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# $k-$type slant helix and generalized Bertrand curve in three dimensional quadratic Lie group

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**Abstract.** In the present article, we introduce $k-$type slant helix and generalized Bertrand curve on $3-$dimensional quadratic Lie group $G^3$. Specifically, we find necessary and sufficient conditions for any curve to become a $k-$type slant helix on $G^3$. In addition, the relation between the Frenet frame of a generalized Bertrand curve and the Frenet frame of its generalized Bertrand mate, is established. Next, we find a condition for generalized Bertrand mate when the generalized Bertrand curve is $1-$type slant helix in $G^3$. Further, we find the energy of the unit vector field generated by the axis of $k-$type slant helix.

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## 1. Introduction

All the advanced studies in helices begin with the result stated by M. A. Lancret in 1802, that is a curve in $E^3$ is a general helix if and only if $\frac{\tau}{\kappa} = constant$, where $\tau$ and $\kappa \neq 0$ are torsion and curvature along the curve. The result was first proved by B. de Saint Venant in 1845 and further re-visited by M. Barros [3] in 1997. Slant helices in $E^3$ are weaker versions of general helices because all the general helices are slant helices, whereas converse need not be true. Slant helices in $E^3$ are those curves having the angle between their principal normal and some fixed direction is constant [8]. The concept of slant helix in $E_1^3$ was introduced by A. T. Ali and R. Lopez in 2011 [1]. A curve $\gamma$ in a Minkowski 3 - space is said to be a slant helix if the scalar product of the principal normal vector of the curve and a fixed direction is constant.

In 2009 [6], the author introduced the notion of general helix in quadratic Lie group $G^3$ as a curve whose tangent vector field makes a constant angle with a left-invariant vector field. The harmonic curvature function of a general helix in $G^3$ is always a constant function. In 2013 [18], authors introduced the slant helices on three-dimensional Lie groups equipped with a bi-invariant metric $g$. A curve $\gamma$ in $(G^3, g)$ is said to be a slant helix if the normal vector

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field of the curve makes a constant angle with a left-invariant vector field of unit length. The arc length parametrized curve $\gamma$ in $(G^3, g)$ is a slant helix iff $K = \frac{\kappa(H^2+1)^{\frac{3}{2}}}{H'}$, is a constant function. Whereas, a curve in Lie group $G^3$ with left invariant metric $g$ is called as a generalized helix of first kind, second kind or third kind if there exist an unit left-invariant vector field $Z$ such that $g(T, Z) = constant$, $g(N, Z) = constant$ or $g(B, Z) = constant$, respectively [16]. Here $\{T, N, B\}$ is a Frenet frame along $\gamma$. The Darboux helices in $(G^3, g)$ were studied in [13].

A curve $\gamma$ in $E^3$ is a Bertrand curve iff there exist a linear relationship with constant coefficient between the curvature $\kappa$ and torsion $\tau$ of $\gamma$ [4]. Bertrand curve in Minkowski space or in three-dimensional sphere was introduced in [5, 2, 11]. In [19], the authors defined the generalized Bertrand curve in $E^3_1$. A curve in $E^3_1$ is said to be a generalized Bertrand curve if there exists another curve in $E^3_1$ such that the principal normal of the initial curve lies in the normal plane of the other curve and the angle between the principal normal of initial curve and principal normal of second curve is constant. In [12], authors defined the Bertrand curve in three dimensional quadratic Lie group. A curve in three-dimensional quadratic Lie group is a Bertrand curve iff there exist constants $\mu$ and $\nu$ such that the condition $\kappa(\mu + \nu H) = 1$, holds. Here $H$ is a harmonic curvature function along the curve.

This paper is organized as follows: in Section 2, we give definitions for $k-$type slant helix and generalized Bertrand curves with some known preliminary results in 3 - dimensional Lie group $G^3$ equipped with bi-invariant metric $g$. In Section 3, the necessary and sufficient conditions for an arc length parametrized curve $\gamma$ to become $k-$type slant helix are obtained. The relation between $1-$type, $2-$type, and $3-$type slant helices is established in the same section. In Section 4, the relation between the Frenet frame of generalized Bertrand curve and its generalized Bertrand mate lying on $(G^3, g)$ is given. Also, we find the condition for generalized Bertrand couple if one of them is $1-$type slant helix. In Section 5, we find the energy of the unit vector field generated from the axis of $k-$type slant helix. After that, we ended our paper with the conclusion by introducing a new frame in Lie group $(G^3, g)$ and corresponding to this frame, we have written the axis of the slant helix in $(G^3, g)$.

## 2. Preliminaries and some results

Let $G^n$ be an $n-$dimensional quadratic Lie group with definite bi-invariant metric $g$, Levi-Civita connection $\nabla$ and Lie algebra $\mathfrak{g}$. Then

$$g([S, K], T) = g(K, [T, S]) = g(S, [K, T]),$$

$$\nabla_S K = \frac{1}{2}[S, K],$$

for all $S, K, T \in \mathfrak{g}$. Let $\gamma$ be an arc length parametrized curve in $G^n$ and $\{b_1, b_2, ...b_n\}$ be the basis of Lie algebra $\mathfrak{g}$. Then for any vector field, $K(s) =$

$\sum_{i=1}^{n} k^i(s)b_i$ and $T(s) = \sum_{i=1}^{n} t^i(s)b_i$ along $\gamma$, we have [7]

$$[K, T] = \sum_{i,j} k^i t^j [b_i, b_j].$$

If $T(s) = \frac{d\gamma}{ds} = \sum_{i=1}^{n} t^i(s)b_i$ is the tangent vector field along $\gamma$, then covariant derivative of vector field $K$ along $\gamma$ is defined as [7],

$$\nabla_T K = \dot{K} - \frac{1}{2}[K, T]. \tag{2.1}$$

Here $\dot{K} = \sum_{i}^{n} \frac{dk^i}{ds} b_i$ and $t^i(s)$, $k^i(s)$ are smooth real valued functions. If $K$ is a left invariant vector field, then $\dot{K} = 0$.

**Definition 2.1.** [6] Let $\{T, N, B\}$ be a Frenet frame along an arc length parametrized curve $\gamma$ on $(G^3, g)$. Then

$$\tau_G = \frac{1}{2} g([T, N], B) = \frac{1}{2\kappa^2\tau} g(\ddot{T}, [T, \dot{T}]) + \frac{1}{4\kappa^2\tau} ||[T, \dot{T}]||^2,$$

where $\kappa$ and $\tau$ are the curvature and torsion of $\gamma$ in $G^3$.

