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Existence of solutions for systems of conformable fractional dynamic equations on time scales

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Abstract. In this article, we study the existence of solutions to systems of conformable fractional dynamic equations with periodic boundary value or initial value conditions. Existence results are obtained by using the method of solution-tube and Schauder's fixed-point theorem.

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1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Fractional differential equations play an important role in describing many phenomena and processes in various fields of science such as physics, chemistry, control systems, population dynamics, etc., see [1, 19, 24, 25]. In [3, 8, 10, 21, 27, 28, 29, 31], the authors studied fractional calculus on time scales and their important properties.

Recently, a new fractional derivative called the conformable fractional derivative, was introduced by Khalil et al. in [20]. Especialy, Benkhettou et al. in [11] introduced a conformable fractional calculus on an arbitrary time scale, which provides a natural extension of the conformable fractional calculus. Furthermore, in [4, 18, 23, 30] the authors introduced a conformable fractional calculus on an arbitrary time scale.

In this paper, we establish existence results for the following system of conformable fractional dynamic equations:

$$\begin{cases} x_{\Delta}^{(\alpha)}(t) = f(t, x^{\sigma}(t)), & \text{for } \Delta\text{-a.e. } t \in I = [a,b]_{\mathbb{T}}, \\ x \in (\mathfrak{BC}). \end{cases} \tag{1.1}$$

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Here $\mathbb{T}$ is an arbitrary time scale, $J = [a, \sigma(b)]_{\mathbb{T}}$ with $a, b \in \mathbb{T}$, $0 \leq a < b$ and $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a $L^1_{\alpha,\Delta}$-Carathéodory function, $x^{(\alpha)}_{\Delta}(t)$ denotes the conformable fractional derivative of x at t of order $\alpha \in (0, 1]$ and $(\mathfrak{BC})$ denotes the initial value or the periodic boundary value conditions:

$$x(a) = x_0, \tag{1.2}$$

$$x(a) = x(\sigma(b)). \tag{1.3}$$

B. Mirandette in [22], used the solution-tube method for the problem (1.1) with $\mathbb{T} = \mathbb{R}$, $\alpha = 1$, $a = 0$ and $b = 1$:

$$x'(t) = f(t, x(t)), \qquad \text{for a.e. } t \in [0, 1], \quad x \in (\mathfrak{BC}), \tag{1.4}$$

where f is a Carathéodory function, $x \in W^{1,1}([0, 1], \mathbb{R}^n)$.

Existence results for problem (1.1), (1.2) with $\mathbb{T} = \mathbb{R}$:

$$x^{(\alpha)}(t) = f(t, x(t)), \qquad \text{for a.e. } t \in I = [a, b], \qquad x(a) = x_0,$$

were obtained in [26], by using the Banach fixed point theorem with f a continuous function.

Existence of solutions were obtained by B. Bendouma et al. in [7] for the problem (1.1) with $\mathbb{T} = \mathbb{R}$:

$$x^{(\alpha)}(t) = f(t, x(t)), \qquad \text{for a.e. } t \in I \text{ and } a = 0, \quad x \in (\mathfrak{BC}), \tag{1.5}$$

by the help of the solution-tube method which generalizes the notions of lower and upper solutions given in [6].

In [17] Gilbert introduced the notion of solution-tube to problem (1.1) with $\alpha = 1$:

$$x^{\Delta}(t) = f(t, x^{\sigma}(t)), \text{ for } \Delta\text{-a.e. } t \in I = [a, b)_{\mathbb{T}}, \ a = \min\mathbb{T}, \ b = \max\mathbb{T}, \ x \in (\mathfrak{BC}),$$

which generalizes the notions of lower and upper solutions.

In the particular case where $n = 1$, existence results for problem (1.1) were obtained in [8] with nonlinear functional boundary conditions $B(x(a), x) = 0$ or $H(x, x^{\sigma}(b)) = 0$, where B and H are continuous functions. Their results were established, for the scalar case, with the method of lower and upper solutions and cover, as particular cases, the boundary conditions (1.2) and (1.3). In [5], the authors solved problem (1.1),(1.2) (for $n = 1$ and $\mathbb{T} = \mathbb{R}$) with f is a continuous function with the help of the solution-tube method which coincides with the lower and upper solutions method.

Motivated by the above works, we consider the existence of solutions for problem (1.1), using the notion of solution-tube of (1.1) which generalizes the notions of lower and upper solutions given in [8]. It is inspired by a notion of solution tube for systems of conformable fractional differential equations (1.5) introduced in [7], (see also [15, 22] and [17] on time scales).

This paper is organized as follows. In Section 2, we introduce the definition of conformable fractional calculus on time scales and their important properties. In Section 3, we prove the existence of solutions to problem (1.1) by using the method of solution-tube and Schauder's fixed-point theorem.

2. Preliminaries

In this section, we recall some notions and results which we will use in this article.

2.1. Calculus on time scales

We briefly recall the necessary concepts from the time-scale calculus. For more detailed discussions on the calculus on time scales, see [2, 9, 12, 13, 14, 17] and the references therein.

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward and backward jump operators $\sigma,\ \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \ \ \textit{and} \ \ \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively. We say that t is right-scattered (resp., left-scattered) if $\sigma(t) > t$ (resp., if $\rho(t) < t$); that t is isolated if it is right-scattered and left-scattered. Also, if $t < \sup \mathbb{T}$ and $t = \sigma(t)$, we say that t is right-dense. If $t > \inf \mathbb{T}$ and $t = \rho(t)$, we say that t is left dense. Points that are right dense and left dense are called dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^{\kappa} = \mathbb{T}\backslash\{M\}$, otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. The backward graininess $\nu : \mathbb{T} \to [0, \infty)$ is defined by $\nu(t) := t - \rho(t)$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_{\kappa} = \mathbb{T}\backslash\{m\}$, otherwise, $\mathbb{T}_{\kappa} = \mathbb{T}$. The set of all right-scattered points of $\mathbb{T}$ is at most countable. We denote it by $\mathcal{R}_{\mathbb{T}} := \{t \in \mathbb{T}, t < \sigma(t)\} = \{t_i : i \in N,\ for\ some\ N \subset \mathbb{N}\}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$.

Definition 2.1. ([32]) The function $f : \mathbb{T} \to \mathbb{R}^n$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and left-sided limits exist at left-dense points in $\mathbb{T}$. We write $f \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$.

Definition 2.2. ([12, 17]) For $f : \mathbb{T} \to \mathbb{R}^n$ and $t \in \mathbb{T}$, the delta derivative of f at t, denoted by $f^{\Delta}(t)$, is defined to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left\|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\right\| \leq \varepsilon\, |\sigma(t) - s|, \qquad for\ all\ s \in U.$$

The set of functions $f : \mathbb{T} \to \mathbb{R}^n$ which are Δ-differentiable and whose Δ-derivative is rd-continuous is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R}^n)$.

