_\omega := \{X \in \Gamma(M) | \omega(X) = 0\}$ be two distributions of rank $(2n+1)$ that are transversal to the characteristic vector fields ρ_1 and ρ_2 , respectively. It is noted that $d\omega = 0$ and $dA = 2\omega \wedge A$ on D_η , implying that (A, ρ_2, ω, g) forms a Kenmotsu structure on D_η . Additionally, $d\eta = A$ on D_ω indicates that (A, ρ_1, ω, g) constitutes a Sasaki structure on D_ω . Consequently, the structure is called a Sasaki-Kenmotsu structure.

Proposition 2.3. [2] If $(A, \rho_1, \rho_2, \eta, \omega, g)$ is a Sasaki-Kenmotsu structure, then

$$(2.6) \quad (D_X A)Y = -g(AX, AY)\rho_1 - \eta(Y)A^2X + g(AX, Y)\rho_2 - \omega(Y)AX,$$

for all X and $Y \in \Gamma(M)$.

Proposition 2.4. [2] *In a Sasaki-Kenmotsu manifold $(M, A, \rho_1, \rho_2, \eta, \omega, g)$, we have*

$$(2.7) \quad \begin{aligned} D_{\rho_1}\rho_1 &= -2\rho_2, D_{\rho_1}\rho_2 = 2\rho_1, D_{\rho_2}\rho_1 = D_{\rho_2}\rho_2 = 0, \\ \omega(D_X\rho_1) &= -2\eta(X), \text{ and } \eta(D_X\rho_2) = 2\eta(X), \end{aligned}$$

$$(2.8) \quad D_X\rho_1 = AX - 2\eta(X)\rho_2 \text{ and } D_X\rho_2 = -A^2X + 2\eta(X)\rho_1,$$

$$(2.9) \quad \begin{aligned} R(X, Y)\rho_1 &= \eta(X)Y - \eta(Y)X + \omega(X)AY - \omega(Y)AX \\ &+ 3\eta(X)\omega(Y)\rho_2 - 3\eta(Y)\omega(X)\rho_2 + 2g(X, AY)\rho_2, \end{aligned}$$

$$\begin{aligned} R(X, Y)\rho_2 &= \omega(X)Y - \omega(Y)X + \eta(Y)AX - \eta(X)AY \\ &+ 3\omega(X)\eta(Y)\rho_1 - 3\eta(X)\omega(Y)\rho_1 + 2g(AX, Y)\rho_1, \end{aligned}$$

$$\begin{aligned} R(X, \rho_1)Y &= g(X, Y)\rho_1 - \eta(Y)X + g(AX, Y)\rho_2 - \omega(Y)AX \\ &+ 3\omega(Y)\omega(X)\rho_1 + 3\omega(Y)\omega(X)\rho_1 - 3\eta(Y)\omega(X)\rho_2 \\ &- 2\omega(X)AY, \end{aligned}$$

$$(2.10) \quad \begin{aligned} R(X, \rho_2)Y &= g(X, Y)\rho_2 - \omega(Y)X - \eta(Y)AX + g(AX, Y)\rho_1 \\ &+ 3\eta(Y)\eta(X)\rho_2 - 3\omega(Y)\eta(X)\rho_1 - 2\eta(X)AY, \end{aligned}$$

$$(2.11) \quad Ric(X, \rho_1) = -2(n + 2)\eta(X) \text{ and } Ric(X, \rho_2) = -2(n + 2)\omega(X),$$

where R denotes the Riemannian curvature of g , given by the formula $R = [D_X, D_Y] - D_{[X, Y]}$, and Ric is the Ricci curvature tensor is defined by

$$Ric(X, Y) = g(QX, Y) = Tr[Z \rightarrow R(Z, X)Y],$$

for all X, Y and $Z \in \Gamma(M)$. The scalar curvature of g is a smooth function defined by $\tau = Tr(Q)$, and the gradient of the scalar curvature $divQ$ is given by

$$(divQ)X = \frac{1}{2}g(X, grad\tau) = \sum_{i=1}^{(2n+2)} g((D_{E_i}Q)X, E_i),$$

for all $X \in \Gamma(M)$, where $\{E_i\}_{i=1}^{(2n+2)}$ is an orthonormal frame on M .