**Definition 2.2.** [18] For an arc length parametrized curve $\gamma$ on $(G^3, g)$ with Frenet frame $\{T, N, B\}$, the harmonic curvature function of $\gamma$ is defined as $H = \frac{\tau - \tau_G}{\kappa}$.

Here we define the $k-$type slant helix and generalized Bertrand curve in three dimensional quadratic Lie group $G^3$ equipped with bi-invariant metric $g$.

**Definition 2.3.** Let $\gamma$ be a curve parametrized by arc length parameter which is lying on 3-dimensional quadratic Lie group $(G^3, g)$ and let $\{T, N, B\}$ be the Frenet frame along $\gamma$. Then $\gamma$ is said to be a $k-$type slant helix in $(G^3, g)$, if there exist a non null left invariant vector field $A$ along $\gamma$ such that $g(V_k, A) = C(constant) \neq 0$, where $V_k \in \{T, N, B\}$ and vector field $A$ is said to be an axis of $k-$type slant helix $\gamma$. Whereas, if $C = 0$, the curve is said to be a $k-$type orthogonal slant helix.

**Definition 2.4.** A curve $\gamma(s)$ in $G^3$ is said to be a generalized Bertrand curve if there exist another curve $\beta(\bar{s} = \phi(s))$ such that at the corresponding points, principal normal of $\gamma(s)$ lies in the normal plane of $\beta(\bar{s})$ such that the angle between principal normal of $\gamma(s)$ and principal normal of $\beta(\bar{s})$ is some constant $\omega$. The curve $\beta(\bar{s})$ is said to be a generalized Bertrand mate of $\gamma(s)$. The pair $(\gamma(s)$,$\beta(\bar{s}))$ is said to be generalized Bertrand couple.

From [18], we have following proposition which is used in our next sections.

**Proposition 2.5.** *Let $\gamma$ be an arc length parameterized curve lying on $(G^3, g)$, then $[T, N] = 2\tau_G B$ and $[T, B] = -2\tau_G N$.*

## 3. $k-$type slant helix in three dimensional quadratic Lie group $G^3$

Let $\gamma$ be a curve parametrized by arc length parameter in $(G^3, g)$ and $\{T, N, B\}$ be the Frenet frame along $\gamma$. Then the covariant derivative of $T$, $N$ and $B$ along $\gamma$ is defined as $\nabla_T T = \kappa N$, $\nabla_T N = -\kappa T + \tau B$ and $\nabla_T B = -\tau N$. Here $\kappa = g(\nabla_T T, N)$ , $\tau = g(\nabla_T B, N)$ and $g$ is bi-invariant metric on $G^3$.

**Theorem 3.1.** *[6] A curve $\gamma$ parametrized by arc length parameter on $(G^3, g)$ with curvature $\kappa(s) \neq 0$, be a $1-$type slant helix if and only if harmonic curvature function along $\gamma$ is a nonzero constant function ($H = C(constant) \neq 0$).*

**Theorem 3.2.** *Let $\gamma$ be a $1-$type slant helix with curvature $\kappa \neq 0$ in $(G^3, g)$, then axis for $\gamma$, is given by*

$$A = C_1(T + \frac{1}{H}B),$$

*where $C_1$ and $H$ both are constants along $\gamma$ in $(G^3, g)$.*

*Proof.* Let $\gamma$ be a $1-$type slant helix with curvature $\kappa \neq 0$ in $(G^3, g)$, then by Definition 2.3 there exists a left invariant vector field $A$ such that $g(T, A) = C(constant)$. Therefore the vector field $A$ can be written as

$$A = CT + f_2 N + f_3 B, \tag{3.1}$$

where $f_2$ and $f_3$ are smooth functions of parameter s along $\gamma$ in Lie group $G^3$. As $A$ is a left invariant vector field therefore after substituting $\dot{A} = 0$ in equation (2.1), $\nabla_T A = \frac{1}{2}[T, A]$. Thus

$$\begin{cases} g(\nabla_T A, T) = g(T, \frac{1}{2}[T, A]), \\ g(\nabla_T A, N) = g(N, \frac{1}{2}[T, A]), \\ g(\nabla_T A, B) = g(B, \frac{1}{2}[T, A]). \end{cases} \tag{3.2}$$

From equation (3.1) and Proposition 2.5, we obtain

$$\begin{cases} g(T, \frac{1}{2}[T, A]) = g(A, \frac{1}{2}[T, T]) = 0, \\ g(\nabla_T A, T) = Tg(A, T) - g(A, \nabla_T T) \\ \qquad\qquad = T(C) - g(CT + f_2 N + f_3 B, \kappa N), \\ g(N, \frac{1}{2}[T, A]) = -\frac{1}{2}g(A, [T, N]) = -\tau_G, \\ g(\nabla_T A, N) = Tg(A, N) - g(A, \nabla_T N) \\ \qquad\qquad = T(f_2) - g(CT + f_2 N + f_3 B, -\kappa T + \tau B), \\ g(B, \frac{1}{2}[T, A]) = -\frac{1}{2}g(A, [T, B]) = \tau_G, \\ g(\nabla_T A, B) = Tg(A, B) - g(A, \nabla_T N) = f_3' + f_2 \tau. \end{cases} \tag{3.3}$$

Equations (3.3) and (3.2), give the relations

$$\begin{cases} f_2 \kappa = 0, \\ f_2' + C\kappa - f_3(\tau - \tau_G) = 0, \\ f_3' + f_2(\tau - \tau_G) = 0. \end{cases} \tag{3.4}$$

As $\kappa \neq 0$, therefore from first part of equation (3.4), $f_2 = 0$. Then substituting $f_2 = 0$ in equation (3.4), we get $f_3 = C_1(constant)$ and $H = \frac{\tau - \tau_G}{\kappa} = \frac{C}{C_1} = m(constant) \neq 0$. Now, if $m = 0$ then $\tau = \tau_G$ and $C = 0$, is a contradiction. □

**Corollary 3.3.** *An arc length parametrized curve* $\gamma$ *with curvature* $\kappa \neq 0$ *is a* $1-$*type orthogonal slant helix in* $(G^3, g)$ *if and only if* $\tau = \tau_G$.

