Novi Sad J. Math. Vol. 55, No. 2, 2025, 123–146
https://doi.org/10.30755/NSJOM.16382

Growth of (α, β, γ)-order solutions of linear differential equations with entire coefficients

Benharrat Belaïdi [1] [2] and Tanmay Biswas [3]

Abstract. The main aim of this paper is to study the growth of solutions of higher order linear differential equations using the concepts of (α, β, γ)-order and (α, β, γ)-type. We obtain some results which improve and generalize some previous results of Kinnunen [22], Long et al. [27] as well as Belaïdi [3], [4].

AMS Mathematics Subject Classification (2010): 30D35; 34M10

Key words and phrases: differential equations, (α, β, γ)-order, (α, β, γ)-type, growth of solutions

1. Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of entire and meromorphic functions and the theory of complex linear differential equations which are available in [14, 23, 35, 36] and therefore we do not explain those in detail. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M(r, f)$, the maximum term $\mu(r, f)$ and central index $\nu(r, f)$ of $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ on $|z| = r$ are defined as $M(r, f) = \max_{|z|=r} |f(z)|$, $\mu(r, f) = \max_{n \geq 0} \{|a_n| r^n\}$ and $\nu(r, f) = \max\{n : \mu(r, f) = |a_n| r^n\}$ respectively. When f is a meromorphic function, one may introduce another function $T(r, f)$ known as Nevanlinna's characteristic function of $f(z)$, playing the same role as $M(r, f)$ defined by

$$T(r, f) = m(r, f) + N(r, f), \ r > 0$$

with

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta,$$

and

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

[1]Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem-(Algeria),
e-mail: benharrat.belaidi@univ-mosta.dz, ORCID iD: orcid.org/0000-0002-6635-2514
[2]Corresponding author
[3]Rajbari, Rabindrapally, R. N. Tagore Road, P.O. Krishnagar, P.S. Kotwali, Dist-Nadia, PIN-741101, West Bengal, India,
e-mail: tanmaybiswas_math@rediffmail.com, ORCID iD: orcid.org/0000-0001-6984-6897

where $\log^{+} x = \max(0, \log x)$ for $x > 0$, and $n(t, f)$ is the number of poles of $f(z)$ lying in $|z| \leq t$, counted according to their multiplicity. To study the generalized growth properties of entire and meromorphic functions, the concepts of different growth indicators such as the iterated p-order (see [22, 29]), the (p, q)-th order (see [19, 20]), (p, q)-φ order (see [30]) etc. are very useful and during the past decades, several authors made close investigations on the generalized growth properties of entire and meromorphic functions related to the above growth indicators in some different directions. The theory of complex linear differential equations has been developed since 1960s. Many authors have investigated the complex linear differential equation

$$(1.1)\quad L(f) := f^{(k)}(z) + A_{k-1}(z)f^{(k-1)}(z) + \cdots + A_1(z)f'(z) + A_0(z)f(z) = 0$$

and achieved many valuable results when the coefficients $A_0(z), ..., A_{k-1}(z)$ $(k \geq 2)$ in (1.1) are entire functions of finite order or finite iterated p-order or (p, q)-th order or (p, q)-φ order; see ([1], [2], [7], [8], [9], [10], [13], [16], [22], [23], [24], [25], [26], [30], [31], [32], [34]).

Chyzhykov and Semochko [11] showed that both definitions of iterated p-order and the (p, q)-th order have the disadvantage that they do not cover arbitrary growth (see [11, Example 1.4]). They used more general scale, called the φ-order (see [11]) and the concept of φ-order is used to study the growth of solutions of complex differential equations which extend and improve many previous results (see [3, 4, 11, 21]). Extending this notion, Long et al. [27] recently introduce the concepts of $[p, q]_{,\varphi}$-order and $[p, q]_{,\varphi}$-type (see [27]) and obtain some interesting results which considerably extend and improve some earlier results. For details one may see [27].

On the other hand, Mulyava et al. [28] have used the concept of (α, β)-order or generalized order of an entire function in order to investigate the properties of solutions of a heterogeneous differential equation of the second order and obtained several remarkable results. For details one may see [28].

The main aim of this paper is to study the growth of solutions of higher order linear differential equations using the concepts of (α, β, γ)-order and (α, β, γ)-type. In fact, some works relating to study the growth of solutions of higher order linear differential equations using the concepts of (α, β, γ)-order have been explored in [5] and [6]. In this paper, we obtain some results which improve and generalize some previous results of Long et al. [27] as well as Belaïdi [3], [4]. Further, we give an answer to the problem of Chyzhykov and Semochko ([11], Remark 1.11).

2. Definitions and Notations

We denote the Lebesgue linear measure of a set $F \subset [0, +\infty)$ by $m(F) = \int_F dt$, and the logarithmic measure of a set $E \subset (1, +\infty)$ by $m_l(E) = \int_E \frac{dt}{t}$. Furthermore, let throughout this paper E represents a set of finite logarithmic measure, I represents a set of infinite logarithmic measure and F represents a set of finite linear measure in proof of our results. For $x \in [0, +\infty)$ and $k \in \mathbb{N}$

where $\mathbb{N}$ is the set of all positive integers, define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$ with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Now, let L be a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a)+\alpha(b)+c$ for all $a, b \geq R_0$ and fixed $c \in (0, +\infty)$. Further, we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x + O(1)) = (1+o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a)+\alpha(b)$ for all $a, b \geq R_0$ i.e., α is subadditive. Clearly $L_3 \subset L_1$.

Particularly, when $\alpha \in L_3$, then one can easily verify that $\alpha(mr) \leq m\alpha(r)$, $m \geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0, +\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0, 1]$,

$$\alpha(tx) = \alpha(tx + (1-t) \cdot 0) \geq t\alpha(x) + (1-t)\alpha(0) \geq t\alpha(x),$$

so that by choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$,

$$\begin{aligned} \alpha(a+b) &= \frac{a}{a+b}\alpha(a+b) + \frac{b}{a+b}\alpha(a+b) \\ &\leq \alpha\Big(\frac{a}{a+b}(a+b)\Big) + \alpha\Big(\frac{b}{a+b}(a+b)\Big) \\ &= \alpha(a) + \alpha(b), \quad a, b \geq 0. \end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$, this yields that $\alpha(r) \sim \alpha(r + R_0)$ as $r \to +\infty$.

