

On stability of finite variable quadratic functional equation in Banach spaces

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Abstract. The purpose of this paper to propose Hyers-Ulam-Rassias stability in Banach spaces for a finite variable quadratic functional equation

$$\begin{aligned} & \sum_{i=1}^n \emptyset \left(-2v_i + \sum_{j=1, i \neq j}^n v_j \right) \\ &= (n-6) \sum_{1 \leq i < j \leq n} \emptyset(v_i + v_j) - (n^2 - 8n + 3) \sum_{i=1}^n \emptyset(v_i) \end{aligned}$$

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1. Introduction

Functional equations play an important part in the study of stability. In 1940, the stability problems of functional equations about group homomorphisms was introduced by Ulam [17]. In 1941, Hyers [7] gave an affirmative answer to Ulam's question for additive groups (under the assumption that groups are Banach-spaces). Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$ for all $\varepsilon > 0$ and $p \in [0, 1)$. Also, Rassias [12] generalization theorem was delivered by Gavruta [6] who replaced $\varepsilon(\|x\|^p + \|y\|^p)$ by a control function $\emptyset(x, y)$. The papers of Rassias have significantly influenced the development of what we now call the Hyers-Ulam-Rassias stability of functional equations. In 1982, Rassias [12] followed the modern approach of the Rassias theorem [13] in which he replaced the factor product of norms instead of sum of norms. Since that time many researchers [1], [3], [4], [5], [8], [9], [10], [11], [14], [16] proved the stability of many functional equations on different type of spaces.

The functional equation $\emptyset(v_1 + v_2) + \emptyset(v_1 - v_2) = 2\emptyset(v_1) + 2\emptyset(v_2)$ is called a quadratic functional equation. Each solution of the quadratic functional equation is called a quadratic mapping. A Hyers-Ulam-Rassias stability problem

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for the quadratic functional equation was proved by Skof [15] for the mappings $\emptyset : V \rightarrow W$, where V is a normed space and W is a Banach space.

The present work introduces finite variable quadratic functional equation as

$$\begin{aligned}
 (1.1) \quad & \sum_{i=1}^n \emptyset \left(-2v_i + \sum_{j=1, i \neq j}^n v_j \right) \\
 & = (n-6) \sum_{1 \leq i < j \leq n} \emptyset(v_i + v_j) - (n^2 - 8n + 3) \sum_{i=1}^n \emptyset(v_i),
 \end{aligned}$$

where $n \geq 5$ and derive its solution. Also, it obtains Hyers-Ulam-Rassias stability in Banach spaces.

2. General solution

Theorem 2.1. *Let V and W be vector spaces. If a function $\emptyset : V \rightarrow W$, satisfies the functional equation (1.1) for all $v_1, v_2, \dots, v_n \in V$, then the function \emptyset is quadratic mapping, that is, it satisfies $\emptyset(v_1 + v_2) + \emptyset(v_1 - v_2) = 2\emptyset(v_1) + 2\emptyset(v_2)$.*

Proof. Let the function $\emptyset : V \rightarrow W$ satisfy (1.1). Taking $v_1 = v_2 = \dots = v_n = 0$ in (1.1), we have $\emptyset(0) = 0$. Replacing (v_1, v_2, \dots, v_n) by $(v, 0, \dots, 0)$ in (1.1), we get

$$\begin{aligned}
 & \emptyset(-2v) + (n-1)\emptyset(v) = (n-6)(n-1)\emptyset(v) - (n^2 - 8n + 3)\emptyset(v) \\
 (2.1) \quad & \emptyset(-2v) = 4\emptyset(v)
 \end{aligned}$$