3. Curvature tensor on a Sasakian-Kenmotsu manifold with respect to QSMC

Let D and D^1 be Levi-Civita and linear connections of the Sasaki-Kenmotsu manifold M , respectively. Then, we have:

$$(3.1) \quad D_X^1Y = D_XY + G(X, Y),$$

where G is a tensor field of type $(1, 1)$. For D^1 to be $QSMC$ on M , we have

$$(3.2) \quad G(X, Y) = \frac{1}{2}(T^1(X, Y) + G^1(X, Y) + G^1(Y, X)),$$

where

$$(3.3) \quad g(G^1(X, Y), Z) = g(T(Z, X), Y).$$

From (1.1) and (3.3), we get

$$(3.4) \quad G^1(X, Y) = g(AY, X)\rho_1 - \eta(X)AY.$$

Using (1.1) and (3.4) in (3.2), we obtain:

$$(3.5) \quad G(X, Y) = -\eta(X)AY.$$

Hence, a $QSMC$ D^1 in a Sasaki-Kenmotsu manifold is given by

$$(3.6) \quad D_X^1 Y = D_X Y - \eta(X)AY.$$

Let R and \mathfrak{R} be the curvature tensor of D and D^1 of a Sasaki-Kenmotsu manifold, respectively. Then, from (3.6), we have

$$(3.7) \quad \begin{aligned} \mathfrak{R}(X, Y)Z &= R(X, Y)Z - (D_X \eta)(Y)AZ - (D_Y \eta)(X)AZ \\ &+ \eta(X)(D_Y A)Z - \eta(Y)(D_X A)Z. \end{aligned}$$

Using (2.6), (2.7) and (2.8) in equation (3.7), we get

$$(3.8) \quad \begin{aligned} \mathfrak{R}(X, Y)Z &= R(X, Y)Z + \eta(Y)\omega(Z)AX - \eta(X)\omega(Z)AY \\ &+ 2(g(AY, X) + \eta(X)\omega(Y) - \eta(Y)\omega(X))AZ \\ &+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + (\omega(Y)\omega(Z)\eta(X) \\ &- \omega(X)\omega(Z)\eta(Y) + g(X, Z)\eta(Y) - g(Y, Z)\eta(X))\rho_1 \\ &+ (g(AY, Z)\eta(X) - g(AX, Z)\eta(Y) + \eta(Y)\eta(Z)\omega(X) \\ &- \eta(X)\eta(Z)\omega(Y))\rho_2. \end{aligned}$$

The relation between the curvature tensors of M with respect to $QSMC$ D^1 and Levi-Civita connection D is given by (3.8). So from (3.8) and (2.4), we have

$$(3.9) \quad \begin{aligned} \mathfrak{R}(X, \rho_1)Y &= 2(g(X, Y)\rho_1 - \eta(Y)X - g(AY, X)\rho_2 + \omega(Y)\omega(X)\rho_1 \\ &+ \omega(Y)\omega(X)\rho_1 - \eta(Y)\omega(X)\rho_2) + \omega(X)AY, \end{aligned}$$

$$(3.10) \quad \begin{aligned} \mathfrak{R}(X, Y)\rho_1 &= 2(\eta(X)Y - \eta(Y)X + \eta(X)\omega(Y)\rho_2 - \eta(Y)\omega(X)\rho_2) \\ &+ \omega(X)AY - \omega(Y)AX + 2g(X, AY)\rho_2. \end{aligned}$$

Now, taking the inner product of (3.8) with an arbitrary vector field U and contracting with the orthonormal basis $\{E_1, E_2, \dots, E_{2n}, \rho_1, \rho_2\}$, we get

$$(3.11) \quad \mathfrak{Ric}(Y, Z) = Ric(Y, Z) + g(Y, Z) + (2n + 1)\eta(Y)\eta(Z) - \omega(Y)\omega(Z),$$

where \mathfrak{Ric} is the Ricci curvature tensor of the connection D^1 . This implies that the Ricci curvature tensor of the Sasaki-Kenmotsu manifold is symmetric with respect to $QSMC$. Contracting over the Y and Z , we get

$$(3.12) \quad \tau = \tau^1,$$

where $\tau^1 = tr(\mathfrak{Q}^1)$ is the scalar curvature of the connection D^1 . Here, \mathfrak{Q}^1 is the Ricci operator with respect to $QSMC$ D^1 .