*Proof.* Let $\gamma$ be a $1-$type orthogonal slant helix with $\kappa \neq 0$ in $(G^3, g)$. Then axis $A$ of $\gamma$, will be

$$A = f_2 N + f_3 B, \tag{3.5}$$

where $f_2$ and $f_3$ are smooth functions of parameter s along $\gamma$ in Lie group $G^3$. Taking $c = 0$ in equation (3.4), gives us

$$\begin{cases} f_2\kappa = 0, \\ f_2' - f_3(\tau - \tau_G) = 0, \\ f_3' + f_2(\tau - \tau_G) = 0. \end{cases} \tag{3.6}$$

Thus from (3.6), we have $f_2 = 0$, $f_3 = C_1(constant)$ and $\tau = \tau_G$.

Conversely, assume that $\gamma$ is a curve with nonzero curvature function and $\tau = \tau_G$ lying on $G^3$, we take a vector field $A = C_1 B$. Now, $\dot{A} = C(\tau - \tau_G) = 0$, implies that $\nabla_T A = \frac{1}{2}[T, A]$. Hence $A$ is a left invariant vector field. Thus $\gamma$ is a $1-$type orthogonal slant helix with the axis $A$. □

**Theorem 3.4.** *Let* $\gamma$ *be an arc length parametrized curve with curvature* $\kappa \neq 0$ *on* $(G^3, g)$. *Then* $\gamma$ *is a* $2-$*type slant helix if and only if it holds that*

$$\left(\frac{\kappa(1 + H^2)}{H'}\right)' = -(\tau - \tau_G).$$

*Proof.* Let $\gamma$ is a $2-$type slant helix with $\kappa \neq 0$ in $(G^3, g)$ then axis $A$ of $\gamma$ can be expressed as

$$A = g_1 T + CN + g_3 B, \tag{3.7}$$

where $C$ is some constant and $g_1$, $g_3$ are smooth functions of parameter s and are given as $g_1 = g(T, A)$ and $g_3 = g(A, B)$ in $G^3$. Differentiating smooth functions $g_1$ and $g_3$ with respect to arc length parameter, we have

$$\begin{cases} g_1' &= Tg(T, A) = g(\nabla_T A, T) + g(\nabla_T T, A) \\ &= \frac{1}{2}g([T, T], A) + g(\kappa N, g_1 T + CN + g_3 B) \\ &= C\kappa. \\ g_3' &= Tg(A, B) = g(\nabla_T A, B) + g(\nabla_T B, A) \\ &= -\frac{1}{2}g([T, B], A) + g(-\tau N, g_1 T + CN + g_3 B) \\ &= C\tau_G - C\tau = -C(\tau - \tau_G). \end{cases} \tag{3.8}$$

Substituting $g_1$ and $g_3$ from equation (3.8) into equation (3.7), we obtain

$$A = C\int \kappa ds T + CN - C\int (\tau - \tau_G) ds B. \tag{3.9}$$

Since $\dot{A} = 0$, hence

$$\begin{cases} 0 = \dot{A} & = (C\kappa)T - C(\tau - \tau_G)B + C(-\kappa T + (\tau - \tau_G)B). \\ & +C\int \kappa ds(\kappa N) + C\int (\tau - \tau_G)ds(\tau - \tau_G)N \\ & = \left(C\kappa \int \kappa ds + C(\tau - \tau_G)\int (\tau - \tau_G)ds\right) N. \end{cases} \tag{3.10}$$

Thus from equation (3.10), we have

$$\int \kappa ds = -H \int \kappa H ds. \tag{3.11}$$

Thus $H \neq 0$, because $H = 0 \implies \kappa = 0$. So differentiating equation (3.11), we get

$$-\int \kappa H ds = \frac{\kappa(1 + H^2)}{H'}, \tag{3.12}$$

which on further differentiation gives the required condition.

Conversely, assume that $\gamma$ is a curve with nonzero curvature function lying on $(G^3, g)$, such that

$$\left(\frac{\kappa(1 + H^2)}{H'}\right)' = -(\tau - \tau_G).$$

Then on taking $A = C\int \kappa dsT + CN - C\int (\tau - \tau_G)dsB$, we have $\dot{A} = 0$, implies that $A$ is an axis of $\gamma$. Thus, $\gamma$ is a $2-$type slant helix with axis $A$. □

**Corollary 3.5.** *An axis for $2-$type slant helix with $\kappa \neq 0$ in $(G^3, g)$, is given by*

$$A = C\left(\frac{\kappa H(1 + H^2)}{H'}T + N + \frac{\kappa(1 + H^2)}{H'}B\right).$$

*Proof.* Substituting equation (3.12) in (3.9), we get the required result. □

**Theorem 3.6.** *If $\gamma$ is an arc length parametrized curve with $\kappa \neq 0$ on $(G^3, g)$, then $\gamma$ is a $2-$type orthogonal slant helix if and only if harmonic curvature function along $\gamma$ is constant.*

*Proof.* Let $\gamma$ be a $2-$type slant helix on $(G^3, g)$ with $\kappa \neq 0$ and axis orthogonal to $N$. Then by Definition 2.3, $g(N, A) = 0$, where $A$ is an axis of $\gamma$ and can be written as

$$A = g_1 T + g_3 B, \tag{3.13}$$

where $g_1$ and $g_3$ are smooth functions along $\gamma$ on $G^3$. Now, we have

$$\begin{aligned} 0 = \dot{A} &= g_1'T + g_3'B + g_1\dot{T} + g_3\dot{B} \\ &= g_1'T + g_3'B + g_1(\kappa N) + g_3(-\tau + \tau_G)N, \end{aligned}$$

which on collecting gives the following set of equations

$$\begin{cases} g_1' = 0, \\ g_1\kappa - g_3(\tau - \tau_G) = 0, \\ g_3' = 0. \end{cases} \tag{3.14}$$

If $\tau \neq \tau_G$, then from equation (3.14) we get $H = \frac{\tau - \tau_G}{\kappa} = constant$. If we take $\tau = \tau_G$ then we have $g_1 = H = 0$ and $g_3 = C_3 = constant$.

Conversely, assume that $\gamma$ be a curve with $\kappa \neq 0$ and $H = C_1(constant)$ lying on Lie group $G^3$.

*(i)* For $C_1 \neq 0$ take a vector field $A = CT + C\frac{\kappa}{(\tau - \tau_G)}B$, along $\gamma$ in $G^3$, here $C$ is some constant. It is easy to see that $A$ is left invariant vector field and $g(A, N) = 0$. Hence $\gamma$ is a $2-$type orthogonal slant helix with the axis $A$.