Now we add two conditions on α, β and γ: (i) Always $\alpha \in L_1$, $\beta \in L_2$ and $\gamma \in L_3$; and (ii) $\alpha(\log^{[p]} x) = o(\beta(\log \gamma(x)))$, $p \geq 2$, $\alpha(\log x) = o(\alpha(x))$ and $\alpha^{-1}(kx) = o\left(\alpha^{-1}(x)\right)$ $(0 < k < 1)$ as $x \to +\infty$.

Throughout this paper, we assume that α, β and γ always satisfy the above two conditions unless otherwise specifically stated.

Heittokangas et al. [17] have introduced a new concept of φ-order of an entire and meromorphic function considering a subadditive function φ. For details one may see [17]. Extending this notion, recently Belaïdi et al. [5] introduce the definition of the (α, β, γ)-order of a meromorphic function in the following way:

Definition 2.1. *([5]) The (α, β, γ)-order denoted by $\sigma_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined by*

$$\sigma_{(\alpha,\beta,\gamma)}[f] = \limsup_{r \to +\infty} \frac{\alpha(\log T(r, f))}{\beta(\log \gamma(r))}.$$

By the inequality in [14]

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f) \ (0 < r < R)$$

for an entire function $f(z)$, one can easily verify that [5]

$$\sigma_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha\left(\log T\left(r,f\right)\right)}{\beta\left(\log\gamma\left(r\right)\right)} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} M(r,f))}{\beta\left(\log\gamma\left(r\right)\right)}.$$

Proposition 2.1. *([5]) If f is an entire function, then*

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T(r,f))}{\beta\left(\log\gamma\left(r\right)\right)} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[3]} M(r,f))}{\beta\left(\log\gamma\left(r\right)\right)},$$

where $(\alpha(\log),\beta,\gamma)$-order denoted by $\sigma_{(\alpha(\log),\beta,\gamma)}[f]$.

Now to compare the relative growth of two meromorphic functions having same non zero finite (α,β,γ)-order, one may introduce the definition of (α,β,γ)-type in the following manner:

Definition 2.2. *The (α,β,γ)-type, denoted by $\tau_{(\alpha,\beta,\gamma)}[f]$, of a meromorphic function f with $0 < \sigma_{(\alpha,\beta,\gamma)}[f] < +\infty$ is defined by*

$$\tau_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha\left(\log T\left(r,f\right)\right))}{\left(\exp\left(\beta\left(\log\gamma\left(r\right)\right)\right)\right)^{\sigma_{(\alpha,\beta,\gamma)}[f]}}.$$

If f is an entire function with $\sigma_{(\alpha,\beta,\gamma)}[f] \in (0,+\infty)$, then the (α,β,γ)-type of f is defined by

$$\tau_{(\alpha,\beta,\gamma),M}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(\log^{[2]} M(r,f)))}{\left(\exp\left(\beta\left(\log\gamma\left(r\right)\right)\right)\right)^{\sigma_{(\alpha,\beta,\gamma)}[f]}}.$$

Similar to Definition 2.2, we can also define the $(\alpha(\log),\beta,\gamma)$-type of a meromorphic function f in the following way:

Definition 2.3. *The $(\alpha(\log),\beta,\gamma)$-type, denoted by $\tau_{(\alpha(\log),\beta,\gamma)}[f]$, of a meromorphic function f with $0 < \sigma_{(\alpha(\log),\beta,\gamma)}[f] < +\infty$ is defined by*

$$\tau_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(\log^{[2]} T(r,f)))}{\left(\exp\left(\beta\left(\log\gamma\left(r\right)\right)\right)\right)^{\sigma_{(\alpha(\log),\beta,\gamma)}[f]}}.$$

If f is an entire function with $\sigma_{(\alpha(\log),\beta,\gamma)}[f] \in (0,+\infty)$, then the (α,β,γ)-type of f is defined by

$$\tau_{(\alpha(\log),\beta,\gamma),M}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(\log^{[3]} M(r,f)))}{\left(\exp\left(\beta\left(\log\gamma\left(r\right)\right)\right)\right)^{\sigma_{(\alpha(\log),\beta,\gamma)}[f]}}.$$

Proposition 2.2. *([5]) Let f_1, f_2 be nonconstant meromorphic functions with $\sigma_{(\alpha,\beta,\gamma)}[f_1]$ and $\sigma_{(\alpha,\beta,\gamma)}[f_2]$ as their (α,β,γ)-order. Then*
(i) $\sigma_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] \le \max\{\sigma_{(\alpha,\beta,\gamma)}[f_1],\ \sigma_{(\alpha,\beta,\gamma)}[f_2]\}$;
(ii) $\sigma_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] \le \max\{\sigma_{(\alpha,\beta,\gamma)}[f_1],\ \sigma_{(\alpha,\beta,\gamma)}[f_2]\}$;
(iii) If $\sigma_{(\alpha,\beta,\gamma)}[f_1] \ne \sigma_{(\alpha,\beta,\gamma)}[f_2]$, *then* $\sigma_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] = \max\{\sigma_{(\alpha,\beta,\gamma)}[f_1],\ \sigma_{(\alpha,\beta,\gamma)}[f_2]\}$;
(iv) If $\sigma_{(\alpha,\beta,\gamma)}[f_1] \ne \sigma_{(\alpha,\beta,\gamma)}[f_2]$, *then*

$$\sigma_{(\alpha,\beta,\gamma)}[f_2 \cdot f_2] = \max\{\sigma_{(\alpha,\beta,\gamma)}[f_1], \sigma_{(\alpha,\beta,\gamma)}[f_2]\}.$$

Proposition 2.3. *([5]) Let f_1, f_2 be nonconstant meromorphic functions with $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1]$ and $\sigma_{(\alpha(\log),\beta,\gamma)}[f_2]$ as their $(\alpha(\log),\beta,\gamma)$-order. Then*
(i) $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] \leq \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1],\ \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(ii) $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \cdot f_2] \leq \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1],\ \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(iii) If $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1] \neq \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]$, *then*

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] = \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1], \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\};$$

(iv) If $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1] \neq \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]$, *then*

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f_2 \cdot f_2] = \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1], \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\}.$$

By using the properties $T(r,f) = T(r,\frac{1}{f}) + O(1)$ and $T(r,af) = T(r,f) + O(1)$, $a \in \mathbb{C} \setminus \{0\}$, one can obtain the following result.