From the above calculations, we obtain the following theorem:

Theorem 3.1. *Let M be a Sasaki-Kenmotsu manifold with a $QSMC$. Then we have:*

- i . The Riemannian curvature tensor \mathfrak{R} is given by (3.8)*
- ii . The Ricci curvature tensor \mathfrak{Ric} is given by (3.11)*
- iii . $\mathfrak{R}(X, Y, Z, U) + \mathfrak{R}(Y, X, Z, U) = 0$*
- iv . $\mathfrak{R}(X, Y, Z, U) + \mathfrak{R}(X, Y, U, Z) = 0$*
- v . $\mathfrak{Ric}(Y, \rho_1) = -4(n + 1)\eta(Y)$ and $\mathfrak{Ric}(Y, \rho_2) = -2(n + 2)\omega(Y)$*
- vi . $\tau^1 = \tau$*
- vii . The Ricci curvature tensor with respect to $QSMC$ is symmetric.*

4. Locally Symmetric Sasaki-Kenmotsu Manifolds with respect to $QSMC$

Let M be a locally symmetric Sasaki-Kenmotsu manifold with respect to the $QSMC$ D^1 . This implies the following condition

$$(4.1) \quad (D_X^1 \mathfrak{R})(Y, Z)W = 0.$$

By performing suitable contractions, we obtain

$$(4.2) \quad \begin{aligned} (D_X^1 \mathfrak{Ric})(Z, W) &= D_X^1 \mathfrak{Ric}(Z, W) - \mathfrak{Ric}(D_X^1 Z, W) - \mathfrak{Ric}(Z, D_X^1 W) \\ &= 0. \end{aligned}$$

Substituting $W = \rho_1$ into the above equation, we derive

$$(4.3) \quad \begin{aligned} (D_X^1 \mathfrak{Ric})(Z, \rho_1) &= D_X^1 \mathfrak{Ric}(Z, \rho_1) - \mathfrak{Ric}(D_X^1 Z, \rho_1) - \mathfrak{Ric}(Z, D_X^1 \rho_1) \\ &= 0. \end{aligned}$$

Using equations (1.4), (2.9) and (3.11), we obtain the scalar curvature τ as follows:

$$(4.4) \quad \tau = -2n(4n + 5) + (4n + 8).$$

Thus, we can state the following theorem:

Theorem 4.1. *Let M be a locally symmetric Sasaki-Kenmotsu manifold of dimension $(2n + 2)$ with respect to the $QSMC$ D^1 . Then the scalar curvature of the Levi-Civita connection of M is given by (4.4).*

5. Non-existence of Certain Types of Sasaki-Kenmotsu Manifolds with Respect to $QSMC$

Definition 5.1. A non-flat n -dimensional Riemannian manifold M , with $n > 3$, is called generalized recurrent if its curvature tensor R satisfies the condition

$$(5.1) \quad (D_X R)(Y, Z)W = \alpha(X)R(Y, Z)W + \beta(X)(g(Z, W)Y - g(Y, W)Z),$$

for all $X, Y, Z, W \in \Gamma(M)$, where $(\beta \neq 0)$. If $\beta = 0$ and $\alpha \neq 0$ then M is called recurrent. In [1], the authors studied generalized recurrent Kenmotsu manifolds.