Definition 2.3. ([12]) The function $p : \mathbb{T} \to \mathbb{R}$ is μ-regressive if

$$1 + \mu(t)p(t) \neq 0,\ for\ all\ t \in \mathbb{T}^k.$$

The set of all μ-regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R}_{\mu} = \mathcal{R}_{\mu}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^{+}{}_{\mu} = \{p \in \mathcal{R}_{\mu} : 1 + \mu(t)p(t) > 0\ for\ all\ t \in \mathbb{T}\}$.

Definition 2.4. ([12]) If $p \in \mathcal{R}_\mu$, then we define the delta exponential function e_p by:

$$e_p(t,s) = \exp\Big(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\Big),$$

for $t, s \in \mathbb{T}$, where the μ-cylinder transformation is as in:

$$\xi_h(z) = \begin{cases} \frac{1}{h}\log(1+zh); & if\ h > 0; \\ z; & if\ \ h = 0. \end{cases}$$

where log is the principal logarithm function.

2.2. Conformable fractional calculus on time scales

We begin by introducing the notion of conformable fractional derivative of order $\alpha \in]0,1]$ for a function defined on arbitrary time scale $\mathbb{T}$.

Definition 2.5. ([20, 30])(Conformable fractional derivative) Given a function $f : [0,\infty) \to \mathbb{R}^n$ and a real constant $\alpha \in (0,1]$, the conformable fractional derivative of f of order α is defined by

$$f^{(\alpha)}(t) := \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{2.1}$$

for all $t > 0$. If $f^{(\alpha)}(t)$ exists and is finite, we say that f is α-differentiable at t.

If f is α-differentiable in some interval $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then the conformable fractional derivative of f of order α at $t = 0$ is defined as

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

Definition 2.6. ([11])(Conformable fractional derivative on time scale) Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$, and $\alpha \in]0,1]$. For $t > 0$, we define $f_\Delta^{(\alpha)}(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a δ-neighborhood $\mathcal{V}_t \subset \mathbb{T}$ (i.e., $\mathcal{V}_t :=]t-\delta, t+\delta[\cap \mathbb{T})$ of t, $\delta > 0$, such that

$$\Big|[f(\sigma(t)) - f(s)]\, t^{1-\alpha} - f_\Delta^{(\alpha)}(t)\,[\sigma(t) - s]\Big| \le \epsilon\,|\sigma(t) - s| \ \ for\ all\ s \in \mathcal{V}_t.$$

We call $f_\Delta^{(\alpha)}(t)$ the conformable fractional derivative of f of order α at t, and we define the conformable fractional derivative at 0 as $f_\Delta^{(\alpha)}(0) = \lim_{t\to 0^+} f_\Delta^{(\alpha)}(t)$.

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Example 2.7. Let $\alpha \in (0,1]$. Functions $f, g, h : \mathbb{T} \to \mathbb{R} : f(t) = t$, $p \in \mathbb{R}$, $g(t) \equiv \lambda$, $\lambda \in \mathbb{R}$, and $h(t) = e_p(t,a)$, $p \in \mathcal{R}_\mu$, are conformable fractional derivatives of order α with

$$f_\Delta^{(\alpha)}(t) = t^{1-\alpha};\ g_\Delta^{(\alpha)}(t) = 0;\ h_\Delta^{(\alpha)}(t) = t^{1-\alpha} p\ e_p(t,a).$$

Definition 2.8. ([30]) Assume $f : \mathbb{T} \to \mathbb{R}^n$, is a function, $f(t) = (f_1(t), f_2(t), ..., f_n(t))$ and let $t \in \mathbb{T}^k$. Then one defines $f_\Delta^{(\alpha)}(t) = ({f_1}_\Delta^{(\alpha)}(t), {f_2}_\Delta^{(\alpha)}(t), \ldots, {f_n}_\Delta^{(\alpha)}(t))$ (*provided it exists*). One calls $f_\Delta^{(\alpha)}(t)$ the conformable fractional derivative of f of order α at $t > 0$. The function f has conformal fractional differentiable of order α provided $f_\Delta^{(\alpha)}(t)$ exists for all t in $\mathbb{T}^k$. The function $f_\Delta^{(\alpha)} : \mathbb{T}^k \to \mathbb{R}^n$ is then called the conformable fractional derivative of f of order α, and we define the conformable fractional derivative at 0 as $f_\Delta^{(\alpha)}(0) = \lim_{t\to 0^+} f_\Delta^{(\alpha)}(t)$.

Remark 2.9. (i) If $\alpha = 1$, we have $f_\Delta^{(\alpha)}(t) = f^\Delta(t)$.

(ii) If $\mathbb{T} = \mathbb{R}$, then $f_\Delta^{(\alpha)} = f^{(\alpha)}$ is the conformable fractional derivative of f of order α.

We introduce the following spaces::

$$\begin{aligned} C^\alpha_{rd}([a,b]_\mathbb{T}, \mathbb{R}^n) &= \{f \text{ is conformal fractional differentiable of order } \alpha \\ &\quad \text{on } [a,b]_\mathbb{T} \text{ and } f_\Delta^{(\alpha)} \in C_{rd}([a,b]_\mathbb{T}, \mathbb{R}^n)\}. \\ C^\alpha_{0;rd}([a,b]_\mathbb{T}, \mathbb{R}^n) &= \{f \in C^\alpha_{rd}([a,b]_\mathbb{T}, \mathbb{R}^n) : f(a) = f(b) = 0\}. \\ C^\alpha_{a,b;rd}([a,b]_\mathbb{T}, \mathbb{R}^n) &= \{f \in C^\alpha_{rd}([a,b]_\mathbb{T}, \mathbb{R}^n) : f(a) = f(b)\}. \end{aligned}$$

Theorem 2.10. *([30]) Let $\alpha \in]0,1]$. Assume $f : \mathbb{T} \to \mathbb{R}^n$ and let $t \in \mathbb{T}^\kappa$. The following properties hold.*

(i) If f is conformal fractional differentiable of order α at $t > 0$, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is conformable fractional differentiable of order α at t with

$$f_\Delta^{(\alpha)}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha} = t^{1-\alpha} f^\Delta(t).$$

(iii) If t is right-dense, then f is conformable fractional differentiable of order α at t if and only if the limit $\lim_{s\to t} \frac{f(t)-f(s)}{(t-s)} t^{1-\alpha}$ exists as a finite number. In this case,

$$f_\Delta^{(\alpha)}(t) = t^{1-\alpha} f'(t).$$

(iv) If f is conformable fractional differentiable of order α at t, then

$$f(\sigma(t)) = f(t) + (\mu(t)) t^{\alpha-1} f_\Delta^{(\alpha)}(t).$$

Theorem 2.11. *([30]) Assume $f, g : \mathbb{T} \to \mathbb{R}^n$ are conformable fractional differentiable of order α. Then,*

(i) the sum $f + g$ is conformable fractional differentiable with $(f+g)_\Delta^{(\alpha)} = f_\Delta^{(\alpha)} + g_\Delta^{(\alpha)}$;

(ii) *for any* $\lambda \in \mathbb{R}$, λf *is conformable fractional differentiable with* $(\lambda f)_{\Delta}^{(\alpha)} = \lambda f_{\Delta}^{(\alpha)}$;

(iii) *if* f *and* g *are continuous, then the product* fg *is conformable fractional differentiable with* $(fg)_{\Delta}^{(\alpha)} = f_{\Delta}^{(\alpha)}\, g + (f \circ \sigma)\, g_{\Delta}^{(\alpha)} = f_{\Delta}^{(\alpha)}(g \circ \sigma) + f\, g_{\Delta}^{(\alpha)}$;

Theorem 2.12. *([11])(Chainrule) Let* $0 < \alpha \leq 1$. *Assume* $g : \mathbb{T} \to \mathbb{R}$ *is continuous and conformable fractional differentiable of order* α *at* $t \in \mathbb{T}^{\kappa}$ *and* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable. Then there exists* c *in the real interval* $[t, \sigma(t)]$ *with*

$$(f \circ g)_{\Delta}^{(\alpha)}(t) = \langle f'(g(c)), g_{\Delta}^{(\alpha)}(t) \rangle.$$

The next result is an adaptation of Theorem 2.12.