Proposition 2.4. *Let f be a nonconstant meromorphic function with finite (α,β,γ)-order. Then*
(i) $\sigma_{(\alpha,\beta,\gamma)}[\frac{1}{f}] = \sigma_{(\alpha,\beta,\gamma)}[f]$ $(f \not\equiv 0)$;
(ii) $\sigma_{(\alpha(\log),\beta,\gamma)}[\frac{1}{f}] = \sigma_{(\alpha(\log),\beta,\gamma)}[f]$ $(f \not\equiv 0)$;
(iii) If $a \in \mathbb{C} \setminus \{0\}$, *then* $\sigma_{(\alpha,\beta,\gamma)}[af] = \sigma_{(\alpha,\beta,\gamma)}[f]$ *and* $\tau_{(\alpha,\beta,\gamma)}[af] = \tau_{(\alpha,\beta,\gamma)}[f]$;
(iii) If $a \in \mathbb{C} \setminus \{0\}$, *then* $\sigma_{(\alpha(\log),\beta,\gamma)}[af] = \sigma_{(\alpha(\log),\beta,\gamma)}[f]$ *and* $\tau_{(\alpha(\log),\beta,\gamma)}[af] = \tau_{(\alpha(\log),\beta,\gamma)}[f]$.

Proposition 2.5. *Let f_1, f_2 be two nonconstant meromorphic functions. Then the following statements hold:*
(i) If $0 < \sigma_{(\alpha,\beta,\gamma)}[f_1] < \sigma_{(\alpha,\beta,\gamma)}[f_2] < +\infty$ *and* $\tau_{(\alpha,\beta,\gamma)}[f_1] < \tau_{(\alpha,\beta,\gamma)}[f_2]$, *then* $\tau_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] = \tau_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] = \tau_{(\alpha,\beta,\gamma)}[f_2]$;
(ii) If $0 < \sigma_{(\alpha,\beta,\gamma)}[f_1] = \sigma_{(\alpha,\beta,\gamma)}[f_2] = \sigma_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] < +\infty$, *then* $\tau_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] \leq \max\{\tau_{(\alpha,\beta,\gamma)}[f_1],\ \tau_{(\alpha,\beta,\gamma)}[f_2]\}$. *Moreover if* $\tau_{(\alpha,\beta,\gamma)}[f_1] \neq \tau_{(\alpha,\beta,\gamma)}[f_2]$, *then* $\tau_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] = \max\{\tau_{(\alpha,\beta,\gamma)}[f_1],\ \tau_{(\alpha,\beta,\gamma)}[f_2]\}$;
(iii) If $0 < \sigma_{(\alpha,\beta,\gamma)}[f_1] = \sigma_{(\alpha,\beta,\gamma)}[f_2] = \sigma_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] < +\infty$, *then* $\tau_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] \leq \max\{\tau_{(\alpha,\beta,\gamma)}[f_1],\ \tau_{(\alpha,\beta,\gamma)}[f_2]\}$. *Moreover if* $\tau_{(\alpha,\beta,\gamma)}[f_1] \neq \tau_{(\alpha,\beta,\gamma)}[f_2]$, *then* $\tau_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] = \max\{\tau_{(\alpha,\beta,\gamma)}[f_1],\ \tau_{(\alpha,\beta,\gamma)}[f_2]\}$.

Proof. (i) The definition of the (α,β,γ)-type implies that for any given $\varepsilon > 0$, there exists a sequence $\{r_n,\ n \geq 1\}$ tending to infinity such that

$$T(r_n, f_2) \geq \exp\left\{\alpha^{-1}(\log((\tau_{(\alpha,\beta,\gamma)}[f_2] - \varepsilon)(\exp(\beta(\log\gamma(r_n))))^{\sigma_{(\alpha,\beta,\gamma)}[f_2]}))\right\}$$

and for sufficiently large values of r

$$T(r, f_1) \leq \exp\left\{\alpha^{-1}(\log((\tau_{(\alpha,\beta,\gamma)}[f_1] + \varepsilon)(\exp\left(\beta\left(\log\gamma\left(r\right)\right)\right))^{\sigma_{(\alpha,\beta,\gamma)}[f_1]}))\right\}.$$

We know that $T(r, f_1 \pm f_2) \geq T(r, f_2) - T(r, f_1) - \log 2$, we obtain from above two inequalities that

$$T(r_n,f_1\pm f_2)\geq\exp\left\{\alpha^{-1}\left(\log\left((\tau_{(\alpha,\beta,\gamma)}[f_2]-\varepsilon)(\exp(\beta(\log\gamma(r_n))))^{\sigma_{(\alpha,\beta,\gamma)}[f_2]}\right)\right)\right\}$$
$$-\exp\left\{\alpha^{-1}\left(\log\left((\tau_{(\alpha,\beta,\gamma)}[f_1]+\varepsilon)(\exp(\beta(\log\gamma(r_n))))^{\sigma_{(\alpha,\beta,\gamma)}[f_1]}\right)\right)\right\}-\log 2$$
$$\geq\exp\left\{\alpha^{-1}\left(\log\left((\tau_{(\alpha,\beta,\gamma)}[f_2]-2\varepsilon)(\exp(\beta(\log\gamma(r_n))))^{\sigma_{(\alpha,\beta,\gamma)}[f_2]}\right)\right)\right\}$$

provided ε is such that $0 < 2\varepsilon < \tau_{(\alpha,\beta,\gamma)}[f_2] - \tau_{(\alpha,\beta,\gamma)}[f_1]$. It follows from Proposition 2.2 that $\sigma_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] = \sigma_{(\alpha,\beta,\gamma)}[f_2]$, and by the monotoncity of α and above, we obtain that

$$\frac{\exp(\alpha(\log T(r_n, f_1 \pm f_2)))}{(\exp\left(\beta\left(\log\gamma\left(r_n\right)\right)\right))^{\sigma_{(\alpha,\beta,\gamma)}[f_1\pm f_2]}} \geq \tau_{(\alpha,\beta,\gamma)}[f_2] - 2\varepsilon,$$

since ε can be arbitrarily chosen such that $0 < 2\varepsilon < \tau_{(\alpha,\beta,\gamma)}[f_2] - \tau_{(\alpha,\beta,\gamma)}[f_1]$, thus

$$\tau_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] \geq \tau_{(\alpha,\beta,\gamma)}[f_2]. \tag{2.1}$$