*(ii)* If $C_1 = 0$ , $\tau = \tau_G$. Then we can consider a vector field $A = C_3B$, along $\gamma$ in $G^3$, here $C_3$ is some constant. Now differentiating the vector field $A$, we get $\dot{A} = 0$ which implies that $\nabla_T A = \frac{1}{2}[T, A]$. Hence $A$ is left invariant vector field. Thus $\gamma$ is a $2-$type orthogonal slant helix with an axis $A$. □

**Corollary 3.7.** *Every $1-$type slant helices with $\kappa \neq 0$ are also $2-$type orthogonal slant helices and $3-$type slant helices whereas all the $2-$type slant helices with $\kappa \neq 0$ are neither $1-$type nor $3-$type slant helices with same axis in $(G^3, g)$.*

*Proof.* From Theorem 3.2 and equation 3.9 of Theorem 3.4 the axis for $1-$type slant helix and $2-$type slant helix are

$$\begin{cases} U^1 = CT + \frac{C\kappa}{\tau - \tau_G}B, \\ U^2 = C\int \kappa(s)dsT + CN - C\int(\tau - \tau_G)dsB, \end{cases} \tag{3.15}$$

where, $C$ is some constant, $U^1$ is an axis of $1-$type slant helix and $U^2$ is an axis of $2-$type slant helix. Thus, we have

$$\begin{cases} g(N, U^1) = 0, \\ g(B, U^1) = C\frac{\kappa}{\tau - \tau_G}, \\ g(T, U^2) = C\int \kappa(s)ds, \\ g(B, U^2) = -C\int(\tau - \tau_G)ds. \end{cases} \tag{3.16}$$

Equation (3.16) concludes the proof for the corollary. □

**Theorem 3.8.** *An arc length parametrized curve $\gamma$ with $\kappa \neq 0$ on $(G^3, g)$ is a $3-$type slant helix if and only if any one of the following conditions holds*

*(1) $H$ is a nonzero constant function.*

*(2) If $H=0$, then there exist smooth functions $h_1 = rsin(\theta + \int \kappa ds)$ and $h_2 = rcos(\theta + \int \kappa(s)ds)$ on $(G^3, g)$, such that*

$$A = rsin(\theta + \int \kappa ds)T + rcos(\theta + \int \kappa ds)N + CB,$$

*is an axis of $\gamma$ on $G^3$.*

*Proof.* Let $\gamma$ be a $3-$type slant helix with $\kappa \neq 0$ in $(G^3, g)$. Then by Definition 2.3, there exists a left invariant vector field $A$ such that $g(B, A) = C(constant)$. Thus vector field can be written as,

$$A = h_1 T + h_2 N + CB, \tag{3.17}$$

where $h_1$ and $h_2$, are smooth function on $(G^3, g)$. Now on differentiating vector field $A$, we obtain

$$\begin{cases} 0 & = \dot{A} = h_1' T + h_1 \dot{T} + h_2' N + h_2 \dot{N} + C\dot{B} \\ & = h_1' T + h_2' N + h_1(\kappa N) + h_2(-\kappa T + (\tau - \tau_G)B) - C(\tau - \tau_G)N. \end{cases} \tag{3.18}$$

From equation (3.18), we get the following set of equations

$$\begin{cases} h_1' - h_2\kappa = 0, \\ h_2' + h_1\kappa - C(\tau - \tau_G) = 0, \\ h_2(\tau - \tau_G) = 0. \end{cases} \tag{3.19}$$

From the third part of the equation (3.19), either $h_2 = 0$ or $\tau = \tau_G$.

**Case *(i)*** Let $\tau \neq \tau_G$ then from system of equations (3.19), $h_2 = 0$ , $h_1 = c_1 = constant$ and $H = \frac{C_1}{C} = constant$.

**Case *(ii)*** If $\tau = \tau_G$ then system of equations (3.19), gives

$$\begin{cases} h_1' - h_2\kappa = 0, \\ h_2' + h_1\kappa = 0. \end{cases} \tag{3.20}$$

The solution of equation ((3.20), gives $h_1$ and $h_2$ as

$$\begin{cases} h_1 = C_1 sin(\int \kappa ds) + C_2 cos(\int \kappa ds), \\ h_2 = C_1 cos(\int \kappa ds) - C_2 sin(\int \kappa ds). \end{cases} \tag{3.21}$$

where $C_1$ and $C_1$ are arbitrary constants. If we take $C_1 = rcos(\theta)$ and $C_2 = rsin(\theta)$, for some constant angle $\theta$, then

$$\begin{cases} h_1 = rsin(\theta + \int \kappa ds), \\ h_2 = rcos(\theta + \int \kappa ds). \end{cases} \tag{3.22}$$

Therefore, a consequence is that

$$h_1^2 + h_2^2 = constant. \tag{3.23}$$

Thus for $3-$type slant helix $\gamma$ with curvature $\kappa = constant \neq 0$ and $\tau = \tau_G$, an axis not orthogonal to $B$ is

$$A = rsin(\theta + \int \kappa ds)T + rcos(\theta + \int \kappa ds)N + CB.$$

Conversely assume that $\gamma$ with $\kappa \neq 0$ and either $H$ is a nonzero constant or satisfies the condition defined in equation (3.23), lying on $(G^3, g)$. Then

*(i)* If harmonic curvature is a non zero constant along $\gamma$ then take the vector field $A = CHT + CB$, which is a left invariant vector field and $g(A, B) = C$. Thus $\gamma$ is a 3- type slant helix with axis $A$.

*(ii)* If the curve $\gamma$ satisfies the condition given in equation (3.23) and $H = 0$, then take the vector field $A = rsin(\theta + \int \kappa ds)T + rcos(\theta + \int \kappa ds)N + CB$, where $C$ and $\theta$ are some constants. Thus $A$ is a left invariant vector field and $g(A, B) = C$. Hence $\gamma$ is a 3 - type slant helix. □

**Corollary 3.9.** *Let $\gamma$ be an arc length parametrized curve on $G^3$ with nonzero curvature function. Then $\gamma$ is a $3-$type orthogonal slant helix if and only if there exists smooth functions $h_1 = rsin(\theta + \int \kappa ds)$ and $h_2 = rcos(\theta + \int \kappa ds)$ on $(G^3, g)$, such that $A = h_1T + h_2N$ is an axis of $\gamma$.*

*Proof.* Let $\gamma$ be a $3-$type orthogonal slant helix with $\kappa \neq 0$ on $(G^3, g)$. Then from Definition 2.3, and equation (3.19), we obtain

$$\begin{cases} h_1' - h_2\kappa = 0, \\ h_2' + h_1\kappa = 0, \\ h_2(\tau - \tau_G) = 0. \end{cases} \tag{3.24}$$

Since $h_2 \neq 0$, therefore equation (3.24), gives $\tau = \tau_G$. Then from the first and second part of the equation (3.24), we obtain the required condition.