If we take $v_1 = v_2 = v$ and $v_3 = \dots = v_n = 0$ in (1.1), and using $\emptyset(-2v) = 4\emptyset(v)$, we get

$$(2.2) \quad \emptyset(-v) = \emptyset(v)$$

which shows that \emptyset is an even mapping. Also, using the evenness property we have $\emptyset(2v) = 2^2\emptyset(v)$, for all $v \in V$ and for any $n \geq 0$. Also, by substituting $(v_1, v_2, v_3, v_4, \dots, v_n)$ with $(v, v, v, 0, \dots, 0)$, we get $\emptyset(3v) = 3^2\emptyset(v)$ and so on for all $v \in V$. By using PMI we get

$$(2.3) \quad \emptyset(kv) = k^2\emptyset(v).$$

Now replacing $v_3 = \dots = v_n = 0$ in (1.1) we get

$$(2.4) \quad \emptyset(2v_1 - v_2) + \emptyset(v_1 - 2v_2) = 9\emptyset(v_1) + 9\emptyset(v_2) - 4\emptyset(v_1 + v_2).$$

Replacing v_2 by $-v_2$ in (2.4),

$$(2.5) \quad \emptyset(2v_1 + v_2) + \emptyset(v_1 + 2v_2) = 9\emptyset(v_1) + 9\emptyset(v_2) - 4\emptyset(v_1 - v_2).$$

Replacing v_1 by $v_1 + v_2$ in (2.4), we get

$$(2.6) \quad \begin{aligned} \emptyset(2v_1 + v_2) + \emptyset(v_1 - v_2) &= 9\emptyset(v_1 + v_2) + 9\emptyset(v_2) - 4\emptyset(v_1 + 2v_2), \\ \emptyset(2v_1 + v_2) &= 9\emptyset(v_1 + v_2) + 9\emptyset(v_2) - 4\emptyset(v_1 + 2v_2) - \emptyset(v_1 - v_2). \end{aligned}$$

Using (2.6) in (2.5), we get

$$(2.7) \quad \emptyset(v_1 + 2v_2) = -3\emptyset(v_1) + \emptyset(v_1 - v_2) + 3\emptyset(v_1 + v_2).$$

Interchanging v_1 and v_2 in (2.7), we get

$$(2.8) \quad \emptyset(2v_1 + v_2) = -3\emptyset(v_2) + \emptyset(v_1 - v_2) + 3\emptyset(v_1 + v_2).$$

By using (2.5), (2.7) and (2.8), we have

$$(2.9) \quad \emptyset(v_1 + v_2) + \emptyset(v_1 - v_2) = 2\emptyset(v_1) + 2\emptyset(v_2).$$

3. Main results

In this section, we prove the Hyers-Ulam-Rassias stability of quadratic functional equation (1.1). The abbreviation for the given mapping $\emptyset : V \rightarrow W$ is given as follows:

$$(3.1) \quad \begin{aligned} D\emptyset(v_1, v_2, \dots, v_n) &= \sum_{i=1}^n \emptyset \left(-2v_i + \sum_{j=1, i \neq j}^n v_j \right) - (n-6) \sum_{1 \leq i < j \leq n} \emptyset(v_i + v_j) \\ &\quad + (n^2 - 8n + 3) \sum_{i=1}^n \emptyset(v_i), \end{aligned}$$

for all $v_1, v_2, \dots, v_n \in V$.

Theorem 3.1. *Assume that V and W are Banach spaces. If a function $\emptyset : V \rightarrow W$ satisfies the inequality*

$$(3.2) \quad \|D\emptyset(v_1, v_2, \dots, v_n)\| < \varepsilon$$

for some $\varepsilon > 0$, for all $v_1, v_2, \dots, v_n \in V$, then the limit

$$(3.3) \quad Q_2(v) = \lim_{m \rightarrow \infty} \frac{\emptyset(2^m v)}{2^{2m}}$$

exists for each $v \in V$ and $Q_2 : V \rightarrow W$ is unique quadratic function such that

$$(3.4) \quad \|\emptyset(v) - Q_2(v)\| \leq \frac{\varepsilon}{3},$$

for any $v \in V$.