It remains to prove the converse inequality. Indeed, by applying (2.1) and since

$$\sigma_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] = \sigma_{(\alpha,\beta,\gamma)}[f_2] > \sigma_{(\alpha,\beta,\gamma)}[f_1] = \sigma_{(\alpha,\beta,\gamma)}[-f_1],$$

we get

$$\tau_{(\alpha,\beta,\gamma)}[f_2] = \tau_{(\alpha,\beta,\gamma)}[f_1 \pm f_2 - f_1] \geq \tau_{(\alpha,\beta,\gamma)}[f_1 \pm f_2]. \tag{2.2}$$

We deduce from (2.1) and (2.2) that $\tau_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] = \tau_{(\alpha,\beta,\gamma)}[f_2]$.

Now, we prove that $\tau_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] = \tau_{(\alpha,\beta,\gamma)}[f_2]$. By the property $T(r, f_1 \cdot f_2) \geq T(r, f_1) - T(r, f_2) + O(1)$ and a similar discussion as in the above proof, one can easily show that

$$\tau_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] \geq \tau_{(\alpha,\beta,\gamma)}[f_2]. \tag{2.3}$$

Since $\sigma_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] = \sigma_{(\alpha,\beta,\gamma)}[f_2] > \sigma_{(\alpha,\beta,\gamma)}[f_1] = \sigma_{(\alpha,\beta,\gamma)}\left(\frac{1}{f_1}\right)$, then by (2.3), we get that

$$\tau_{(\alpha,\beta,\gamma)}[f_2] = \tau_{(\alpha,\beta,\gamma)}\left(f_1 \cdot f_2 \frac{1}{f_1}\right) \geq \tau_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2]$$

and therefore $\tau_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] = \tau_{(\alpha,\beta,\gamma)}[f_2]$.

(ii) The definition of the (α, β, γ)-type implies that for any given $\varepsilon > 0$ and for all r sufficiently large, we have

$$\begin{aligned} &T(r, f_i) \\ &\leq \exp\left\{\alpha^{-1}(\log((\tau_{(\alpha,\beta,\gamma)}[f_i] + \varepsilon)(\exp\left(\beta\left(\log\gamma\left(r\right)\right)\right))^{\sigma_{(\alpha,\beta,\gamma)}[f_i]}))\right\},\ i = 1, 2. \end{aligned} \tag{2.4}$$

By the assumption $0<\sigma_{(\alpha,\beta,\gamma)}[f_1]=\sigma_{(\alpha,\beta,\gamma)}[f_2]=\sigma_{(\alpha,\beta,\gamma)}[f_1\pm f_2]<+\infty$, we get that

$$T(r,f_1\pm f_2)\le T(r,f_1)+T(r,f_2)+\log 2$$

$$\begin{aligned}\le\ & \exp\left\{\alpha^{-1}(\log((\tau_{(\alpha,\beta,\gamma)}[f_1]+\varepsilon)(\exp(\beta(\log\gamma(r))))^{\sigma_{(\alpha,\beta,\gamma)}[f_1]}))\right\}\\ & +\exp\left\{\alpha^{-1}(\log((\tau_{(\alpha,\beta,\gamma)}[f_2]+\varepsilon)(\exp(\beta(\log\gamma(r))))^{\sigma_{(\alpha,\beta,\gamma)}[f_2]}))\right\}+\log 2\end{aligned}$$

$$\begin{aligned}\le\exp\{&\alpha^{-1}(\log((\max(\tau_{(\alpha,\beta,\gamma)}[f_1],\tau_{(\alpha,\beta,\gamma)}[f_2])+3\varepsilon)\\ &\times(\exp(\beta(\log\gamma(r))))^{\sigma_{(\alpha,\beta,\gamma)}[f_1\pm f_2]}))\Big\}.\end{aligned}$$

By the monotonicity of α and above, we obtain that

$$\frac{\exp(\alpha(\log T(r,f_1\pm f_2)))}{(\exp(\beta(\log\gamma(r))))^{\sigma_{(\alpha,\beta,\gamma)}[f_1\pm f_2]}}\le\max(\tau_{(\alpha,\beta,\gamma)}[f_1],\tau_{(\alpha,\beta,\gamma)}[f_2])+3\varepsilon.$$

Since $\varepsilon>0$ can be chosen arbitrarily, then we get that

$$\tau_{(\alpha,\beta,\gamma)}[f_1\pm f_2]\le\max\{\tau_{(\alpha,\beta,\gamma)}[f_1],\tau_{(\alpha,\beta,\gamma)}[f_2]\}. \tag{2.5}$$

Without loss of generality, we may suppose $\tau_{(\alpha,\beta,\gamma)}[f_1]<\tau_{(\alpha,\beta,\gamma)}[f_2]$. Then by (2.5) and since $\sigma_{(\alpha,\beta,\gamma)}[f_1\pm f_2]=\sigma_{(\alpha,\beta,\gamma)}[f_1]=\sigma_{(\alpha,\beta,\gamma)}[-f_1]$, it follows that

$$\begin{aligned}\tau_{(\alpha,\beta,\gamma)}[f_2] &= \tau_{(\alpha,\beta,\gamma)}[f_1\pm f_2-f_1]\\ &\le \max\{\tau_{(\alpha,\beta,\gamma)}[f_1\pm f_2],\tau_{(\alpha,\beta,\gamma)}[f_1]\}=\tau_{(\alpha,\beta,\gamma)}[f_1\pm f_2].\end{aligned}\tag{2.6}$$

We deduce from (2.5) and (2.6) that

$$\tau_{(\alpha,\beta,\gamma)}[f_1\pm f_2]=\max\{\tau_{(\alpha,\beta,\gamma)}[f_1],\tau_{(\alpha,\beta,\gamma)}[f_2]\}.$$