The converse part can be proved in similar way as we proved in second case of the converse part of Theorem 3.8. □

**Corollary 3.10.** *A $3-$type slant helix with $\kappa \neq 0$ and $\tau = \tau_G$ is neither $1-$type nor $2-$type slant helix with same axis in $(G^3, g)$.*

*Proof.* From Theorem 3.8 the axis of $3-$type slant helix $\gamma$ with $\kappa \neq 0$ and $\tau = \tau_G$ in $(G^3, g)$ is

$$A = rsin(\theta + \int \kappa ds)T + rcos(\theta + \int \kappa ds)N + CB, \tag{3.25}$$

where $r$ and $\theta$ are some constants. Therefore from equation (3.25), we have

$$\begin{cases} g(A, T) = rsin(\theta + \int \kappa ds), \\ g(A, N) = rcos(\theta + \int \kappa ds). \end{cases} \tag{3.26}$$

Thus from equation (3.26), we can conclude that a $3-$type slant helix with $\kappa(s) \neq 0$ and $\tau = \tau_G$ is neither $1-$type nor $3-$type slant helix in $(G^3, g)$. □

## 4. Generalized Bertrand curves in three k dimensional quadratic Lie group $G^3$

In this section, we give some theorems for the generalized Bertrand curve on $(G^3, g)$. Generalized Bertrand curves are the generalization of Bertrand curves in $G^3$ which are defined in [12]. In this section for generalized Bertrand couple $(\gamma, \beta)$, we will consider arc length parameter $(s)$ for $\gamma$ and arc length parameter $(\bar{s})$ for $\beta$ on $(G^3, g)$.

**Theorem 4.1.** *Let $(\gamma, \beta)$ be a generalized Bertrand couple with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ along $\gamma$ and $\{\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}\}$ along $\beta$ on $(G^3, g)$. Then the relation between these apparatuses is*

$$\begin{bmatrix} \bar{T} \\ \bar{N} \\ \bar{B} \end{bmatrix} = \frac{1}{\sqrt{\kappa_1^2 + \kappa_2^2}} \begin{bmatrix} \kappa_1 & 0 & \kappa_2 \\ \kappa_2 \sin\omega & \cos\omega\sqrt{\kappa_1^2+\kappa_2^2} & -\kappa_1 \sin\omega \\ -\kappa_2 \cos\omega & \sin\omega\sqrt{\kappa_1^2+\kappa_2^2} & \kappa_1 \cos\omega \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

*where $\kappa_1 = 1 + \chi\bar{\kappa}\cos\omega$, $\kappa_2 = \chi(\bar{\tau} - \bar{\tau}_G)$ and $\chi = constant$, and $\omega$ is a constant angle between $N$ and $\bar{N}$.*

*Proof.* Let $(\gamma, \beta)$ be a generalized Bertrand couple with Frenet apparatus $\{T, N, B, \kappa, \tau\}$ along $\gamma$, and $\{\bar{T}, \bar{N}, \bar{B}, \bar{\kappa}, \bar{\tau}\}$ along $\beta$, then we have [16, 12],

$$\beta(\bar{s}) = \gamma(s) + \chi(s)N(s), \tag{4.1}$$

where $\bar{s} = \bar{s}(s)$ and $\chi(s)$ is giving the distance between generalized Bertrand couple at corresponding points. Also by the definition of generalized Bertrand curve

$$N(s) = cos\omega\bar{N} + \sin\omega\bar{B}. \tag{4.2}$$

On substituting (4.2) in equation (4.1), and then differentiating the equation, we obtain

$$\begin{cases} \bar{T}\frac{d\bar{s}}{ds} & = T + \chi'(s)\cos\omega\bar{N} + \chi'(s)\sin\omega\bar{B} + \chi(s)\frac{d\bar{s}}{ds}\Big(\cos\omega(-\bar{\kappa}\bar{T} \\ & \quad +(\bar{\tau} - \bar{\tau}_G)\bar{B}) - \sin\omega(\bar{\tau} - \bar{\tau}_G)\bar{N}\Big). \end{cases} \tag{4.3}$$

As $N$ is lying in the normal plane of $\beta$, therefore from equation (4.3) and equation (4.2), we have

$$\begin{cases} 0 & = g(N, \bar{T}\frac{d\bar{s}}{ds}) = g\Big(N, T + \chi'(s)\cos\omega\bar{N} + \chi'(s)\sin\omega\bar{B}\Big) \\ & \quad +g\Big(N, \chi(s)\frac{d\bar{s}}{ds}\{(\bar{\tau} - \bar{\tau}_G)\bar{B} - \sin\omega(\bar{\tau} - \bar{\tau}_G)\bar{N}\}\Big) \\ & \quad +g\Big(N, \chi(s)\frac{d\bar{s}}{ds}\cos\omega(-\bar{\kappa}\bar{T})\Big) \\ & = \chi'(s)(cos^2\omega + \sin^2\omega) + \chi(s)\left(\cos\omega\sin\omega(\bar{\tau} - \bar{\tau}_G)\right) \\ & \quad -\chi(s)\left(\cos\omega\sin\omega(\bar{\tau} - \bar{\tau}_G)\right) \\ & = \chi'(s). \end{cases} \tag{4.4}$$

Thus substituting equation (4.4), in equation (4.3), we get

$$T = \{(1 + \chi\bar{\kappa}\cos\omega)\bar{T} + \chi\sin\omega(\bar{\tau} - \bar{\tau}_G)\bar{N} - \chi\cos\omega(\bar{\tau} - \bar{\tau}_G)\bar{B}\}\frac{d\bar{s}}{ds}. \tag{4.5}$$