Proof. Replace (v_1, v_2, \dots, v_n) by $(v, 0, \dots, 0)$ in (3.2), we have

$$(3.5) \quad \begin{aligned} & \|\emptyset(2v) - 2^2\emptyset(v)\| < \varepsilon, \\ & \left\| \frac{\emptyset(2v)}{2^2} - \emptyset(v) \right\| < \frac{\varepsilon}{2^2}. \end{aligned}$$

Replace v by $2^t v$ in (3.5), we have

$$(3.6) \quad \left\| \frac{\emptyset(2^{t+1}v)}{2^2} - \emptyset(2^t v) \right\| < \frac{\varepsilon}{2^2},$$

$$(3.7) \quad \left\| \frac{\emptyset(2^{t+1}v)}{2^{2(t+1)}} - \frac{\emptyset(2^t v)}{2^{2t}} \right\| < \frac{\varepsilon}{2^{2(t+1)}},$$

for all $v \in V$ and all $\varepsilon > 0$. Since

$$(3.8) \quad \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) = \sum_{i=0}^{m-1} \left(\frac{\emptyset(2^{i+1}v)}{2^{2(i+1)}} - \frac{\emptyset(2^i v)}{2^{2i}} \right).$$

So,

$$(3.9) \quad \begin{aligned} \left\| \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) \right\| & \leq \sum_{i=0}^{m-1} \left\| \frac{\emptyset(2^{i+1}v)}{2^{2(i+1)}} - \frac{\emptyset(2^i v)}{2^{2i}} \right\| \\ & < \sum_{i=0}^{m-1} \frac{\varepsilon}{2^{2(i+1)}} \\ & = \frac{\varepsilon}{3} \left(1 - \frac{1}{2^{2m}} \right). \end{aligned}$$

Replace v by $2^n v$, we get

$$(3.10) \quad \left\| \frac{\emptyset(2^{m+n}v)}{2^{2(m+n)}} - \frac{\emptyset(2^n v)}{2^{2n}} \right\| < \frac{\varepsilon}{3} \left(\frac{1}{2^{2n}} - \frac{1}{2^{2(m+n)}} \right),$$

for all $v \in V$ and all $\varepsilon > 0$. As R.H.S $\rightarrow 0$ as $m, n \rightarrow \infty$ then $\left\{ \frac{\emptyset(2^n v)}{2^{2n}} \right\}$ is a Cauchy sequence in W . Since W is a Banach space, thus sequence $\left\{ \frac{\emptyset(2^n v)}{2^{2n}} \right\}$ converges to some $Q_2(v) \in W$. For $v \in V$,

$$(3.11) \quad \begin{aligned} \|Q_2(v) - \emptyset(v)\| & = \left\| Q_2(v) - \frac{\emptyset(2^m v)}{2^{2m}} + \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) \right\| \\ & \leq \left\| Q_2(v) - \frac{\emptyset(2^m v)}{2^{2m}} \right\| + \left\| \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) \right\| \\ & < \left\| Q_2(v) - \frac{\emptyset(2^m v)}{2^{2m}} \right\| + \frac{\varepsilon}{3} \left(1 - \frac{1}{2^{2m}} \right). \end{aligned}$$

For all $v \in V$ and all $\varepsilon > 0$. Taking the limit $m \rightarrow \infty$,

$$(3.12) \quad \|Q_2(v) - \emptyset(v)\| < \frac{\varepsilon}{3}.$$

Replacing (v_1, v_2, \dots, v_n) by $(2^m v_1, 2^m v_2, \dots, 2^m v_n)$ in (3.2), we have

$$(3.13) \quad \begin{aligned} & \|D\emptyset(2^m v_1, 2^m v_2, \dots, 2^m v_n)\| < \varepsilon, \\ & \left\| D\emptyset \left(\frac{2^m v_1}{2^{2m}}, \frac{2^m v_2}{2^{2m}}, \dots, \frac{2^m v_n}{2^{2m}} \right) \right\| < \frac{\varepsilon}{2^{2m}}. \end{aligned}$$

Applying $m \rightarrow \infty$, show that Q_2 satisfies the functional equation (1.1).