(iii) By a similar discussion as in the above proof and the fact that $T(r,f_1\cdot f_2)\le T(r,f_2)+T(r,f_1)$, one can prove that

$$\tau_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]\le\max\{\tau_{(\alpha,\beta,\gamma)}[f_1],\tau_{(\alpha,\beta,\gamma)}[f_2]\}. \tag{2.7}$$

On the other hand, if we suppose that $\tau_{(\alpha,\beta,\gamma)}[f_1]<\tau_{(\alpha,\beta,\gamma)}[f_2]$, then by (2.7) and since $\sigma_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]=\sigma_{(\alpha,\beta,\gamma)}[f_1]=\sigma_{(\alpha,\beta,\gamma)}\left(\frac{1}{f_1}\right)$, we get

$$\begin{aligned}\tau_{(\alpha,\beta,\gamma)}[f_2]&=\tau_{(\alpha,\beta,\gamma)}\left(f_1\cdot f_2\frac{1}{f_1}\right)\\ &\le\max\{\tau_{(\alpha,\beta,\gamma)}[f_1\cdot f_2],\tau_{(\alpha,\beta,\gamma)}[f_1]\}=\tau_{(\alpha,\beta,\gamma)}[f_1\cdot f_2].\end{aligned}\tag{2.8}$$

It follows from (2.7) and (2.8) that $\tau_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]=\max\{\tau_{(\alpha,\beta,\gamma)}[f_1],\tau_{(\alpha,\beta,\gamma)}[f_2]\}$.

□

Corollary 2.1. *Let f_1, f_2 be two nonconstant meromorphic functions. Then the following statements hold:*
(i) If $0 < \sigma_{(\alpha,\beta,\gamma)}[f_1] = \sigma_{(\alpha,\beta,\gamma)}[f_2] = \sigma_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] < +\infty$, then $\tau_{(\alpha,\beta,\gamma)}[f_1] \leq \max\{\tau_{(\alpha,\beta,\gamma)}[f_1 \pm f_2],\ \tau_{(\alpha,\beta,\gamma)}[f_2]\}$;
(ii) If $0 < \sigma_{(\alpha,\beta,\gamma)}[f_1] = \sigma_{(\alpha,\beta,\gamma)}[f_2] = \sigma_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] < +\infty$, then $\tau_{(\alpha,\beta,\gamma)}[f_1] \leq \max\{\tau_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2],\ \tau_{(\alpha,\beta,\gamma)}[f_2]\}$.

Proof. The proof follows immediately from Proposition 2.5. Indeed, since

$$\sigma_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] = \sigma_{(\alpha,\beta,\gamma)}[f_2] = \sigma_{(\alpha,\beta,\gamma)}[-f_2],$$

then

$$\tau_{(\alpha,\beta,\gamma)}[f_1] = \tau_{(\alpha,\beta,\gamma)}[f_2 \pm f_1 - f_2] \leq \max\{\tau_{(\alpha,\beta,\gamma)}[f_1 \pm f_2], \tau_{(\alpha,\beta,\gamma)}[f_2]\}.$$

Similarly, since $\sigma_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] = \sigma_{(\alpha,\beta,\gamma)}[f_2] = \sigma_{(\alpha,\beta,\gamma)}\left(\frac{1}{f_2}\right)$, then

$$\tau_{(\alpha,\beta,\gamma)}[f_1] = \tau_{(\alpha,\beta,\gamma)}\left(f_1 \cdot f_2 \frac{1}{f_2}\right) \leq \max\{\tau_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2], \tau_{(\alpha,\beta,\gamma)}[f_2]\}.$$

□

Proposition 2.6. *Let f_1, f_2 be two nonconstant meromorphic functions. Then the following statements hold:*
(i) If $0 < \sigma_{(\alpha(\log),\beta,\gamma)}[f_1] < \sigma_{(\alpha(\log),\beta,\gamma)}[f_2] < +\infty$ and $\tau_{(\alpha(\log),\beta,\gamma)}[f_1] < \tau_{(\alpha(\log),\beta,\gamma)}[f_2]$, then $\tau_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] = \tau_{\alpha(\log),\beta,\gamma)}[f_1 \cdot f_2] = \tau_{(\alpha(\log),\beta,\gamma)}[f_2]$;
(ii) If $0 < \sigma_{(\alpha(\log),\beta,\gamma)}[f_1] = \sigma_{(\alpha(\log),\beta,\gamma)}[f_2] = \sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] < +\infty$, then $\tau_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] \leq \max\{\tau_{(\alpha(\log),\beta,\gamma)}[f_1],\ \tau_{(\alpha(\log),\beta,\gamma)}[f_2]\}$. Moreover if $\tau_{(\alpha(\log),\beta,\gamma)}[f_1] \neq \tau_{(\alpha(\log),\beta,\gamma)}[f_2]$, then $\tau_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] = \max\{\tau_{(\alpha(\log),\beta,\gamma)}[f_1],\ \tau_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(iii) If $0 < \sigma_{(\alpha(\log),\beta,\gamma)}[f_1] = \sigma_{(\alpha(\log),\beta,\gamma)}[f_2] = \sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \cdot f_2] < +\infty$, then $\tau_{(\alpha(\log),\beta,\gamma)}[f_1 \cdot f_2] \leq \max\{\tau_{(\alpha(\log),\beta,\gamma)}[f_1],\ \tau_{(\alpha(\log),\beta,\gamma)}[f_2]\}$. Moreover if $\tau_{(\alpha(\log),\beta,\gamma)}[f_1] \neq \tau_{(\alpha(\log),\beta,\gamma)}[f_2]$, then $\tau_{(\alpha(\log),\beta,\gamma)}[f_1 \cdot f_2] = \max\{\tau_{(\alpha(\log),\beta,\gamma)}[f_1],\ \tau_{(\alpha(\log),\beta,\gamma)}[f_2]\}$.

Corollary 2.2. *Let f_1, f_2 be two nonconstant meromorphic functions. Then the following statements hold:*
(i) If $0 < \sigma_{(\alpha(\log),\beta,\gamma)}[f_1] = \sigma_{(\alpha(\log),\beta,\gamma)}[f_2] = \sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] < +\infty$, then $\tau_{(\alpha(\log),\beta,\gamma)}[f_1] \leq \max\{\tau_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2],\ \tau_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(ii) If $0 < \sigma_{(\alpha(\log),\beta,\gamma)}[f_1] = \sigma_{(\alpha(\log),\beta,\gamma)}[f_2] = \sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \cdot f_2] < +\infty$, then $\tau_{(\alpha(\log),\beta,\gamma)}[f_1] \leq \max\{\tau_{(\alpha(\log),\beta,\gamma)}[f_1 \cdot f_2],\ \tau_{(\alpha(\log),\beta,\gamma)}[f_2]\}$.