Theorem 2.13. *Let* $0 < \alpha \leq 1$ *and* W *be an open set of* $\mathbb{R}^n$ *and* $t \in \mathbb{T}$ *a right-dense point. If* $g : \mathbb{T} \to \mathbb{R}^n$ *is conformable fractional differentiable of order* α *at* $t > 0$ *and if* $f : W \to \mathbb{R}^n$ *is differentiable at* $g(t) \in W$, *then* $f \circ g$ *is conformable fractional differentiable of order* α *at* t *and*

$$(f \circ g)_{\Delta}^{(\alpha)}(t) = \langle f'(g(t)), g_{\Delta}^{(\alpha)}(t) \rangle.$$

Example 2.14. Let $\alpha \in (0, 1]$ and $x : \mathbb{T} \to \mathbb{R}^n$ is conformable fractional differentiable of order α at $t > 0$. We know that $\|.\| : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$ is differentiable. If $t = \sigma(t)$, by the previous theorem, we have

$$\|x(t)\|_{\Delta}^{(\alpha)} = \frac{< x(t), x_{\Delta}^{(\alpha)}(t) >}{\|x(t)\|}.$$

Now we introduce the α-conformable fractional integral (or α-fractional integral) on time scales.

Definition 2.15. ([11]) Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. Then the α-fractional integral of f, $0 < \alpha \leq 1$, is defined by $\int f(t)\Delta^{\alpha} t := \int f(t) t^{\alpha - 1} \Delta t$.

Definition 2.16. ([11]) Suppose $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Denote the indefinite α-fractional integral of f of order α, $\alpha \in (0,\ 1]$, as follows: $F(t) = \int f(t)\Delta^{\alpha} t$. Then, for all $a, b \in \mathbb{T}$, we define the Cauchy α-fractional integral by $\int_a^b f(t)\Delta^{\alpha} t = F(b) - F(a)$.

Theorem 2.17. *[11] Let* $\alpha \in (0,\ 1]$. *Then, for any rd-continuous function* $f : \mathbb{T} \to \mathbb{R}$, *there exist a function* $F : \mathbb{T} \to \mathbb{R}$ *such that* $F_{\Delta}^{(\alpha)}(t) = f(t)$ *for all* $t \in \mathbb{T}^{\kappa}$. *The function* F *is said to be an* α*-antiderivative of* f.

Theorem 2.18. *([11]) If* $f : \mathbb{T}^{\kappa} \to \mathbb{R}$ *is a rd-continuous function and* $t \in \mathbb{T}^{\kappa}$, *then*

$$\int_t^{\sigma(t)} f(s)\Delta^{\alpha} s = f(t)\mu(t)t^{\alpha - 1}.$$

The notions of Δ-measurable and Δ-integrable functions $f : \mathbb{T} \to \mathbb{R}$ are defined the same as those in [16].

Definition 2.19. ([30]) Assume $f : \mathbb{T} \to \mathbb{R}$, is a function. Let A be a Δ-measurable subset of $\mathbb{T}$. f is α-integrable on A if and only if $t^{\alpha-1}f(t)$ is integrable on A, and $\int_A f(t)\Delta^\alpha t = \int_A t^{\alpha-1}f(t)\Delta t$.

Definition 2.20. ([30]) Assume $f : \mathbb{T} \to \mathbb{R}^n$, is a function and $f(t) = (f_1(t), f_2(t), ..., f_n(t))$. Let A be a Δ-measurable subset of $\mathbb{T}$. Then f is α-integrable on A if and only if $f_i (i = 1, 2, ..., n)$ are α-integrable on A, and $\int_A f(t)\Delta^\alpha t = \Big(\int_A f_1(t)\Delta^\alpha t, \int_A f_2(t)\Delta^\alpha t, ..., \int_A f_n(t)\Delta^\alpha t \Big)$.

Theorem 2.21. *([30]) Let $\alpha \in (0,\ 1]$, $a, b, c \in \mathbb{T}$, $\lambda, \gamma \in \mathbb{R}$, and $f, g : \mathbb{T} \to \mathbb{R}^n$ be two rd-continuous functions. Then,*

(i) $$\int_a^b [\lambda f(t) + \gamma g(t)]\Delta^\alpha t = \lambda \int_a^b f(t)\Delta^\alpha t + \gamma \int_a^b g(t)\Delta^\alpha t;$$

(ii) $$\int_a^b f(t)\Delta^\alpha t = -\int_b^a f(t)\Delta^\alpha t;$$

(iii) $$\int_a^b f(t)\Delta^\alpha t = \int_a^c f(t)\Delta^\alpha t + \int_c^b f(t)\Delta^\alpha t;$$

(iv) $$\int_a^a f(t)\Delta^\alpha t = 0;$$

(v) if there exist $g : \mathbb{T} \to \mathbb{R}$ with $\|f(t)\| \leq |g(t)|$ for all $t \in [a,\ b]$, then
$$\Big\| \int_a^b f(t)\Delta^\alpha t \Big\| \leq \int_a^b \Big| g(t) \Big| \Delta^\alpha t.$$

Now we introduce the concept of an absolutely continuous function.

Definition 2.22. ([30]) A function $f : \mathbb{T} \to \mathbb{R}^n$ is said to be absolutely continuous on $[a, b]_\mathbb{T}$ (*i.e.*, $f \in AC([a, b]_\mathbb{T}, \mathbb{R}^n)$) if for every $\varepsilon > 0$, there exists a $\eta > 0$ such that if $\{[a_k, b_k]_\mathbb{T}\}_{k=1}^m$ is a finite pairwise disjoint family of subintervals of $[a, b]_\mathbb{T}$ satisfying
$$\sum_{k=1}^{k=m} (b_k - a_k) < \eta, \ then \ \sum_{k=1}^{k=m} \|f(\rho(b_k)) - f(a_k)\| < \varepsilon.$$
$$(i.e.,\ f \in AC([a, b]_\mathbb{T}, \mathbb{R}^n) \ if \ and \ only \ if \ f_i \in AC([a, b]_\mathbb{T}, \mathbb{R}) \ i = 1, ..., n)).$$

Theorem 2.23. *([30]) Assume function $f : [a, b]_\mathbb{T} \to \mathbb{R}^n$ is absolutely continuous on $[a, b]_\mathbb{T}$, then f is conformable fractional differentiable of order α Δ-a.e. on $[a, b]_\mathbb{T}$ and the following equality is valid:*
$$f(t) = f(a) + \int_{[a,t]_\mathbb{T}} f_\Delta^{(\alpha)}(s)\Delta^\alpha s \quad for \ all \ t \in [a, b]_\mathbb{T}.$$

The following Proposition can be proved analogously to Proposition 2.20 in [17].