Suppose that there exists a generalized recurrent Sasaki-Kenmotsu manifold M with respect to $QSMC D^1$. Then, from equation (5.1), we get

$$(5.2) \quad (D_X^1 \mathfrak{R})(Y, Z)W = \alpha(X)\mathfrak{R}(Y, Z)W + \beta(X)(g(Z, W)Y - g(Y, W)Z).$$

Taking $Y = W = \rho_1$ in (5.2), we obtain

$$(5.3) \quad (D_X^1 \mathfrak{R})(\rho_1, Z)\rho_1 = \alpha(X)\mathfrak{R}(\rho_1, Z)\rho_1 + \beta(X)(\eta(Z)\rho_1 - Z).$$

Setting $X = \rho_1$ in (3.10) and using the result, we get

$$(5.4) \quad \begin{aligned} (D_X^1 \mathfrak{R})(\rho_1, Z)\rho_1 &= (2\alpha(X) - \beta(X))Z + (\beta(X) - 2\alpha(X))\eta(Z)\rho_1 \\ &+ 2\alpha(X)\omega(Z)\rho_2. \end{aligned}$$

On the other hand, by using equations (1.4), (2.9), (2.10) and (3.10), we have

$$(5.5) \quad \begin{aligned} (D_X^1 \mathfrak{R})(\rho_1, Z)\rho_1 &= 2\eta(X)AZ - 2(g(X, Z) - \eta(X)\eta(Z) \\ &- \omega(X)\omega(Z))\rho_2. \end{aligned}$$

By equating (5.4) and (5.5) and then taking the inner product with ρ_2 , we finally obtain

$$(5.6) \quad \begin{aligned} 2(g(X, Z) - \eta(X)\eta(Z) - \omega(X)\omega(Z)) &= (\beta(X) - 2\alpha(X))\omega(Z) \\ &- 2\alpha(X)\omega(Z). \end{aligned}$$

By taking $Z = \rho_2$ in (5.6), it reduces to $4\alpha(X) - \beta(X) = 0$. If $\beta = 0$, it implies $\alpha = 0$. Thus, we can state the following theorem:

Theorem 5.2. *There exists only a generalized recurrent Sasaki-Kenmotsu manifold with respect to $QSMC D^1$ and no recurrent Sasaki-Kenmotsu manifold with respect to $QSMC D^1$.*

Definition 5.3. A non-flat n -dimensional Riemannian manifold M , with $n > 3$ is called A -recurrent [5] if its curvature tensor R satisfies the condition

$$(5.7) \quad A^2((D_X R)(Y, Z)W) = \alpha(X)R(Y, Z)W,$$

for all $X, Y, Z, W \in \Gamma(M)$, where A is a $(1, 1)$ tensor field and α is a non-zero 1-form.

A. Basuri et al. [12] studied the more general case of A -recurrent Kenmotsu manifolds as generalized A -recurrent Kenmotsu manifolds.

Suppose that there exists an A -recurrent Sasaki-Kenmotsu manifold M with respect to the $QSMC D^1$. Then (5.7) can be written as

$$(5.8) \quad A^2((D_X^1 \mathfrak{R})(Y, Z)W) = \alpha(X)\mathfrak{R}(Y, Z)W.$$

Using (2.11) in (5.8), we obtain

$$(5.9) \quad \begin{aligned} -(D_X \mathfrak{R})(Y, Z)W + \eta((D_X^1 \mathfrak{R})(Y, Z)W)\rho_1 &+ \omega((D_X^1 \mathfrak{R})(Y, Z)W)\rho_2 \\ &= \alpha(X)\mathfrak{R}(Y, Z)W. \end{aligned}$$

Setting $Y = W = \rho_1$ in (5.9), we have

$$(5.10) \quad \begin{aligned} - (D_X^1 \mathfrak{R})(\rho_1, Z)\rho_1 + \eta((D_X^1 \mathfrak{R})(\rho_1, Z)\rho_1)\rho_1 \\ + \omega((D_X^1 \mathfrak{R})(\rho_1, Z)\rho_1)\rho_2 = \alpha(X)\mathfrak{R}(\rho_1, Z)\rho_1. \end{aligned}$$

Using (5.5) in (5.9), we arrive at

$$(5.11) \quad \begin{aligned} -2(g(X, Z) - \eta(X)\eta(Z) - \omega(X)\omega(Z))\rho_2 \\ + \eta(X)AZ = \alpha(X)(Z - \eta(Z)\rho_1 + \omega(Z)\rho_2). \end{aligned}$$