The proofs of Proposition 2.6 and Corollary 2.2 would run parallel to that of Proposition 2.5 and Corollary 2.1, respectively. We omit the details.

3. Main Results

In this section we present our main results which considerably extend the results of Kinnunen [22], Long et al. [27] as well as Belaïdi [3], [4]. Moreover, the results obtained give an answer to the problem of Chyzhykov and Semochko ([11], Remark 1.11).

Theorem 3.1. *Let $A_0(z)$, $A_1(z), ..., A_{k-1}(z)$ be entire functions. Then all nontrivial solutions f of* (1.1) *satisfy*

$$\sup\{\sigma_{(\alpha(\log),\beta,\gamma)}[f] \mid L(f)=0\} = \sup\{\sigma_{(\alpha,\beta,\gamma)}[A_j] \mid j=0,...,k-1\}.$$

Theorem 3.2. *Let $A_0(z)$, $A_1(z), ..., A_{k-1}(z)$ be entire functions, $m = \max\{j \mid \sigma_{(\alpha,\beta,\gamma)}[A_j] \geq \lambda,\ j=0,...,k-1\}$. Then* (1.1) *processes at most m entire linearly independent solutions f with $\sigma_{(\alpha(\log),\beta,\gamma)}[f] < \lambda$.*

Theorem 3.3. *Let $A_0(z)$, $A_1(z), ..., A_{k-1}(z)$ be entire functions such that $\sigma_{(\alpha,\beta,\gamma)}[A_0] > \max\{\sigma_{(\alpha,\beta,\gamma)}[A_j],\ j=1,...,k-1\}$. Then every solution $f(z) \not\equiv 0$ of* (1.1) *satisfies $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha,\beta,\gamma)}[A_0]$.*

Theorem 3.4. *Let $A_0(z)$, $A_1(z), ..., A_{k-1}(z)$ be entire functions. Assume that*

$$\max\{\sigma_{(\alpha,\beta,\gamma)}[A_j], j=1,...,k-1\} \leq \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma_0 < +\infty$$

and

$$\max\{\tau_{(\alpha,\beta,\gamma),M}[A_j] : \sigma_{(\alpha,\beta,\gamma)}[A_j] = \sigma_{(\alpha,\beta,\gamma)}[A_0] > 0\} < \tau_{(\alpha,\beta,\gamma),M}[A_0] = \tau_M.$$

Then every solution $f(z) \not\equiv 0$ of (1.1) *satisfies $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha,\beta,\gamma)}[A_0]$.*

By combining Theorem 3.3 and Theorem 3.4, we obtain the following result.

Corollary 3.1. *Let $A_0(z)$, $A_1(z), ..., A_{k-1}(z)$ be entire functions. Assume that*

$$\max\{\sigma_{(\alpha,\beta,\gamma)}[A_j], j=1,...,k-1\} < \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma_0 < +\infty,$$

or

$$\max\{\sigma_{(\alpha,\beta,\gamma)}[A_j], j=1,...,k-1\} \leq \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma_0 < +\infty \ (0 < \sigma_0 < +\infty)$$

and

$$\max\{\tau_{(\alpha,\beta,\gamma),M}[A_j] : \sigma_{(\alpha,\beta,\gamma)}[A_j] = \sigma_{(\alpha,\beta,\gamma)}[A_0] > 0\}$$
$$< \tau_{(\alpha,\beta,\gamma),M}[A_0] = \tau_M \ (0 < \tau_M < +\infty).$$

Then every solution $f(z) \not\equiv 0$ of (1.1) *satisfies $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha,\beta,\gamma)}[A_0]$.*

Remark 3.1. *Theorems 3.1-3.3 are counterparts of Theorems 1.8–1.10 of Chyzhykov and Semochko [11] for the (α,β,γ)-order.*

4. Some Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 4.1. *Let f be a meromorphic function of order $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma$, $k \in \mathbb{N}$. Then, for any given $\varepsilon > 0$,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\exp\{\alpha^{-1}((\sigma+\varepsilon)\beta(\log\gamma(r)))\}),$$

outside, possibly, an exceptional set $F \subset [0,+\infty)$ of finite linear measure.

Proof. Let $k=1$. The definition of $\sigma_{(\alpha(\log),\beta,\gamma)}[f]$ implies that for any given $\varepsilon>0$, there exists $r_0>1$, such that for all $r>r_0$,

$$T(r,f)\leq \exp^{[2]}\left\{\alpha^{-1}((\sigma+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}. \tag{4.1}$$

It follows from (4.1) and the lemma of logarithmic derivative and the condition $\alpha(\log^{[2]}x)=o(\beta(\log\gamma(x)))$ as $x\to+\infty$ that

$$\begin{aligned} m\left(r,\frac{f'}{f}\right) &= O(\log T(r,f)+\log r) \\ &= O\left(\exp\left\{\alpha^{-1}((\sigma+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),\ r\notin F, \end{aligned} \tag{4.2}$$

where $F\subset[0,+\infty)$ is of finite linear measure.