Proposition 2.24. *Let $u : \mathbb{T} \to \mathbb{R}$ be an absolutely continuous function, then the Δ-measure of the set $\{t \in \mathbb{T}^{\kappa} \backslash \mathcal{R}_{\mathbb{T}^{\kappa}} : u(t) = 0$ and $u_{\Delta}^{(\alpha)}(t) \neq 0\}$ is zero.*

Next, we develop the fractional Sobolev's spaces via conformable fractional calculus and their important properties. The basic definitions and relations based on [30] are given:

Definition 2.25. ([30]) Let $E \subset \mathbb{T}$ be a Δ-measurable set and let $\varphi : \mathbb{T} \longrightarrow \overline{\mathbb{R}}$ be a Δ-measurable function. Say that φ belongs to $L^1_{\alpha,\Delta}(E, \mathbb{R})$ provided that either

$$\int_E |\varphi(s)| \Delta^{\alpha} s < +\infty$$

We say that a Δ-measurable function $f : E \to \mathbb{R}^n$ is in the set $L^1_{\alpha,\Delta}(E, \mathbb{R}^n)$ provided

$$\int_E \|f(s)\| \, \Delta^{\alpha} s = \int_E \|f(s)\| \, s^{\alpha-1} \Delta s < +\infty.$$

i.e. $\displaystyle\int_E |f_i(s)| \, \Delta^{\alpha} s < +\infty, \quad \textit{for each of its components } f_i : E \to \mathbb{R}, i = 1, ..., n$

Proposition 2.26. *([30]) The set $L^1_{\alpha,\Delta}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ is a Banach space together with the norm defined for $\varphi \in L^1_{\alpha,\Delta}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ as*

$$\|\varphi\|_{L^1_{\alpha,\Delta}([a,b]_{\mathbb{T}},\mathbb{R}^n)} := \int_{[a,b]_{\mathbb{T}}} \|\varphi(t)\| \Delta^{\alpha} t.$$

Definition 2.27. ([30]) Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$.

One says that $f \in W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ if and only if $f \in L^1_{\alpha,\Delta}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$, and there exists $g : [a,b)_{\mathbb{T}} \to \mathbb{R}^n$ such that $g \in L^1_{\alpha,\Delta}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ and
(2.2)

$$\int_{[a,b]_{\mathbb{T}}} f(t) \; \phi^{(\alpha)}_{\Delta}(t) \Delta^{\alpha} t = -\int_{[a,b]_{\mathbb{T}}} g(t) \; \phi^{\sigma}(t) \Delta^{\alpha} t, \text{ for all } \phi \in C^{\alpha}_{a,b;rd}([a,b]_{\mathbb{T}}, \mathbb{R}^n).$$

We denote

$$V^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}},\mathbb{R}^n) = \{u \in AC([a,b]_{\mathbb{T}},\mathbb{R}^n), u^{(\alpha)}_{\Delta} \in L^1_{\alpha,\Delta}([a,b]_{\mathbb{T}},\mathbb{R}^n) : u(a) = u(b)\}.$$

Theorem 2.28. *([30]) Assume that $f \in W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ and and that (2.2) holds for $g \in L^1_{\alpha,\Delta}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$. Then, there exists a unique function $x \in V^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ such that*

$$x = f, \; x^{(\alpha)}_{\Delta} = g \quad \Delta\text{-a.e. on } [a,b]_{\mathbb{T}}.$$

Theorem 2.29. *([30]) The set $W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$ is a Banach space together with the norm defined as*

$$\|\varphi\|_{W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}},\mathbb{R}^n)} := \int_{[a,b]_{\mathbb{T}}} |\varphi(t)| \Delta^{\alpha} t + \int_{[a,b]_{\mathbb{T}}} |(\varphi)^{(\alpha)}_{\Delta}(t)| \Delta^{\alpha} t,$$

for every $\varphi \in W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$.

Proposition 2.30. *Let $x \in W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$. Then x is conformable fractional differentiable of order α Δ-a.e. on $[a,b]_{\mathbb{T}}$ and*

$$\|x(t)\|^{(\alpha)}_{\Delta} = \frac{< x(t), x^{(\alpha)}_{\Delta}(t) >}{\|x(t)\|}, \ \Delta\text{-}a.e. \ on \ \{t \in [a,b]_{\mathbb{T}} : t = \sigma(t)\}.$$

Proof. If $x \in W^{\alpha,1}_{\Delta;a,b}([a,b]_{\mathbb{T}}, \mathbb{R}^n)$. By Theorems 2.28 and 2.23, x is conformable fractional differentiable of order α Δ-a.e. on $[a,b]_{\mathbb{T}}$. From Example 2.14, we obtain

$$\|x(t)\|^{(\alpha)}_{\Delta} = \frac{< x(t), x^{(\alpha)}_{\Delta}(t) >}{\|x(t)\|}, \ \Delta\text{- a.e. on } \{t \in [a,b]_{\mathbb{T}} : t = \sigma(t)\}.$$

□

We now define a notion of $L^1_{\alpha,\Delta}$-Carathéodory functions on a compact time scale.

Definition 2.31. A function $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is called a $L^1_{\alpha,\Delta}$-Carathéodory function if the following three conditions hold.

1. for every $x \in \mathbb{R}^n$, the function $t \mapsto f(t,x)$ is Δ-measurable;
2. the function $x \mapsto f(t,x)$ is continuous for Δ-almost every $t \in I$;
3. for every $r > 0$, there exists a function $h_r \in L^1_{\alpha,\Delta}(I,[0,\infty))$ such that $\|f(t,x)\| \leq h_r(t)$ for Δ-almost every $t \in I$ and for all $x \in \mathbb{R}^n$ such that $\|x\| \leq r$.

3. MAIN RESULT

In this section, we establish an existence result for the problem (1.1). A solution of problem (1.1) will be a function $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J, \mathbb{R}^n)$ for which (1.1) is satisfied. We introduce the notion of solution-tube of this problem as follows.

Definition 3.1. Let $(v, M) \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J, \mathbb{R}^n) \times W^{\alpha,1}_{\Delta;a,\sigma(b)}(J, [0,\infty))$. We say that (v, M) is a solution tube to problem (1.1) if

(i) $\langle x - v^{\sigma}(t), f(t,x) - v^{(\alpha)}_{\Delta}\rangle \leq M^{\sigma}(t) M^{(\alpha)}_{\Delta}(t)$ Δ-a.e. $t \in I$ and for every $x \in \mathbb{R}^n$ such that $\|x - v^{\sigma}(t)\| = M^{\sigma}(t)$,

(ii) $v^{(\alpha)}_{\Delta}(t) = f(t, v^{\sigma}(t))$ Δ-a.e. $t \in I$ *such that* $M^{\sigma}(t) = 0$,

(iii) $M(t) = 0$ for every $t \in I$ *such that* $M^{\sigma}(t) = 0$,

(iv) - if $(\mathfrak{BC})$ denotes (1.2), then $\|x_0 - v(a)\| \leq M(a)$,
- if $(\mathfrak{BC})$ denotes (1.3), then $\|v(\sigma(b)) - v(a)\| \leq M(a) - M(\sigma(b))$.