To prove the uniqueness of the quadratic mapping Q_2 , assume that there exists another quadratic mapping Q'_2 , which satisfies inequality (3.12). Fix $v \in V$. Clearly, $Q_2(2^t v) = 2^{2t} Q_2(v)$ and $Q'_2(2^t v) = 2^{2t} Q'_2(v)$ for all $v \in V$, from (3.12), we have

$$(3.14) \quad \begin{aligned} \|Q_2(v) - Q'_2(v)\| &= \left\| \frac{Q_2(2^m v)}{2^{2m}} - \frac{\emptyset(2^m v)}{2^{2m}} + \frac{\emptyset(2^m v)}{2^{2m}} - \frac{Q'_2(2^m v)}{2^{2m}} \right\| \\ &< \frac{1}{2^{2m-1}} \cdot \frac{\varepsilon}{3}. \end{aligned}$$

Taking $m \rightarrow \infty$, we have $Q_2(v) = Q'_2(v)$. □

Theorem 3.2. *Assume that V and W are Banach spaces. If a function $\emptyset : V \rightarrow W$ satisfies the inequality*

$$(3.15) \quad \|D\emptyset(v_1, v_2, \dots, v_n)\| \leq \theta \sum_{i=1}^n \|v_i\|^p.$$

For some $p < 2$, for all $v_1, v_2, \dots, v_n \in V$, then the limit

$$(3.16) \quad Q_2(v) = \lim_{m \rightarrow \infty} \frac{\emptyset(2^m v)}{2^{2m}}$$

exists for each $v \in V$ and $Q_2 : V \rightarrow W$ is the unique quadratic function such that

$$(3.17) \quad \|\emptyset(v) - Q_2(v)\| \leq \frac{\theta \|v\|^p}{(2^2 - 2^p)},$$

for any $v \in V$.

Proof. Replace (v_1, v_2, \dots, v_n) by $(v, 0, \dots, 0)$ in (3.15), we have

$$(3.18) \quad \|\emptyset(2v) - 2^2 \emptyset(v)\| \leq \theta \|v\|^p,$$

$$(3.19) \quad \left\| \frac{\emptyset(2v)}{2^2} - \emptyset(v) \right\| \leq \frac{\theta \|v\|^p}{2^2}.$$

Replace v by $2^t v$ in (3.19), we have

$$(3.20) \quad \begin{aligned} & \left\| \frac{\emptyset(2^{t+1} v)}{2^2} - \emptyset(2^t v) \right\| \leq \frac{\theta \|2^t v\|^p}{2^2}, \\ & \left\| \frac{\emptyset(2^{t+1} v)}{2^{2(t+1)}} - \frac{\emptyset(2^t v)}{2^{2t}} \right\| \leq \frac{\theta \|v\|^p}{2^{2(t+1)-2t}} \end{aligned}$$

for all $v \in V$. Since

$$(3.21) \quad \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) = \sum_{i=0}^{m-1} \frac{\emptyset(2^{i+1} v)}{2^{2(i+1)}} - \frac{\emptyset(2^i v)}{2^{2i}}.$$

So,

$$(3.22) \quad \begin{aligned} \left\| \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) \right\| &\leq \sum_{i=0}^{m-1} \left\| \frac{\emptyset(2^{i+1} v)}{2^{2(i+1)}} - \frac{\emptyset(2^i v)}{2^{2i}} \right\| \\ &\leq \sum_{i=0}^{m-1} \frac{\theta \|v\|^p}{2^{2(i+1)-pi}} \\ &= \frac{\theta \|v\|^p}{(2^2 - 2^p)} \left(1 - \frac{1}{2^{m(2-p)}} \right). \end{aligned}$$