Taking the inner product of $T$ with itself, we obtain

$$\frac{ds}{d\bar{s}} = \sqrt{\kappa_1^2 + \kappa_2^2} \neq 0, \tag{4.6}$$

where $\kappa_1 = 1 + \chi\bar{\kappa}\cos\omega$ and $\kappa_2 = \chi(\bar{\tau} - \bar{\tau}_G)$. As we know $B = T \times N$, therefore

$$B = \{\chi(\bar{\tau} - \bar{\tau}_G)\bar{T} - \sin\omega(1 + \chi\bar{\kappa}\cos\omega)\bar{N} + cos\omega(1 + \chi\bar{\kappa}\cos\omega)\bar{B}\}\frac{d\bar{s}}{ds}. \tag{4.7}$$

Multiplying equation (4.7) by factor $-(1 + \chi\bar{\kappa}\cos\omega)$ and equation (4.5) with factor $\chi(\bar{\tau} - \bar{\tau}_G)$, and then adding together the following relation is obtained

$$-\kappa_1 B\frac{ds}{d\bar{s}} + \kappa_2 T\frac{ds}{d\bar{s}} = (\sin\omega\kappa_1^2 + \sin\omega\kappa_2^2)\bar{N} - (\cos\omega\kappa_1^2 + \cos\omega\kappa_2^2)\bar{B}. \tag{4.8}$$

Equation (4.8) and equation (4.2), gives us

$$\begin{cases} \bar{B} = \frac{1}{\sqrt{\kappa_1^2+\kappa_2^2}}\{-\kappa_2\cos\omega T + \sqrt{\kappa_1^2 + \kappa_2^2}\sin\omega N + \kappa_1\cos\omega B\}, \\ \bar{N} = \frac{1}{\sqrt{\kappa_1^2+\kappa_2^2}}\{\kappa_2\sin\omega T + \sqrt{\kappa_1^2 + \kappa_2^2}\cos\omega N - \kappa_1\sin\omega B\}. \end{cases} \tag{4.9}$$

By using equation (4.9), tangent vector field along curve $\beta$ can be found by taking the cross product of normal and binormal of $\beta$ i.e., $\bar{T} = \bar{N} \times \bar{B}$

$$\bar{T} = \frac{1}{\sqrt{\kappa_1^2 + \kappa_2^2}}\{\kappa_1 T + \kappa_2 B\}. \tag{4.10}$$

Therefore equation (4.9) and equation (4.10) give the relation between Frenet frame along $\gamma$ and $\beta$. □

**Theorem 4.2.** *If* $(\gamma, \beta)$ *is a generalized Bertrand couple with Frenet apparatus as defined in Theorem 4.1, then harmonic curvature along the curve* $\gamma$ *is given by*

$$H = \frac{\bar{\tau} - \bar{\tau}_G}{\bar{\kappa}\cos\omega(1 + \chi\bar{\kappa}\cos\omega) + \chi(\bar{\tau} - \bar{\tau}_G)^2}.$$

*Proof.* Let $(\gamma, \beta)$ be a generalized Bertrand couple with Frenet apparatus as defined in Theorem 4.1, then

$$N(s) = cos\omega\bar{N} + \sin\omega\bar{B}. \tag{4.11}$$

Now differentiating equation (4.11), we obtain

$$\begin{aligned} -\kappa T + (\tau - \tau_G)B = \dot{N} &= cos\omega\dot{\bar{N}}\frac{d\bar{s}}{ds} + \sin\omega\dot{\bar{B}}\frac{d\bar{s}}{ds} \\ &= \{cos\omega(-\bar{\kappa}\bar{T} + (\bar{\tau} - \bar{\tau}_G)\bar{B}) - \sin\omega(\bar{\tau} - \bar{\tau}_G)\bar{N}\}\frac{d\bar{s}}{ds}, \end{aligned} \tag{4.12}$$

where $\dot{\bar{N}} = \nabla_{\bar{T}}\bar{N} - \frac{1}{2}[\bar{T}, \bar{N}]$, $\dot{\bar{B}} = \nabla_{\bar{T}}\bar{B} - \frac{1}{2}[\bar{T}, \bar{B}]$ and $2\bar{\tau}_G = g([\bar{T}, \bar{N}], \bar{B})$. If we substitute $\bar{T}$, $\bar{N}$, and $\bar{B}$ from Theorem 4.1 in equation (4.12), then

$$\begin{aligned} -\kappa T + (\tau - \tau_G)B = & \frac{1}{\kappa_1^2 + \kappa_2^2}\left[-\bar{\kappa} cos\omega(\kappa_1 T + \kappa_2 B) - \kappa_2 cos^2\omega(\bar{\tau} - \bar{\tau}_G)T\right] \\ & + \frac{1}{\kappa_1^2 + \kappa_2^2}\left[cos\omega(\bar{\tau} - \bar{\tau}_G)(\sqrt{\kappa_1^2 + \kappa_2^2}\sin\omega N + \kappa_1\cos\omega B)\right] \\ & + \frac{1}{\kappa_1^2 + \kappa_2^2}\left[-\sin\omega(\bar{\tau} - \bar{\tau})(\kappa_2\sin\omega T + \sqrt{\kappa_1^2 + \kappa_2^2}\cos\omega N)\right] \\ & + \frac{1}{\kappa_1^2 + \kappa_2^2}\left[\kappa_1\sin^2\omega(\bar{\tau} - \bar{\tau}_G)B\right]. \end{aligned}$$

Thus, we have

$$\begin{cases} -\kappa T + (\tau - \tau_G)B & = \frac{1}{\kappa_1^2+\kappa_2^2}\{(-\bar{\kappa}\kappa_1 cos\omega - \kappa_2(\bar{\tau} - \bar{\tau}_G))T \\ & +(-\bar{\kappa}\kappa_2\cos\omega + \kappa_1(\bar{\tau} - \bar{\tau}_G))B. \end{cases} \tag{4.13}$$

On comparing both sides of equation (4.13), we get

$$\begin{cases} \kappa = \frac{\bar{\kappa}\cos\omega(1+\chi\bar{\kappa}\cos\omega)+\chi(\bar{\tau}-\bar{\tau}_G)^2}{\kappa_1^2+\kappa_2^2}, \\ \tau - \tau_G = \frac{(1+\chi\bar{\kappa}\cos\omega)(\bar{\tau}-\bar{\tau}_G)-\bar{\kappa}\cos\omega\chi(\bar{\tau}-\bar{\tau}_G)}{\kappa_1^2+\kappa_2^2}. \end{cases} \tag{4.14}$$