We denote

$$\mathbf{T}(v, M) := \left\{x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J, \mathbb{R}^n) : \|x(t) - v(t)\| \leq M(t), \ for \ all \ t \in J\right\}.$$

Remark 3.2. (i) If $\alpha = 1$ ($resp., \mathbb{T}$ is a real interval $[a, b]$), our definition of solution tube is equivalent to the notion of solution tube introduced in [17] (resp.,in [7]).

(ii) If $\alpha = 1$ and $\mathbb{T}$ is a real interval $[a, b]$, our definition of solution tube is equivalent to the notion of solution tube introduced in [22].

The existence of a solution-tube insures the existence of a solution to (1.1). We consider the following modified problem:

$$
\begin{cases} x_{\Delta}^{(\alpha)}(t) + \alpha\, t^{1-\alpha}\, x(\sigma(t)) = f(t, \overline{x}(\sigma(t))) + \alpha\, t^{1-\alpha}\, \overline{x}(\sigma(t)), \quad t \in I, \\ x \in (\mathfrak{B}\mathfrak{S}). \end{cases} \tag{3.1}
$$

where

$$
\overline{x}(t) = \begin{cases} \frac{M(t)}{\|x-v(t)\|}(x - v(t)) + v(t), & \text{if } \|x - v(t)\| > M(t), \\ x(t), & \text{if } \|x - v(t)\| \le M(t). \end{cases} \tag{3.2}
$$

We need the following auxiliary lemmas, which are direct generalizations of [8, Corollary 3.2 and Corollary 3.4], and we omit the proofs.

Lemma 3.3. *For every* $g \in L^1_{\alpha,\Delta}(I, \mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, $0 < \alpha \le 1$ *and* $-p \in \mathcal{R}_\mu$, *the problem*

$$
\begin{cases} x_{\Delta}^{(\alpha)}(t) - t^{1-\alpha} p(t)\, x(\sigma(t)) = g(t), \quad \Delta\text{-a.e. } t \in I; \\ x(a) = x_0. \end{cases} \tag{3.3}
$$

has a unique solution $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J, \mathbb{R}^n)$, *given by the following expression*

$$
x(t) := \int_{[a,\sigma(b)]_{\mathbb{T}}} G_I(t, s) g(s) \Delta^\alpha s + x_0 e_{-p}(a, t), \quad t \in J, \tag{3.4}
$$

where

$$
G_I(t, s) = e_{-p}(s, t) \begin{cases} 1, & a \le s \le t \le \sigma(b), \\ 0, & a \le t \le s \le \sigma(b). \end{cases} \tag{3.5}
$$

Lemma 3.4. *For every* $g \in L^1_{\alpha,\Delta}(I, \mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, $0 < \alpha \le 1$, $p \in \mathbb{R}\backslash\{0\}$, $-p \in \mathcal{R}_\mu$, *and* $e_{-p}(\sigma(b), a) \ne 1$, *the problem*

$$
\begin{cases} x_{\Delta}^{(\alpha)}(t) - t^{1-\alpha} p(t)\, x(\sigma(t)) = g(t), \quad \Delta\text{-a.e. } t \in I; \\ x(a) = x(\sigma(b)). \end{cases} \tag{3.6}
$$

has a unique solution $x \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R}^n)$, *given by the following expression*

$$x(t) := \int_{[a,\sigma(b)]_{\mathbb{T}}} G_P(t,s)g(s)\Delta^\alpha s, \quad t \in J, \tag{3.7}$$

where

$$G_P(t,s) = \frac{e_{-p}(s,t)}{e_{-p}(\sigma(b),a)-1}\begin{cases} e_{-p}(\sigma(b),a), & a \le s \le t \le \sigma(b), \\ 1, & a \le t \le s \le \sigma(b), \end{cases} \tag{3.8}$$

Similar to Lemma 2.24 in [17], we give the following Lemma of maximum principle.

Lemma 3.5. *Let* $r \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R})$, *such that* $r^{(\alpha)}_\Delta(t) < 0$ Δ*-a.e. on* $\{t \in I : r(\sigma(t)) > 0\}$. *If one of the two following conditions holds,*

(i) $r(a) \le 0$,

(ii) $r(a) \le r(\sigma(b))$,

then $r(t) \le 0$ *for every* $t \in J$.

Proof. Suppose the conclusion is false. Then, there exists $t_0 \in J$ such that $r(t_0) = \max_{t\in J} r(t) > 0$, since r is continuous on J. If $\rho(t_0) < t_0$, then $r^{(\alpha)}_\Delta(\rho(t_0))$ exists, since $\mu(\rho(t_0)) = t_0 - \rho(t_0) > 0$ and because $r \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R})$. Then,

$$r^{(\alpha)}_\Delta(\rho(t_0)) = \frac{r(t_0) - r(\rho(t_0))}{t_0 - \rho(t_0)}(\rho(t_0))^{1-\alpha} \ge 0,$$

which is a contradiction since $r(t_0) = r(\sigma(\rho(t_0))) > 0$.
If $t_0 = \rho(t_0) > a$, then there exists an interval $[t_1, \rho(t_0))$ such that $r(\sigma(t)) > 0$ for all $t \in [t_1, \rho(t_0)) \cap \mathbb{T}$. Thus,

$$0 \le r(t_0) - r(t_1) = r(\rho(t_0)) - r(t_1) = \int_{[t_1,\rho(t_0))\cap\mathbb{T}} r^{(\alpha)}_\Delta(s)\Delta^\alpha s < 0$$

by hypothesis and by Theorem 2.23. Hence, we get a contradiction. The case $t_0 = a$ is impossible if hypothesis (i) holds and if $r(a) \le r(\sigma(b))$, we must have $r(a) = r(\sigma(b))$. If we take $t_0 = \sigma(b)$, by using previous steps of this proof, one can check that $r(\sigma(b)) \le 0$, and then, the lemma is proved. □

Let us define the operators $\mathcal{N}_1, \mathcal{N}_2 : C(J,\mathbb{R}^n) \to C(J,\mathbb{R}^n)$ by

$$\mathcal{N}_1(x)(t) = \int_{[a,\sigma(b)]_{\mathbb{T}}} G_I(t,s)\left(f(s,\overline{x}(\sigma(s))) + \alpha\, s^{1-\alpha}\, \overline{x}(\sigma(s))\right)\Delta^\alpha s + x_0 e_\alpha(a,t)$$

and

$$\mathcal{N}_2(x)(t) = \int_{[a,\sigma(b)]_{\mathbb{T}}} G_P(t,s)\left(f(s,\overline{x}(\sigma(s))) + \alpha\, s^{1-\alpha}\, \overline{x}(\sigma(s))\right)\Delta^\alpha s$$

where G_I (resp., G_P) is Green's function related to the initial problem (3.3)(resp., periodic problem (3.6)) and is given by expression (3.5)(resp.,(3.8)) with $p = -\alpha$.