Now, we assume that for some $k\in\mathbb{N}$,

$$m\left(r,\frac{f^{(k)}}{f}\right)=O\left(\exp\left\{\alpha^{-1}((\sigma+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),\ r\notin F. \tag{4.3}$$

Since $N(r,f^{(k)})\leq(k+1)N(r,f)$, we deduce

$$\begin{aligned} T(r,f^{(k)}) &= m(r,f^{(k)})+N(r,f^{(k)}) \\ &\leq m\left(r,\frac{f^{(k)}}{f}\right)+m(r,f)+(k+1)N(r,f) \\ &\leq (k+1)T(r,f)+O\left(\exp\left\{\alpha^{-1}((\sigma+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right) \\ &\leq O\left(\exp^{[2]}\left\{\alpha^{-1}((\sigma+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right). \end{aligned} \tag{4.4}$$

It follows from (4.2) and (4.4) that

$$m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)=m\left(r,\frac{\left(f^{(k)}\right)'}{f^{(k)}}\right)=O(\log T(r,f^{(k)})+\log r)$$

$$=O\left(\exp\left\{\alpha^{-1}((\sigma+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),r\notin F.$$

Thus

$$\begin{aligned} m\left(r,\frac{f^{(k+1)}}{f}\right) &\leq m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)+m\left(r,\frac{f^{(k)}}{f}\right) \\ &= O\left(\exp\left\{\alpha^{-1}((\sigma+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),\ r\notin F. \end{aligned}$$

□

Lemma 4.2. *([13, 23]) Let $g:[0,+\infty)\to\mathbb{R}$ and $h:[0,+\infty)\to\mathbb{R}$ be monotone nondecreasing functions such that $g(r)\leq h(r)$ outside of an exceptional set E of finite linear measure or finite logarithmic measure. Then, for any $d>1$, there exists $r_0>0$ such that $g(r)\leq h(dr)$ for all $r>r_0$.*

Lemma 4.3. *Let f be a meromorphic function. Then $\sigma_{(\alpha,\beta,\gamma)}[f']=\sigma_{(\alpha,\beta,\gamma)}[f]$.*

Proof. Set $\sigma_{(\alpha,\beta,\gamma)}[f] = \sigma$. From the definition of (α, β, γ)-order, for any given $\varepsilon > 0$, there exists $r_0 > 1$, such that for all $r \geq r_0$,

$$\log T(r, f) \leq \alpha^{-1}((\sigma + \varepsilon)\beta(\log \gamma(r))).$$

Obviously, $T(r, f') \leq 2T(r, f) + m\left(r, \frac{f'}{f}\right)$. By the lemma of logarithmic derivative (p. 34 in [14]) and the condition $\alpha(\log^{[2]} x) = o(\beta(\log \gamma(x)))$ as $x \to +\infty$, we have

$$\begin{aligned} \log T(r, f') &\leq \log T(r, f) + \log\{O(\log r + \log T(r, f))\} + O(1) \\ &\leq \log T(r, f) + \log(\log r) + \log\log T(r, f) + O(1) \\ &\leq \alpha^{-1}((\sigma + 4\varepsilon)\beta(\log \gamma(r))), \ r \notin F, \end{aligned}$$

where $F \subset [0, +\infty)$ is a set of finite linear measure. By Lemma 4.2, for $d = 2$ and all $r > r_0$, we have

$$\log T(r, f') \leq \alpha^{-1}((\sigma + 4\varepsilon)\beta(\log \gamma(2r))).$$

Therefore, by using $\gamma(2r) \leq 2\gamma(r)$ and $\beta(x+O(1)) = (1+o(1))\beta(x)$ as $x \to +\infty$

$$\alpha(\log T(r, f')) \leq (\sigma + 4\varepsilon)\beta(\log \gamma(2r)) \leq (\sigma + 4\varepsilon)\beta(\log(2\gamma(r)))$$

$$= (\sigma + 4\varepsilon)\beta(\log 2 + \log \gamma(r)) = (\sigma + 4\varepsilon)(1 + o(1))\beta(\log \gamma(r)).$$

By the arbitrariness of $\varepsilon > 0$, we get that $\sigma_{(\alpha,\beta,\gamma)}[f] \geq \sigma_{(\alpha,\beta,\gamma)}[f']$.

On the other hand, we prove the inequality $\sigma_{(\alpha,\beta,\gamma)}[f] \leq \sigma_{(\alpha,\beta,\gamma)}[f']$. The definition of $\sigma_{(\alpha,\beta,\gamma)}[f'] = \sigma'$ implies that for any given above $\varepsilon > 0$ and for all r sufficiently large, we have

$$T(r, f') \leq \exp\left\{\alpha^{-1}((\sigma' + \varepsilon)\beta(\log \gamma(r)))\right\}.$$

Since $T(r, f) \leq O(T(2r, f') + \log r)$, $r \to +\infty$ (cf. [36]), then by using $\gamma(2r) \leq 2\gamma(r)$, $\beta(x+O(1)) = (1+o(1))\beta(x)$ and $\alpha(\log^{[2]} x) = o(\beta(\log \gamma(x)))$ as $x \to +\infty$, we can get that

$$\begin{aligned} T(r, f) &\leq O\left(\exp\left\{\alpha^{-1}((\sigma' + \varepsilon)\beta(\log \gamma(2r)))\right\} + \log r\right) \\ &\leq \exp\left\{\alpha^{-1}((\sigma' + 2\varepsilon)\beta(\log \gamma(2r)))\right\} \\ &\leq \exp\left\{\alpha^{-1}((\sigma' + 2\varepsilon)(1 + o(1))\beta(\log \gamma(r)))\right\}, \end{aligned}$$

$$i.e., \ \log T(r, f)) \leq \alpha^{-1}((\sigma' + 2\varepsilon)(1 + o(1))\beta(\log \gamma(r))),$$

$$i.e., \ \alpha(\log T(r, f)) \leq (\sigma' + 2\varepsilon)(1 + o(1))\beta(\log \gamma(r)), \ r \to +\infty.$$

Since $\varepsilon > 0$ is an arbitrary number, we obtain that $\sigma_{(\alpha,\beta,\gamma)}[f] \leq \sigma' = \sigma_{(\alpha,\beta,\gamma)}[f']$ and therefore $\sigma_{(\alpha,\beta,\gamma)}[f'] = \sigma_{(\alpha,\beta,\gamma)}[f]$. Hence the proof follows. □

Remark 4.1. *In the line of Lemma 4.3 one can easily deduce that*

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f'] = \sigma_{(\alpha(\log),\beta,\gamma)}[f],$$

where f is a meromorphic function.

Lemma 4.4. *([15, 18, 33]) Let f be a transcendental entire function, let $0 < \delta < \frac{1}{4}$ and z such that $|z| = r$ and $|f(z)| > M(r,f)\nu(r,f)^{-\frac{1}{4}+\delta}$. Then there exists a set $E \subset \mathbb{R}_+$ of finite logarithmic measure such that*

$$f^{(m)}(z) = \left(\frac{\nu(r,f)}{z}\right)^m (1 + o(1))f(z)$$

holds for integer $m \geq 0$ and $r \notin E$, where $\nu(r,f)$ is the central index of f.