Thus, the harmonic curvature along the curve $\gamma$ is

$$H = \frac{\bar{\tau} - \bar{\tau}_G}{\bar{\kappa}\cos\omega(1 + \chi\bar{\kappa}\cos\omega) + \chi(\bar{\tau} - \bar{\tau}_G)^2}.$$

□

**Corollary 4.3.** *Let $(\gamma, \beta)$ be a generalized Bertrand couple with $\kappa \neq 0$, $\bar{\kappa} \neq 0$ and normal of $\gamma$ does not coincide with the binormal of $\beta$ then $\gamma$ is a $1-$type orthogonal slant helix iff $\beta$ is $1-$type orthogonal slant helix in $(G^3, g)$.*

*Proof.* Proof followed directly from Theorem 4.2 and Corollary 3.3. □

**Corollary 4.4.** *Let $(\gamma, \beta)$ be a generalized Bertrand couple with $\kappa \neq 0$, $\bar{\kappa} \neq 0$ and let the normal of $\gamma$ coincide with the binormal of $\beta$. Then the generalized Bertrand curve $\gamma$ is a $1-$type slant helix with $\kappa \neq 0$ iff $\beta$ is a $1-$type slant helix with $\bar{\kappa} = constant \neq 0$.*

*Proof.* Since the normal of $\gamma$ is coinciding with the binormal of $\beta$, then from equation (4.11), $\cos\omega = 0$ and $\sin\omega = 1$. On substituting $\cos\omega = 0$ in Theorem 4.2, we find

$$H = \frac{1}{\chi\left(\bar{\kappa}\bar{H}\right)}. \tag{4.15}$$

Thus (4.15) proves the statement. □

**Theorem 4.5.** *If* $\varphi$ *is the angle between the tangent vector fields* $(T, \bar{T})$ *of generalized Bertrand couple* $(\gamma, \beta)$ *on* $(G^3, g)$*, then* $\varphi = \tan^{-1}\left(\frac{\chi \bar{H}}{\bar{\kappa} + \chi \cos\omega}\right)$.

*Proof.* From equation (4.2), it is easy to see that $\bar{T}$ is orthogonal to $N$, therefore $\bar{T}$ must lie in the plane $\{T, B\}$. Now if $\varphi$ is the angle between $\bar{T}$ and $T$, then

$$\bar{T} = \cos\varphi(s)T + \sin\varphi(s)B. \tag{4.16}$$

Thus comparing the equation (4.10) and equation (4.16), we obtain

$$\begin{cases} \cos\varphi = \frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}}, \\ \sin\varphi = \frac{\kappa_2}{\sqrt{\kappa_1^2 + \kappa_2^2}}, \end{cases} \tag{4.17}$$

where $\kappa_1 = 1 + \chi\bar{\kappa}\cos\omega$ and $\kappa_2 = \chi(\bar{\tau} - \bar{\tau}_G)$. Hence from equation (4.17), $\varphi = \tan^{-1}\left(\frac{\chi \bar{H}}{\bar{\kappa} + \chi \cos\omega}\right)$. □

## 5. Energy of the unit vector field

A vector field $\xi$ on Riemannian manifold $(M, g)$ is said to be a unit vector field if $g_p(\xi_p, \xi_p) = 1$ for all $p \in M$. Any non null vector field generates a unit vector field by dividing the vector field by its norm. Using these basic ideas we will find the energy of unit vector field which we are constructing from the axis of $k-$type slant helix obtained in Section 3.

**Definition 5.1.** Let $\xi$ be a differentiable map from the Riemannian manifold $(M_1^n, g_1)$ to the Riemannian manifold $(M_2, g_2)$. Then the energy of $\xi$ is defined by [9, 15],

$$energy(\xi) = \frac{1}{2}\int_{M_1} \sum_{x=1}^{n} g_2\big(d\xi(Z_x), d\xi(Z_x)\big)v,$$

where $\{Z_x\}$ is a basis element of $T_xM_1$ and $v$ is a canonical volume form in $M_1$.

**Definition 5.2.** [9, 15] Let $\xi_1$, $\xi_2$ be unit vector fields in double tangent bundle $T_\xi(T^1M)$, where $T^1M$ is unit tangent bundle of $TM$, then the Sasaki metric in $T_\xi(T^1M)$ is defined as

$$g_s(\xi_1, \xi_2) = g(d\Pi(\xi_1), d\Pi(\xi_2)) + g(\Phi(\xi_1), \Phi(\xi_2)),$$

where the projection map $\Pi : T^1M \to M$ is a Riemannian submersion and $\Phi : T(T^1M) \to T^1M$ is a connection map.

**Proposition 5.3.** *[9, 15] For a connection map* $\Phi : T(T^1M) \to T^1M$*, we have*

*(i)* $\Pi \circ \Phi = \Phi \circ d\Pi$ *and* $\Pi \circ \Phi = \Phi \circ \tilde{\Pi}$*, where* $\tilde{\Pi} : T(T^1M) \to T^1M$*, is a tangent bundle projection.*

*(ii) For* $T \in T_xM$ *and a section* $\xi : M \to T^1M$*, we have* $\Phi(d\xi(T)) = \nabla_T\xi$.

**Theorem 5.4.** *[10] Let $\gamma$ be a smooth curve parametrized by arc length parameter lying on Lie group $G$, then the energy of Frenet vector using Sasaki metric is given as,*

$$\begin{cases} energy(T) = \frac{1}{2}\int_0^s (1+\kappa^2)ds, \\ energy(N) = \frac{1}{2}\int_0^s (1+\kappa^2(1+H^2))ds, \\ energy(B) = \frac{1}{2}\int_0^s (1+\kappa^2 H^2)ds. \end{cases} \tag{5.1}$$

**Theorem 5.5.** *Let $\gamma$ be a $k-$type slant helix with axis $U$ in $(G^3, g)$ and $X = \frac{U}{g(U,U)^{\frac{1}{2}}}$ is a unit vector field, then*
*(i) For $1-$type slant helix, $2-$type orthogonal slant helix having $\tau \neq \tau_G$, and $3-$type slant with $\tau \neq \tau_G$ the energy of $X$ is*

$$energy(X) = \frac{1}{2}\int_0^s \left(1 + \frac{H'^2}{(1+H^2)^2} + \frac{\tau_G^2}{1+H^2}\right) ds.$$