Clearly, from Lemma 3.3 (resp. Lemma 3.4) with $p = -\alpha$, the solutions of problem $(1.1), (1.2)$ (resp.,$(1.1), (1.3)$), coincide with the fixed points of operator $\mathcal{N}_1$ (resp., $\mathcal{N}_2$).

Proposition 3.6. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a $L^1_{\alpha,\Delta}$-Carathéodory function. Assume there exists $(v, M) \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J, \mathbb{R}^n) \times W^{\alpha,1}_{\Delta;a,\sigma(b)}(J, [0, \infty))$ a solution tube of $(1.1), (1.3)$, then the operator $\mathcal{N}_2$ is compact.*

Proof. We first observe that, from Definitions 2.31 and 3.1, there exists a function $h \in L^1_{\alpha,\Delta}(I, [0, \infty))$ such that

$$\|f(t, \overline{x}(\sigma(t))) + \alpha\ t^{1-\alpha}\ \overline{x}(\sigma(t))\| \leq h(t),$$

for Δ-a.e. $t \in I$ and all $x \in C(J, \mathbb{R}^n)$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $C(J, \mathbb{R}^n)$ converging to $x \in C(J, \mathbb{R}^n)$. In this case, it is clear that

$$\begin{aligned}
&\Big\|\mathcal{N}_2(x_n(t)) - \mathcal{N}_2(x(t))\Big\| \\
&\quad \leq \int_{[a,\sigma(b)]_{\mathbb{T}}} s^{\alpha-1}|G_P(t, s)|\Big\|\Big(f(s, \overline{x_n}(\sigma(s))) + \alpha\ s^{1-\alpha}\ \overline{x_n}(\sigma(s))\Big) \\
&\qquad -\Big(f(s, \overline{x}(\sigma(s))) + \alpha\ s^{1-\alpha}\ \overline{x}(\sigma(s))\Big)\Big\|\Delta s \\
&\quad \leq M \int_{[a,\sigma(b)]_{\mathbb{T}}} s^{\alpha-1}\Big\|\Big(f(s, \overline{x_n}(\sigma(s))) + \alpha\ s^{1-\alpha}\ \overline{x_n}(\sigma(s))\Big) \\
&\qquad -\Big(f(s, \overline{x}(\sigma(s))) + \alpha\ s^{1-\alpha}\ \overline{x}(\sigma(s))\Big)\Big\|\Delta s.
\end{aligned}$$

where $M := \max_{s,t\in J} |G_P(t, s)|$.

The continuity of operator $\mathcal{N}_2$ follows from the continuous dependence with respect to x of function f, the definition of $\overline{x}$ and Lebesgue's Dominated convergence Theorem.

To see that the set $\mathcal{N}_2(C(J, \mathbb{R}^n))$ is relatively compact on $C(J, \mathbb{R}^n)$, consider $x \in C(J, \mathbb{R}^n)$. Therefore,

$$\Big\|\mathcal{N}_2(x)(t)\Big\| \leq M\ \|h\|_{L^1_{\alpha,\Delta}(I,\mathbb{R}^n)}.$$

So, $\mathcal{N}_2(C(J, \mathbb{R}^n))$ is uniformly bounded. This set is also equicontinuous since for every $t_1 < t_2 \in J$,

$$\begin{aligned}
&\left\|\mathcal{N}_2(x)(t_2)-\mathcal{N}_2(x)(t_1)\right\| \\
&\leq \int_{[a,t_1]_{\mathbb{T}}} |G_P(t_2,s)-G_P(t_1,s)|\left\|\left(f(s,\overline{x}(\sigma(s)))+\alpha\ s^{1-\alpha}\ \overline{x}(\sigma(s))\right)\right\|\Delta^\alpha s \\
&\quad +\int_{[t_2,\sigma(b)]_{\mathbb{T}}} |G_P(t_2,s)-G_P(t_1,s)|\left\|\left(f(s,\overline{x}(\sigma(s)))+\alpha\ s^{1-\alpha}\ \overline{x}(\sigma(s))\right)\right\|\Delta^\alpha s \\
&\quad +\int_{[t_1,t_2]_{\mathbb{T}}} |G_P(t_2,s)-G_P(t_1,s)|\left\|\left(f(s,\overline{x}(\sigma(s)))+\alpha\ s^{1-\alpha}\ \overline{x}(\sigma(s))\right)\right\|\Delta^\alpha s \\
&\leq K|e_\alpha(s,t_2)-e_\alpha(s,t_1)|\Big(\int_{[a,t_1]_{\mathbb{T}}} h(s)\Delta^\alpha s+\int_{[t_2,\sigma(b)]_{\mathbb{T}}} h(s)\Delta^\alpha s\Big) \\
&\quad +2M\int_{[t_1,t_2]_{\mathbb{T}}} h(s)\Delta^\alpha s,
\end{aligned}$$

where

$$K:=\max_{s\in I}\{\frac{e_\alpha(\sigma(b),a)}{e_\alpha(\sigma(b),a)-1},\frac{1}{e_\alpha(\sigma(b),a)-1}\}=\frac{e_\alpha(\sigma(b),a)}{e_\alpha(\sigma(b),a)-1}>0.$$

By the Arzelà-Ascoli Theorem, we conclude that the set $\mathcal{N}_2(C(J,\mathbb{R}^n))$ is relatively compact in $C(J,\mathbb{R}^n)$. Hence, $\mathcal{N}_2$ is compact. □

The following result can be proved as the previous one.

Proposition 3.7. *Let $f : I\times\mathbb{R}^n\to\mathbb{R}^n$ be a $L^1_{\alpha,\Delta}$-Carathéodory function. Assume there exists $(v,M)\in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R}^n)\times W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,[0,\infty))$ a solution tube of* (1.1), (1.2)*, then the operator $\mathcal{N}_1$ is compact.*

Now, we can obtain our main theorem. The proof is based on the one given in [17].

Theorem 3.8. *Let $f : I\times\mathbb{R}^n\to\mathbb{R}^n$ be a $L^1_{\alpha,\Delta}$-Carathéodory function. Assume there exists $(v,M)\in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R}^n)\times W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,[0,\infty))$ a solution tube of* (1.1)*. Then, problem* (1.1) *has a solution $x\in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R}^n)\cap\mathrm{T}(v,M)$.*

Proof. We will do the proof for the initial case (1.2). As we will see, the proof for the periodic problem (1.3) is analogous.

By Proposition 3.7 the operator $\mathcal{N}_1$ is compact. It has a fixed point by the Schauder fixed-point Theorem. Lemma 3.3 implies that this fixed point is a solution for the problem (3.1). Then, it suffices to show that for every solution x of (3.1), $x\in\mathbf{T}(v,M)$.