Replace v by $2^n v$, we get

$$(3.23) \quad \left\| \frac{\emptyset(2^{m+n} v)}{2^{2(m+n)}} - \frac{\emptyset(2^n v)}{2^{2n}} \right\| \leq \frac{\theta \|v\|^p}{(2^{2-p} - 1)} \left(\frac{1}{2^{2n}} - \frac{1}{2^{2(m+n)-mp}} \right)$$

for all $v \in V$. As R.H.S $\rightarrow 0$ as $m, n \rightarrow \infty$ then $\left\{ \frac{\emptyset(2^n v)}{2^{2n}} \right\}$ is a Cauchy sequence in W . Since W is Banach space, thus sequence $\left\{ \frac{\emptyset(2^n v)}{2^{2n}} \right\}$ converges to some $Q_2(v) \in W$. For $v \in V$,

$$\begin{aligned} \|Q_2(v) - \emptyset(v)\| &= \left\| Q_2(v) - \frac{\emptyset(2^m v)}{2^{2m}} + \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) \right\| \\ &\leq \left\| Q_2(v) - \frac{\emptyset(2^m v)}{2^{2m}} \right\| + \left\| \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) \right\| \\ &\leq \left\| Q_2(v) - \frac{\emptyset(2^m v)}{2^{2m}} \right\| + \frac{\theta \|v\|^p}{(2^2 - 2^p)} \left(1 - \frac{1}{2^{m(2-p)}} \right) \end{aligned}$$

for all $v \in V$. Taking the limit $m \rightarrow \infty$,

$$(3.24) \quad \|Q_2(v) - \emptyset(v)\| \leq \frac{\theta \|v\|^p}{(2^2 - 2^p)}.$$

Replacing (v_1, v_2, \dots, v_n) by $(2^m v_1, 2^m v_2, \dots, 2^m v_n)$ in (3.15), we have

$$\begin{aligned} \|D\emptyset(2^m v_1, 2^m v_2, \dots, 2^m v_n)\| &\leq \theta \sum_{i=1}^n \|2^m v_i\|^p, \\ \left\| D\emptyset \left(\frac{2^m v_1}{2^{2m}}, \frac{2^m v_2}{2^{2m}}, \dots, \frac{2^m v_n}{2^{2m}} \right) \right\| &\leq \frac{\theta}{2^{2m-mp}} \sum_{i=1}^n \|v_i\|^p. \end{aligned}$$

Applying $m \rightarrow \infty$ shows that Q_2 satisfies the functional equation (1.1).

To prove the uniqueness of the quadratic mapping Q_2 , assume that there exists another quadratic mapping Q'_2 , which satisfies inequality (3.24). Fix $v \in V$. Clearly, $Q_2(2^t v) = 2^{2t} Q_2(v)$ and $Q'_2(2^t v) = 2^{2t} Q'_2(v)$ for all $v \in V$. We have

$$\begin{aligned} \|Q_2(v) - Q'_2(v)\| &= \left\| \frac{Q_2(2^m v)}{2^{2m}} - \frac{\emptyset(2^m v)}{2^{2m}} + \frac{\emptyset(2^m v)}{2^{2m}} - \frac{Q'_2(2^m v)}{2^{2m}} \right\| \\ &\leq \frac{\theta \|v\|^p}{2^{2m-mp-1}(2^2 - 2^p)}. \end{aligned}$$

Taking $m \rightarrow \infty$, we have $Q_2(v) = Q'_2(v)$. □

Theorem 3.3. *Assume that V and W are Banach spaces. Let $\emptyset : V^n \rightarrow R^+$ be a function such that $\sum_{i=0}^{\infty} \frac{\emptyset(2^i v, 0, \dots, 0)}{2^{2i}}$ converges and*

$$(3.25) \quad \lim_{i \rightarrow \infty} \frac{\emptyset(2^i v, 0, \dots, 0)}{2^{2i}} = 0.$$