*(ii) For $1-$type orthogonal slant helix , and $2-$type orthogonal slant helix with $\tau = \tau_G$, the energy of $X$ is*

$$energy(X) = \frac{1}{2}\int_0^s \left(1+\tau^2\right) ds.$$

*(iii) For $2-$type slant helix with the condition $\left(\frac{\kappa(1+H^2)}{H'}\right)' = -(\tau - \tau_G)$, the energy of $X$ is given by*

$$\begin{aligned} &energy(X) \\ &= \frac{1}{2}\int_0^s \left(1 + \left[\left(\frac{\kappa H(1+H^2)}{H'}\right)\left(\frac{H'}{\sqrt{\kappa^2(1+H^2)+H'^2}}\right)'\right]^2\right) \\ &+ \left(\frac{H'}{\sqrt{\kappa^2(1+H^2)+H'^2}} + \frac{H'^2 - \tau_G(1+H^2)}{H'}\right)^2 \\ &+ \left[\left(\frac{\kappa(1+H^2)}{H'}\right)\left(\frac{H'}{\sqrt{\kappa^2(1+H^2)+H'^2}}\right)' + \frac{\tau_G H'}{\sqrt{\kappa^2(1+H^2)+H'^2}}\right]^2 ds. \end{aligned}$$

*(iv ) The energy of $3-$type slant helix with $\tau = \tau_G$, is given by*

$$energy(X) = \frac{1}{2}\int_0^s \left(1 + \tau^2\left(1 - \cos(\theta + \int \kappa(s)ds)\right)^2\right) ds.$$

*(v ) For $3-$type slant helix with orthogonal axis the energy of $X$ is*

$$energy(X) = \frac{1}{2}\int_0^s \left(1 + \tau^2\cos^2\int \kappa(s)ds\right) ds.$$

*Proof.* Let $\gamma$ be $1-$type slant helix with non orthogonal axis or $2-$type orthogonal slant helix with $\tau \neq \tau_G$ or $3-$type slant helix with non orthogonal axis and $\tau \neq \tau_G$. Then from Theorem 3.2, Theorem 3.6 and case $(i)$ of Theorem 3.8, we have

$$X = \frac{U}{g(U,U)^{\frac{1}{2}}} = \frac{H}{\sqrt{1+H^2}}T + \frac{1}{\sqrt{1+H^2}}B. \tag{5.2}$$

From Definition 5.1, we have

$$energy(X) = \frac{1}{2}\int_0^s g_s(dX(T), dX(T))ds \tag{5.3}$$

Now from Definition 5.2, the Sasaki metric is

$$\begin{aligned} g_s(dX(T), dX(T)) &= g(d\Pi(dX(T)), d\Pi(dX(T))) \\ &\quad + g\big(\Phi(dX(T)), \Phi(dX(T))\big). \end{aligned} \tag{5.4}$$

Since $X$ is a section, therefore we have

$$d\Pi \circ dX = d(\Pi \circ X) = d(id_\gamma) = id_{T\gamma}$$

On using second part of Proposition 5.3, in equation (5.4) we have

$$g_s(dX(T), dX(T)) = g(T,T) + g(\nabla_T X, \nabla_T X). \tag{5.5}$$

Thus from equation (5.2), we obtain

$$\begin{cases} \nabla_T X &= \left(\frac{H}{\sqrt{1+H^2}}\right)' T + \left(\frac{1}{\sqrt{1+H^2}}\right)' B + \left(\frac{H}{\sqrt{1+H^2}}\right)\nabla_T T \\ &\quad + \left(\frac{1}{\sqrt{1+H^2}}\right)\nabla_T B. \\ &= \left(\frac{H'}{(1+H^2)^{\frac{3}{2}}}\right) T - \left(\frac{HH'}{(1+H^2)^{\frac{3}{2}}}\right) B + \left(\frac{H}{\sqrt{1+H^2}}\right)(\kappa N) \\ &\quad + \left(\frac{1}{\sqrt{1+H^2}}\right)(-\tau N). \\ &= \left(\frac{H'}{(1+H^2)^{\frac{3}{2}}}\right) T - \left(\frac{\tau_G}{\sqrt{1+H^2}}\right) N - \left(\frac{HH'}{(1+H^2)^{\frac{3}{2}}}\right) B. \end{cases} \tag{5.6}$$

Hence using equation (5.6) and equation (5.5) in equation (5.3), we obtain the first part of the equation. To prove the remaining parts, we find $X$ from Section 3 and then proceed in the same way. □

**Conclusion :** *(i)* Let $\gamma(s)$ be a curve parametrized by the arc length parameter lying on $(G^3, g)$ and $\beta(\bar{s})$ representing any of the tangent indicatrices, binormal indicatrices or the involute of $\gamma(s)$. Then the relation between $\gamma(s)$ and $\beta(\bar{s})$, is given by [18]

$$\tau_\beta = \frac{H^|}{\kappa(1+H^2)} + \tau_{G_\beta},$$

where $\tau_\beta$ is curvature of $\beta(\bar{s})$ and $\tau_{G_\beta} = \frac{1}{2}g([T_\beta, N_\beta], B_\beta)$. Therefore, if $\gamma(s)$ is a $1-$type slant helix, then its tangent indicatrices, binormal indicatrix or the involute of $\beta(\bar{s})$ is also $1-$type slant helix with orthogonal axis. Moreover, if $\gamma(s)$ have $\tau = \tau_G$, then $\beta(\bar{s})$ is $1-$type slant helix with orthogonal axis.

*(ii)* If instead of Frenet frame $\{T, N, B\}$, we introduce the new frame [14, 17] $\{N, C, W\}$ in $(G^3, g)$, where $N$ is unit principal normal vector field along $\gamma(s)$, $C = \frac{\dot{N}}{g(\dot{N},\dot{N})^{\frac{1}{2}}}$ and $W = \frac{H}{\sqrt{1+H^2}}T + \frac{1}{\sqrt{1+H^2}}B$, then for slant helix $g(N, U) = cos\theta$ the curve $\gamma(s)$ will be slant helix iff $\frac{\kappa(1+H^2)^{\frac{3}{2}}}{H'} = \tan\theta$. And the left invariant vector field $U$ in frame $\{N, C, W\}$ will be $U = \left(N - \frac{\kappa(1+H^2)^{\frac{3}{2}}}{H'}\right) cos\theta$.

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