Consider the set $\mathcal{B}:=\{t\in I:\|x(\sigma(t))-v(\sigma(t))\|>M(\sigma(t))\}$. By Proposition 2.30, Δ-a.e. on the set $\widehat{\mathcal{B}}=\{t\in\mathcal{B}:t=\sigma(t)\}$, we have

(3.9)

$$(\|x(t)-v(t)\|-M(t))^{(\alpha)}_\Delta=\frac{<x(\sigma(t))-v(\sigma(t)),x^{(\alpha)}_\Delta(t)-v^{(\alpha)}_\Delta(t)>}{\|x(\sigma(t))-v(\sigma(t))\|}-M^{(\alpha)}_\Delta(t).$$

If $t \in \mathcal{B}$ is right scattered, then $\mu(t) = \sigma(t) - t > 0$ and by Theorem 2.10, we have

$$\begin{aligned}
&(\|x(t)-v(t)\| - M(t))_{\Delta}^{(\alpha)} n \\
&= \frac{\|x(\sigma(t))-v(\sigma(t))\| - \|x(t)-v(t)\|}{\mu(t)} t^{1-\alpha} \\
&\quad - M_{\Delta}^{(\alpha)}(t) \\
&= \frac{\|x(\sigma(t))-v(\sigma(t))\|^2 - \|x(\sigma(t))-v(\sigma(t))\|\|x(t)-v(t)\|}{\mu(t)\|x(\sigma(t))-v(\sigma(t))\|} t^{1-\alpha} - M_{\Delta}^{(\alpha)}(t) \\
&\leq \frac{\langle x(\sigma(t))-v(\sigma(t)),\ x(\sigma(t))-v(\sigma(t))-(x(t)-v(t))\rangle}{\mu(t)\|x(\sigma(t))-v(\sigma(t))\|} t^{1-\alpha} - M_{\Delta}^{(\alpha)}(t) \\
&= \frac{\langle x(\sigma(t))-v(\sigma(t)),\ x_{\Delta}^{(\alpha)}(t)-v_{\Delta}^{(\alpha)}(t)\rangle}{\|x(\sigma(t))-v(\sigma(t))\|} - M_{\Delta}^{(\alpha)}(t).
\end{aligned}$$

Therefore, since (v, M) is a solution tube of problem (1.1), we have Δ-a.e. on $\{t \in \mathcal{B} : M(\sigma(t)) > 0\}$, that

$$\begin{aligned}
&(\|x(t)-v(t)\| - M(t))_{\Delta}^{(\alpha)} \\
&= \frac{\langle x(\sigma(t))-v(\sigma(t)),\ f(t,\overline{x}(\sigma(t))) + \alpha\, t^{1-\alpha}\,(\overline{x}(\sigma(t))-x(\sigma(t))) - v_{\Delta}^{(\alpha)}(t)\rangle}{\|x(\sigma(t))-v(\sigma(t))\|} \\
&\quad - M_{\Delta}^{(\alpha)}(t) \\
&= \frac{\langle \overline{x}(\sigma(t))-v(\sigma(t)),\ f(t,\overline{x}(\sigma(t))) - v_{\Delta}^{(\alpha)}(t)\rangle}{M(\sigma(t))} \\
&\quad + \alpha\, t^{1-\alpha}(M(\sigma(t)) - \|x(\sigma(t))-v(\sigma(t))\|) - M_{\Delta}^{(\alpha)}(t) \\
&< \frac{M(\sigma(t))M_{\Delta}^{(\alpha)}(t)}{M(\sigma(t))} - M_{\Delta}^{(\alpha)}(t) < 0.
\end{aligned}$$

On the other hand, we have Δ-a.e. on $\{t \in \mathcal{B} : M(\sigma(t)) = 0\}$, that

$$\begin{aligned}
&(\|x(t)-v(t)\| - M(t))_{\Delta}^{(\alpha)} \\
&\leq \frac{\langle x(\sigma(t))-v(\sigma(t)),\ f(t,\overline{x}(\sigma(t))) + \alpha\, t^{1-\alpha}\,(\overline{x}(\sigma(t))-x(\sigma(t))) - v_{\Delta}^{(\alpha)}(t)\rangle}{\|x(\sigma(t))-v(\sigma(t))\|} \\
&\quad - M_{\Delta}^{(\alpha)}(t) \\
&= \frac{\langle x(\sigma(t))-v(\sigma(t)),\ f(t,v(\sigma(t))) - v_{\Delta}^{(\alpha)}(t)\rangle}{\|x(\sigma(t))-v(\sigma(t))\|} \\
&\quad - \alpha\, t^{1-\alpha}(\|x(\sigma(t))-v(\sigma(t))\|) - M_{\Delta}^{(\alpha)}(t) \\
&< -M_{\Delta}^{(\alpha)}(t) \leq 0.
\end{aligned}$$

This last equality follows from Definition 3.1(iii) and Proposition 2.24.

If we set, $r(t) := \|x(t)-v(t)\| - M(t)$, then $r_{\Delta}^{(\alpha)} < 0$ Δ-a.e. on $\mathcal{B} := \{t \in I : r(\sigma(t)) > 0\}$. Moreover, since (v, M) is a solution tube to problem (1.1) and x

satisfies (1.2), then $r(a) < 0$ and, as consequence, Lemma 3.5 (i) implies that $\mathcal{B} = \emptyset$. So, $x \in T(v, M)$ and the result holds for this case.

When the periodic case is studied, we follow the same steps with operator $\mathcal{N}_2$ and we arrive to the fact that

$$r(a) - r(\sigma(b)) \leq \|v(a) - v(\sigma(b))\| - (M(a) - M(\sigma(b))) \leq 0,$$

and the result follows from Lemma 3.5 (ii). □

Next, we present another existence result which will follow from Theorem 3.8. The proof is based on the one given in [17]

Theorem 3.9. *Let $f : J^\kappa \times \mathbb{R}^n \to \mathbb{R}^n$ be an unbounded continuous function. If there exist nonnegative constants L and N such that*

$$\|f(t,p)\| \leq -2L \langle p, f(t,p)\rangle + N$$

for every $t \in J$, and every $p \in \mathbb{R}^n$, then the problem (1.1), (1.2) has at least one solution.

Proof. Observe that $L > 0$ since f is unbounded. By hypothesis, there exists a constant $K := N/2L$ such that $\langle p, f(t,p)\rangle \leq K$. Let us define $M : J \to [0,\infty)$ by

$$M(t) := \|x_0\| + 1 + \int_{[a,t)\cap\mathbb{T}} K\Delta^\alpha s.$$

Then, $M_\Delta^{(\alpha)}(t) = K$ for every $t \in J$ and, thus,

$$\langle p, f(t,p)\rangle \leq K \leq M^\sigma(t) M_\Delta^{(\alpha)}(t)$$

for every $t \in J$ and every $p \in \mathbb{R}^n$. Then, if we take $v = 0$, we get a solution tube (v, M) for our problem and by Theorem 3.8, the problem has a solution x such that $\|x(t)\| \leq M(t)$ for every $t \in J$.