Taking the inner product of equation (5.11) with ρ_2 , we get

$$(5.12) \quad -g(X, Z) + \eta(X)\eta(Z) + \omega(X)\omega(Z) = \alpha(X)\omega(Z).$$

Replacing $Z = \rho_2$ in (5.12) gives $\alpha(X) = 0$. This contradicts the fact that $\alpha \neq 0$. Thus, we can state the following theorem:

Theorem 5.4. *There is no A -recurrent Sasaki-Kenmotsu manifold with respect to $QSMC D^1$.*

Definition 5.5. A non-flat n -dimensional Riemannian manifold (M, g) , $n > 3$ is called pseudosymmetric if there exists a 1-form α on M such that:

$$(5.13) \quad \begin{aligned} (D_X R)(T, Z)U &= 2\alpha(X)R(Y, Z)U + \alpha(Y)R(X, Z)U \\ &+ \alpha(Z)R(Y, X)U + \alpha(U)R(Y, Z)X \\ &+ g(R(Y, Z)U, X)V. \end{aligned}$$

Here $X, Y, Z, U \in \Gamma(M)$ are arbitrary vector fields, α is a non-zero 1-form on M and $V \in \Gamma(M)$ is a vector field corresponding to the 1-form α such that $g(X, V) = \alpha(X)$ [3].

Suppose that there exists a pseudosymmetric Sasaki-Kenmotsu manifold M with respect to $QSMC D^1$. Then, equation (5.13) takes the form

$$(5.14) \quad \begin{aligned} (D_X^1 \mathfrak{R})(Y, Z)U &= 2\alpha(X)\mathfrak{R}(Y, Z)U + \alpha(Y)\mathfrak{R}(X, Z)U \\ &+ \alpha(Z)\mathfrak{R}(Y, X)U + \alpha(U)\mathfrak{R}(Y, Z)X \\ &+ g(\mathfrak{R}(Y, Z)U, X)V. \end{aligned}$$

Setting $Y = E_i$, we take the inner product with E_i and sum over i from 1 to $2n + 2$, yielding

$$(5.15) \quad \begin{aligned} (D_X^1 \mathfrak{Ric})(Z, U) &= 2\alpha(X)\mathfrak{Ric}(Z, U) + \alpha(\mathfrak{R}(X, Z), U) \\ &+ \alpha(Z)\mathfrak{Ric}(X, U) + \alpha(U)\mathfrak{Ric}(Z, X) \\ &+ \alpha(\mathfrak{R}(X, U)Z). \end{aligned}$$

Setting $U = \rho_1$ on both sides of (5.15) and using equations (3.9)-(3.11), we get

$$(5.16) \quad \begin{aligned} (D_X^1 \mathfrak{Ric})(Z, \rho_1) &= -(8n + 12)\eta(Z)\alpha(X) - (4n + 2)\alpha(Z)\eta(X) \\ &+ 2\omega(X)\alpha(AZ) - \omega(Z)\alpha(AX) \\ &+ (Ric(Y, Z) + 3g(Y, Z) - (2n + 1)\eta(X)\eta(Z) \\ &+ \omega(X)\omega(Z))\alpha(\rho_1) + (2\eta(X)\omega(Z) \\ &+ 4g(X, AZ) - 4\eta(Z)\omega(X))\alpha(\rho_2). \end{aligned}$$

Covariant differentiation of the Ricci tensor \mathfrak{Ric} with respect to $QSMC D^1$ is expressed as:

$$(5.17) \quad (D_X^1 \mathfrak{Ric})(Z, U) = D_X^1 \mathfrak{Ric}(Z, U) - \mathfrak{Ric}(D_X^1 Z, U) - \mathfrak{Ric}(Z, D_X^1 U).$$

By setting $U = \rho_1$ in (5.17) and using (3.6) and (3.11), we derive

$$(5.18) \quad (D_X^1 \mathfrak{Ric})(Z, \rho_1) = 4n\eta(X)\omega(Z) - Ric(Z, AX) - (4n + 5)g(Z, AX).$$