Also, if a function $\emptyset : V \rightarrow W$ satisfies the inequality

$$(3.26) \quad \|D\emptyset(v_1, v_2, \dots, v_n)\| \leq \emptyset(v_1, v_2, \dots, v_n)$$

for all $v_1, v_2, \dots, v_n \in V$, then the limit

$$(3.27) \quad Q_2(v) = \lim_{m \rightarrow \infty} \frac{\emptyset(2^m v)}{2^{2m}}$$

exists for each $v \in V$ and $Q_2 : V \rightarrow W$ is the unique quadratic function such that

$$(3.28) \quad \|\emptyset(v) - Q_2(v)\| \leq \sum_{i=0}^{\infty} \frac{\emptyset(2^i v, 0, \dots, 0)}{2^{2(i+1)}}$$

for any $v \in V$.

Proof. Replace (v_1, v_2, \dots, v_n) by $(v, 0, \dots, 0)$ in (3.26), we have

$$(3.29) \quad \|\emptyset(2v) - 2^2 \emptyset(v)\| \leq \emptyset(v, 0, \dots, 0),$$

$$(3.30) \quad \left\| \frac{\emptyset(2v)}{2^2} - \emptyset(v) \right\| \leq \frac{\emptyset(v, 0, \dots, 0)}{2^2}.$$

Replace v by $2^t v$ in (3.30), we have

$$(3.31) \quad \begin{aligned} \left\| \frac{\emptyset(2^{t+1} v)}{2^2} - \emptyset(2^t v) \right\| &\leq \frac{\emptyset(2^t v, 0, \dots, 0)}{2^2}, \\ \left\| \frac{\emptyset(2^{t+1} v)}{2^{2(t+1)}} - \frac{\emptyset(2^t v)}{2^{2t}} \right\| &\leq \frac{\emptyset(2^t v, 0, \dots, 0)}{2^{2(t+1)}} \end{aligned}$$

for all $v \in V$. Since

$$(3.32) \quad \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) = \sum_{i=0}^{m-1} \left(\frac{\emptyset(2^{i+1} v)}{2^{2(i+1)}} - \frac{\emptyset(2^i v)}{2^{2i}} \right).$$

So,

$$(3.33) \quad \begin{aligned} \left\| \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) \right\| &\leq \sum_{i=0}^{m-1} \left\| \frac{\emptyset(2^{i+1} v)}{2^{2(i+1)}} - \frac{\emptyset(2^i v)}{2^{2i}} \right\| \\ &\leq \sum_{i=0}^{m-1} \frac{\emptyset(2^i v, 0, \dots, 0)}{2^{2(i+1)}}. \end{aligned}$$

Replace v by $2^n v$, we get

$$(3.34) \quad \begin{aligned} \left\| \frac{\emptyset(2^{m+n} v)}{2^{2(m+n)}} - \frac{\emptyset(2^n v)}{2^{2n}} \right\| &\leq \sum_{i=0}^{m+n-1} \frac{\emptyset(2^i v, 0, \dots, 0)}{2^{2(i+1)}} + \sum_{i=0}^{n-1} \frac{\emptyset(2^i v, 0, \dots, 0)}{2^{2(i+1)}} \\ &\leq \sum_{i=n}^{m+n-1} \frac{\emptyset(2^i v, 0, \dots, 0)}{2^{2(i+1)}} \end{aligned}$$

for all $v \in V$. As R.H.S $\rightarrow 0$ as $n \rightarrow \infty$ then $\left\{ \frac{\emptyset(2^n v)}{2^{2n}} \right\}$ is a Cauchy sequence in W . Since W is a Banach space, thus sequence $\left\{ \frac{\emptyset(2^n v)}{2^{2n}} \right\}$ converges to some $Q_2(v) \in W$. For $v \in V$