□

The following example is a modified version, considering a periodic condition, of Example 4.6 in [17]:

Example 3.10. Consider the periodic problem:

(3.10)
$$\begin{cases} x_\Delta^{(\frac{1}{3})}(t) = a_1\|x(t)\|^2 x(t) - a_2 x(t) + a_3\varphi(t), & \Delta\text{-a.e. } t \in I = [0,2]_\mathbb{T}, \\ x(0) = x(\sigma(2)). \end{cases}$$

where $a_1, a_2, a_3 \in \mathbb{R}_+$ such that $a_2 \geq a_1 + a_3$ and $\varphi : I \to \mathbb{R}^n$ is a continuous function satisfying $\|\varphi(t)\| = 1$ for every $t \in I$. It is easy to check that $v = 0$ and $M = 1$, is a tube solution. By Theorem 3.8, problem (3.10) has a solution $x \in W^{\frac{1}{3},1}_{\Delta;0,\sigma(2)}([0,\sigma(2)]_\mathbb{T}, \mathbb{R}^n)$ such that $\|x(t)\| \leq 1$ for every $t \in [0,\sigma(2)]_\mathbb{T}$.

Remark 3.11. If $n = 1$, Definition 3.1 generalizes the notion of lower and upper solutions introduced in [8]. We recall this definition for problem (1.1):

Definition 3.12. A function $\gamma \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R})$ is called a lower solution of (1.1), if

(i) $\gamma^{(\alpha)}_{\Delta}(t) \geq f(t,\gamma^{\sigma}(t))$, for Δ-a.e. $t \in I$;

(ii) - if $(\mathfrak{BC})$ denotes (1.2), then $\gamma(a) \geq x_0$,
- if $(\mathfrak{BC})$ denotes (1.3), then $\gamma(a) \geq \gamma(\sigma(b))$.

A function $\delta \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R})$ is called an upper solution of (1.1) if it satisfies $(i),(ii)$ with the reversed inequalities.

If $\gamma,\ \delta \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R})$ are, respectively, lower and upper solutions of (1.1) such that $\delta(t) \leq \gamma(t)$ for every $t \in J$, then $v = (\gamma+\delta)/2$ and $M = (\gamma-\delta)/2$ is a solution tube for this problem. Conversely, if $(v,M) \in W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,\mathbb{R}) \times W^{\alpha,1}_{\Delta;a,\sigma(b)}(J,[0,\infty))$ is a solution tube of (1.1), then $\delta = v - M$ and $\gamma = v + M$ are, respectively, upper and lower solutions of (1.1).

Example 3.13. Let $\mathbb{T} = [-1,\frac{1}{5}] \cup [\frac{1}{3},1]$. Consider the periodic problem:

$$\begin{cases} x^{(\frac{1}{3})}_{\Delta}(t) = \dfrac{1-2t-x^5(\sigma(t))}{2\sqrt{t}}, & \Delta\text{-a.e. } t \in I = [0,1]_{\mathbb{T}}, \\ x(0) = x(\sigma(1)). \end{cases} \tag{3.11}$$

This problem is a particular case of (1.1), (1.3) with $n = 1$, $\alpha = \frac{1}{3}$ and $f(t,x^{\sigma}(t)) = \dfrac{1-2t-x^5(\sigma(t))}{2\sqrt{t}}$. It is clear that f is a $L^1_{\frac{1}{3},\Delta}$-Carathéodory function. Verify that with $v = 0$ and $M = 1$, (v,M) is a solution-tube of (3.11). By Theorem 3.8, the problem (3.11) has a solution $x \in W^{\frac{1}{3},1}_{\Delta;0,\sigma(1)}([0,\sigma(1)]_{\mathbb{T}})$, such that $|x(t)| \leq 1$ for every $t \in [0,\sigma(1)]_{\mathbb{T}}$.
Observe that $\delta = v - M = -1$ and $\gamma = v + M = 1$ are, respectively, upper and lower solutions of (3.11). This follows from the fact that

$$\begin{cases} \delta^{(\frac{1}{3})}_{\Delta}(t) = 0 \leq f(t,\delta^{\sigma}(t)) = \dfrac{1-t}{\sqrt{t}}, \text{ for } \Delta\text{-a.e. } t \in I = [0,1]_{\mathbb{T}},\ \delta(0) \leq \delta(\sigma(1)), \\ \gamma^{(\frac{1}{3})}_{\Delta}(t) = 0 \geq f(t,\gamma^{\sigma}(t)) = -\sqrt{t}, \text{ for } \Delta\text{-a.e. } t \in I = [0,1]_{\mathbb{T}},\ \gamma(0) \geq \gamma(\sigma(1)), \end{cases}$$

such that $-1 \leq x(t) \leq 1$, for all $t \in [0,\sigma(1)]_{\mathbb{T}}$.

Next, we present the following example is a modified version of Example 4.6 in [8], where we apply Theorem 3.8:

Example 3.14. Consider the initial problem:

$$\begin{cases} x_{\Delta}^{(\frac{1}{2})}(t) = -2\sin(x(t+1)) - \dfrac{e^t}{\sqrt{t}}x(t+1), & \Delta\text{-a.e. } t \in I = [0,b]_{\mathbb{Z}},\ b \in \mathbb{Z}, \\ x(0) = \frac{2}{3}. \end{cases} \tag{3.12}$$

This problem is a particular case of (1.1), (1.3) with $n = 1$, $\alpha = \frac{1}{2}$, $\mathbb{T} = \mathbb{Z}$ and $f(t, x^\sigma(t)) = -2\sin(x^\sigma(t)) - \dfrac{e^t}{\sqrt{t}}x^\sigma(t)$. It is clear that f is a $L^1_{\frac{1}{2},\Delta}$-Carathéodory function. Verify that with $v = 0$ and $M = 1$, (v, M) is a solution-tube of (3.12). By Theorem 3.8, the problem (3.12) has a solution $x \in W^{\frac{1}{2},1}_{\Delta;0,1}([0,\sigma(b)]_{\mathbb{Z}})$, such that $|x(t)| \leq 1$ for every $t \in [0, b+1]_{\mathbb{Z}}$.
Observe that $\delta = v - M = -1$ and $\gamma = v + M = 1$ are, respectively, upper and lower solutions of (3.12) follows from the fact that

$$\begin{cases} \delta_{\Delta}^{(\frac{1}{2})}(t) = 0 \leq f(t, \delta^\sigma(t)) = 2\sin(1) + \dfrac{e^t}{\sqrt{t}}, \text{ for } \Delta\text{-a.e. } t \in I = [0,b]_{\mathbb{Z}}, \delta(0) \leq \frac{2}{3}, \\ \gamma_{\Delta}^{(\frac{1}{2})}(t) = 0 \geq f(t, \gamma^\sigma(t)) = -2\sin(1) - \dfrac{e^t}{\sqrt{t}}, \text{ for } \Delta\text{-a.e. } t \in I = [0,b]_{\mathbb{Z}}, \gamma(0) \geq \frac{2}{3}, \end{cases}$$

such that $-1 \leq x(t) \leq 1$, for all $t \in [0, b+1]_{\mathbb{Z}}$.

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