Equating (5.16) and (5.18), we obtain

$$(5.19) \quad \begin{aligned} &-4(2n+3)\alpha(X)\eta(Z) - 2(2n+1)\alpha(Z)\eta(X) + 2\omega(X)\alpha(AZ) \\ &- \omega(Z)\alpha(AX) + (Ric(X, Z) + 3g(X, Z) - (2n+1)\eta(X)\eta(Z) \\ &+ \omega(X)\omega(Z))\alpha(\rho_1) + (2\eta(X)\omega(Z) + 4g(X, AZ) - 4\eta(Z)\omega(X))\alpha(\rho_2) \\ &= 4n\eta(X)\omega(Z) - Ric(Z, AX) - (4n+5)g(Z, AX). \end{aligned}$$

By setting $Z = X = \rho_1$ in (5.19), we obtain $(n + 1)\alpha(\rho_1) = 0$. Since $n > 3$, this implies $\alpha(\rho_1) = 0$. Similarly, putting $U = \rho_2$ in (5.15) and using (3.8) and (3.11) we get

$$(5.20) \quad \begin{aligned} (D_X^1 \mathfrak{Ric})(Z, \rho_2) &= -(4n + 10)\alpha(X)\omega(Z) - (2n - 3)\omega(X)\alpha(Z) \\ &+ \eta(Z)\alpha(AX) - 2\eta(X)\alpha(AZ) + (Ric(X, Z) \\ &+ 2g(X, Z) - 2(n - 1)\eta(X)\eta(Z) - \omega(X)\omega(Z))\alpha(\rho_2). \end{aligned}$$

Again setting $U = \rho_2$ in (5.17), we derive

$$(5.21) \quad \begin{aligned} (D_X^1 \mathfrak{Ric})(Z, \rho_2) &= -Ric(X, Z) - (2n + 5)g(Z, X) + 4n\eta(X)\eta(Z) \\ &+ \omega(X)\omega(Z). \end{aligned}$$

By equating expressions (5.20) and (5.21) with $X = Z = \rho_2$, we obtain $(4n + 5)\alpha(\rho_2) = 0$. Since $n > 3$ this implies $\alpha(\rho_2) = 0$.

Thus, by putting $Z = \rho_1$ in (5.19), we get $(4n + 3)\alpha(X) = 0$, with the fact $n > 3$, we obtain $\alpha(X) = 0$, for all $X \in \Gamma(M)$. This means $\alpha = 0$ on M . This contradicts the fact that $\alpha \neq 0$ in the definition of pseudosymmetry. Thus, we can state the following theorem:

Theorem 5.6. *There is no pseudosymmetric Sasaki-Kenmotsu manifold with respect to the QSMC D^1 .*

Definition 5.7. [13] Let M be a Riemannian manifold of dimension m . The conformal curvature tensor \mathfrak{C} is given by

$$(5.22) \quad \mathfrak{C} = R(X, Y)Z - \frac{1}{m-2}(Ric(Y, Z)X - Ric(X, Z)Y + g(Y, Z)QX - g(X, Z)QY) + \frac{\tau}{(m-1)(m-2)}(g(Y, Z)X - g(X, Z)Y).$$

In particular, the manifold M is called as:

- i . conformally flat if $\mathfrak{C}(X, Y)Z = 0$,
- ii . ρ_1 -conformally flat if $\mathfrak{C}(X, Y)\rho_1 = 0$, for all $X, Y, Z \in \Gamma(M)$.

Suppose that a Sasaki-Kenmotsu manifold M of dimension $(2n + 2)$ is ρ_1 -conformally flat. Then (5.22) gives

$$(5.23) \quad R(X, Y)\rho_1 = \frac{1}{2n}(Ric(Y, \rho_1)X - Ric(X, \rho_1)Y + \eta(Y)Q\rho_1 - QY) - (\eta(Y)\rho_1 - Y).$$

Replacing X by ρ_1 in (5.23) and making use of (2.9), (2.11), we get

$$(5.24) \quad Y - \eta(Y)\rho_1 + 3\omega(Y)\rho_2 = \frac{1}{2n}(2(n+2)Y - 4(n+2)\eta(Y)\rho_1 - QY) - \frac{r}{2n(2n+1)}(\eta(Y)\rho_1 - Y).$$