$$\begin{aligned} \|Q_2(v) - \emptyset(v)\| &= \left\| Q_2(v) - \frac{\emptyset(2^m v)}{2^{2m}} + \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) \right\| \\ &\leq \left\| Q_2(v) - \frac{\emptyset(2^m v)}{2^{2m}} \right\| + \left\| \frac{\emptyset(2^m v)}{2^{2m}} - \emptyset(v) \right\| \\ &\leq \left\| Q_2(v) - \frac{\emptyset(2^m v)}{2^{2m}} \right\| + \sum_{i=0}^{m-1} \frac{\emptyset(2^i v, 0, \dots, 0)}{2^{2(i+1)}} \end{aligned}$$

for all $v \in V$. Taking the limit $m \rightarrow \infty$,

$$(3.35) \quad \|Q_2(v) - \emptyset(v)\| \leq \sum_{i=0}^{\infty} \frac{\emptyset(2^i v, 0, \dots, 0)}{2^{2(i+1)}}.$$

Replacing (v_1, v_2, \dots, v_n) by $(2^m v_1, 2^m v_2, \dots, 2^m v_n)$ in (3.26), we have

$$\begin{aligned} \|D\emptyset(2^m v_1, 2^m v_2, \dots, 2^m v_n)\| &\leq \emptyset(2^m v_1, 2^m v_2, \dots, 2^m v_n) \\ \left\| D\emptyset \left(\frac{2^m v_1}{2^{2m}}, \frac{2^m v_2}{2^{2m}}, \dots, \frac{2^m v_n}{2^{2m}} \right) \right\| &\leq \frac{\emptyset(2^m v_1, 2^m v_2, \dots, 2^m v_n)}{2^{2m}}. \end{aligned}$$

Applying $m \rightarrow \infty$, show that Q_2 satisfies the functional equation (1.1).

To prove the uniqueness of the quadratic mapping Q_2 , assume that there exists another quadratic mapping Q'_2 , which satisfies inequality (3.35). Fix $v \in V$. Clearly, $Q_2(2^t v) = 2^{2t} Q_2(v)$ and $Q'_2(2^t v) = 2^{2t} Q'_2(v)$ for all $v \in V$. We have

$$\begin{aligned} \|Q_2(v) - Q'_2(v)\| &= \left\| \frac{Q_2(2^m v)}{2^{2m}} - \frac{\emptyset(2^m v)}{2^{2m}} + \frac{\emptyset(2^m v)}{2^{2m}} - \frac{Q'_2(2^m v)}{2^{2m}} \right\| \\ &\leq \sum_{i=m}^{\infty} \frac{\emptyset(2^i v, 0, \dots, 0)}{2^{2(i+1)}} + \sum_{i=m}^{\infty} \frac{\emptyset(2^i v, 0, \dots, 0)}{2^{2(i+1)}}. \end{aligned}$$

Taking $m \rightarrow \infty$, we have $Q_2(v) = Q'_2(v)$. □

4. Counter example

We provide a counter example for the non-stability to the functional equation (1.1) in real normed space as:

Example 4.1. Let a mapping $\phi : R \rightarrow R$ be defined as

$$(4.1) \quad \phi(v) = \sum_{t=0}^{\infty} \frac{g(2^t v)}{2^{2t}},$$

where $g(v) = \begin{cases} \theta v^2, & |v| < 1, \\ \theta, & \text{else,} \end{cases}$

then the mapping $\phi : R \rightarrow R$ satisfies the inequality

$$(4.2) \quad |D\phi(v_1, v_2, \dots, v_n)| \leq (3n^3 - 25n^2 + 28n - 12) \frac{16}{3} \theta \sum_{i=1}^n |v_i|^2,$$

for all $v_1, v_2, \dots, v_n \in R, n \geq 8$, but there does not exist a quadratic mapping $Q_2 : R \rightarrow R$ satisfies $|\phi(v) - Q_2(v)| \leq \varepsilon |v|^2$ for all $v \in R$.

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