Taking the inner product with Z in (5.24) and making suitable rearrangements, we get

$$(5.25) \quad Ric(Y, Z) = \left(\frac{r}{2n+1} + 4\right)g(Y, Z) - \left(\frac{r}{2n+1} + 2n + 8\right)\eta(Y)\eta(Z) - 6n\omega(Y)\omega(Z).$$

This shows that the manifold M is a $(\eta - \omega)$ -Einstein manifold. By generalizing this result, we have that if the Sasaki-Kenmotsu manifold is said to be a $(\eta - \omega)$ -Einstein manifold, it satisfies the condition:

$$(5.26) \quad Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) + c\omega(X)\omega(Y),$$

for all X and $Y \in \Gamma(M)$, where a, b and c are smooth functions on M .

Let \mathfrak{C}^1 be the conformal curvature tensor on a Sasaki-Kenmotsu manifold of dimension $(2n + 2)$ with respect to the QSMC. Then, this tensor is given by

$$(5.27) \quad \mathfrak{C}^1(X, Y)Z = \mathfrak{R}(X, Y)Z - \frac{1}{2n}(\mathfrak{R}ic(Y, Z)X - \mathfrak{R}ic(X, Z)Y) + g(Y, Z)\mathfrak{Q}^1X - g(X, Z)\mathfrak{Q}^1Y + \frac{\tau^1}{2n(2n+1)}(g(Y, Z)X - g(X, Z)Y).$$

Taking the inner product with U in (5.27) and using equations (3.8), (3.11) and (3.12) in (5.27), we get

$$\begin{aligned}
 (5.28) \quad \mathfrak{C}^1(X, Y, Z, U) &= \mathfrak{C}(X, Y, Z, U) + \eta(Y)\omega(Z)g(AX, Y) \\
 &- \eta(X)\omega(Z)g(AY, U) + 2(g(AY, X) + \eta(X)\omega(Y) \\
 &- \eta(Y)\omega(X))g(AZ, U) + \eta(X)\eta(Z)g(Y, U) \\
 &- \eta(Y)\eta(Z)g(X, U) + (\omega(Y)\omega(Z)\eta(X) \\
 &- \omega(X)\omega(Z)\eta(Y) + g(X, Z)\eta(Y) - g(Y, Z)\eta(Y))\eta(U) \\
 &+ (g(AY, Z)\eta(X) - g(AX, Z)\eta(Y) + \eta(Y)\eta(Z)\omega(X) \\
 &- \eta(X)\eta(Z)\omega(Y))\omega(U) - \frac{1}{2n}(2(g(Y, Z)g(X, U) \\
 &- g(X, Z)g(Y, U)) + (2n + 1)(g(X, Z)\eta(Y)\eta(U) \\
 &- g(Y, Z)\eta(X)\eta(U) - \eta(Y)\eta(Z)g(X, U) \\
 &+ \eta(X)\eta(Z)g(Y, U)) + g(X, Z)\omega(Y)\omega(U) \\
 &- g(Y, Z)\omega(X)\omega(U) + g(Y, U)\omega(X)\omega(Z) \\
 &- g(X, U)\omega(Y)\omega(Z)),
 \end{aligned}$$

where $\mathfrak{C}^1(X, Y, Z, U) = g(\mathfrak{C}^1(X, Y)Z, U)$ and $\mathfrak{C}(X, Y, Z, U) = g(\mathfrak{C}(X, Y)Z, U)$; for all X, Y, Z and $U \in \Gamma(M)$.

Interchanging the roles of X and Y in (5.28) and adding with (5.28), we finally get

$$(5.29) \quad \mathfrak{C}^1(X, Y, Z, U) + \mathfrak{C}^1(Y, X, Z, U) = 0.$$

Similarly, interchanging the roles of Z and U in (5.28) and summing with (5.28), we obtain

$$(5.30) \quad \mathfrak{C}^1(X, Y, Z, U) + \mathfrak{C}^1(Y, X, U, Z) = 0.$$

Thus, we can state the following theorem:

Theorem 5.8. *Let M be a $(2n + 2)$, $(n > 1)$ dimensional Sasaki-Kenmotsu manifold with respect to the QSMC D^1 . Then, we have:*

1. $\mathfrak{C}^1(X, Y, Z, U) + \mathfrak{C}^1(Y, X, Z, U) = 0,$
2. $\mathfrak{C}^1(X, Y, Z, U) + \mathfrak{C}^1(X, Y, U, Z) = 0.$

Now suppose that a Sasaki-Kenmotsu manifold M is conformally flat with respect to QSMC D^1 , meaning that $\mathfrak{C}^1(X, Y)Z = 0$. By (5.27), we obtain

$$\begin{aligned}
 (5.31) \quad \mathfrak{R}(X, Y)Z &= \frac{1}{2n}(\mathfrak{Ric}(Y, Z)X - \mathfrak{Ric}(X, Z)Y + g(Y, Z)\Omega^1 X \\
 &- g(X, Z)\Omega^1 Y) - \frac{\tau^1}{2n(2n + 1)}(g(Y, Z)X - g(X, Z)Y).
 \end{aligned}$$

Taking the inner product with ρ_1 in (5.31), we get

$$(5.32) \quad g(\mathfrak{R}(X, Y), Z, \rho_1) = \frac{1}{2n}(\mathfrak{R}ic(Y, Z)\eta(X) - \mathfrak{R}ic(X, Z)\eta(Y)) \\ + g(Y, Z)\mathfrak{R}ic(X, \rho_1) - g(X, Z)\mathfrak{R}ic(Y, \rho_1) \\ - \frac{\tau^1}{2n(2n+1)}(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)).$$

Again, replacing X by ρ_1 in (5.32) and using (3.9), (3.11) and (3.12), we have

$$(5.33) \quad \mathfrak{R}ic(Y, Z) = \frac{\tau}{(2n+1)}(g(Y, Z) - \eta(Y)\eta(Z)) - 4n(g(Y, Z) - \eta(Y)\eta(Z)) \\ + \omega(Y)\omega(Z) - 8(n+1)\eta(Y)\eta(Z) + 4(n+1)g(Y, Z).$$

Using (3.11) in (5.33), we get

$$(5.34) \quad Ric(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z) + c\omega(Y)\omega(Z),$$

where $a = \frac{\tau}{(2n+1)} + 3$, $b = -(\frac{\tau}{2n+1} + 2n + 7)$ and $c = -(4n + 1)$.

Next, we consider that a Sasaki-Kenmotsu manifold M of dimension $(2n + 2)$ is ρ_1 -conformally flat with respect to $QSMC$, i.e., $\mathfrak{C}^1 = 0$. Then (5.27) gives

$$(5.35) \quad \mathfrak{R}(X, Y)\rho_1 = \frac{1}{2n}(\mathfrak{R}ic(Y, \rho_1)X - \mathfrak{R}ic(X, \rho_1)Y + \eta(Y)\Omega^1 X) \\ - \eta(X)\Omega^1 Y - \frac{\tau^1}{2n(2n+1)}(\eta(Y)X - \eta(X)Y).$$

Using equations (3.10)-(3.12) in (5.35), then taking the inner product with Z , we get

$$(5.36) \quad 2(g(Y, Z)\eta(X) - g(X, Z)\eta(Y) + \eta(X)\omega(Y)\omega(Z) - \eta(Y)\omega(X)\omega(Z)) \\ + \omega(X)g(AY, Z) - \omega(Y)g(AX, Z) + 2g(X, AY)\omega(Z) \\ = \frac{1}{2n}(4(n+1)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)) \\ + \mathfrak{R}ic(X, Z)\eta(Y) - \mathfrak{R}ic(Y, Z)\eta(X)) \\ - \frac{\tau}{2n(2n+1)}(g(X, Z)\eta(Y) - g(Y, Z)\eta(X)).$$

Putting $Y = \rho_1$ in (5.36) and using (3.11), we finally obtain (5.34). Thus, we can state the following theorem:

Theorem 5.9. *Let M be a Sasaki-Kenmotsu manifold that is conformally flat or ρ_1 -conformally flat with respect to $QSMC D^1$. Then it is a $(\eta - \omega)$ -Einstein manifold.*

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