Lemma 4.5. *([23, p. 10]) Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial, where $a_n \neq 0$. Then all zeros of $P(z)$ lie in the disc $D(0,r)$ of radius*

$$r \leq 1 + \max_{0 \leq k \leq n-1}\left(\left|\frac{a_k}{a_n}\right|\right).$$

Lemma 4.6. *Let f be a meromorphic function with $\sigma_{(\alpha,\beta,\gamma)}[f] = \sigma_0 \in (0, +\infty)$. Then for all μ $(< \sigma_0)$, there exists a set $I \subset (1, +\infty)$ of infinite logarithmic measure such that $\alpha(\log T(r,f)) > \mu\beta(\log\gamma(r))$ holds for all $r \in I$.*

Proof. The definition of (α, β, γ)-order implies that there exists a sequence $(R_j)_{j=1}^{+\infty}$ satisfying

$$\left(1 + \frac{1}{j}\right) R_j < R_{j+1}, \qquad \lim_{j\to+\infty} \frac{\alpha(\log T(R_j,f))}{\beta(\log\gamma(R_j))} = \sigma_0.$$

From the equality above, for any given $\varepsilon \in (0, \sigma_0 - \mu)$, there exists an integer j_1 such that for $j \geq j_1$,

$$\alpha(\log T(R_j,f)) > (\sigma_0 - \varepsilon)\beta(\log\gamma(R_j)). \tag{4.5}$$

Since $\mu < \sigma_0 - \varepsilon$, there exists an integer j_2 such that for $j \geq j_2$,

$$\frac{\sigma_0 - \varepsilon}{\mu} > \frac{\beta\left(\log\gamma\left(\left(1 + \frac{1}{j}\right) R_j\right)\right)}{\beta(\log\gamma(R_j))}.$$

It follows from this inequality and (4.5) that for $j \geq j_3 = \max\{j_1, j_2\}$ and for any $r \in \left[R_j, \left(1 + \frac{1}{j}\right) R_j\right]$,

$$\begin{aligned}
\alpha\left(\log T\left(r,f\right)\right) &\geq \alpha(\log T(R_j,f)) > (\sigma_0 - \varepsilon)\beta(\log\gamma(R_j)) \\
&= \frac{\sigma_0 - \varepsilon}{\mu}\mu\frac{\beta(\log\gamma(R_j))}{\beta\left(\log\gamma\left(r\right)\right)}\beta\left(\log\gamma\left(r\right)\right) \\
&\geq \frac{\sigma_0 - \varepsilon}{\mu}\frac{\beta(\log\gamma(R_j))}{\beta\left(\log\gamma\left(\left(1 + \frac{1}{j}\right) R_j\right)\right)}\mu\beta\left(\log\gamma\left(r\right)\right) \\
&> \mu\beta\left(\log\gamma\left(r\right)\right).
\end{aligned}$$

Set $I = \bigcup_{j=j_3}^{+\infty} \left[R_j, \left(1+\frac{1}{j}\right) R_j\right]$. It is easy to show that I is of infinite logarithmic measure,

$$m_l I := \int_I \frac{dr}{r} = \sum_{j=j_3}^{+\infty} \int_{R_j}^{\left(1+\frac{1}{j}\right)R_j} \frac{dr}{r} = \sum_{j=j_3}^{+\infty} \log\left(1+\frac{1}{j}\right) = +\infty.$$

□

We can also prove the following result by using similar reason as in the proof of Lemma 4.6.

Lemma 4.7. *Let f be an entire function with $\sigma_{(\alpha,\beta,\gamma)}[f] = \sigma_0 \in (0,+\infty)$ and $\tau_{(\alpha,\beta,\gamma),M}[f] \in (0,+\infty)$. Then for any given $\omega < \tau_{(\alpha,\beta,\gamma),M}[f]$, there exists a set $I \subset (1,+\infty)$ of infinite logarithmic measure such that for all $r \in I$,*

$$\exp\left\{\alpha(\log^{[2]} M(r,f))\right\} > \omega\left(\exp\left\{\beta\left(\log\gamma\left(r\right)\right)\right\}\right)^{\sigma_0}.$$

Lemma 4.8. *([16]) Let f be a solution of* (1.1)*, and let $1 \le p < +\infty$. Then for all $0 < r < R$, where $0 < R < +\infty$,*

$$m_p(r,f)^p \le C\left(\sum_{j=0}^{k-1} \int_0^{2\pi} \int_0^r \left|A_j(se^{i\theta})\right|^{\frac{p}{k-j}} ds d\theta + 1\right),$$

where $C > 0$ is a constant which depends on p and the initial value of f in a point z_0, where $A_j \ne 0$ for some $j = 0, ..., k-1$, and where

$$m_p(r,f)^p = \frac{1}{2\pi}\int_0^{2\pi} (\log^+ \left|f(re^{i\theta})\right|)^p d\theta.$$

The following logarithmic derivative estimation was found in [12] from Gundersen.

Lemma 4.9. *([12]) Let f be a transcendental meromorphic function, and let $\xi > 1$ be a given constant. Then there exists a set $E \subset (1,+\infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on ξ, and i, j, $0 \le i < j \le k-1$, such that for all z satisfying $|z| = r \notin [0,1] \cup E$,*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \le B\left\{\frac{T(\xi r, f)}{r}(\log^{\xi} r)\log T(\xi r, f)\right\}^{j-i}.$$

Lemma 4.10. *Let $A_0(z)$, $A_1(z), ..., A_{k-1}(z)$ be entire functions. Then every nontrivial solution f of* (1.1) *satisfies*

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f] \le \max\{\sigma_{(\alpha,\beta,\gamma)}[A_j] : j = 0, 1, ..., k-1\}.$$

Proof. Set

$$\varpi = \max\{\sigma_{(\alpha,\beta,\gamma)}[A_j] : j = 0, 1, ..., k-1\}.$$

By the definition of $\sigma_{(\alpha,\beta,\gamma)}[A_j]$, for any given $\varepsilon > 0$ and for sufficiently large r,

$$M(r, A_j) \leq \exp^{[2]}\left\{\alpha^{-1}((\varpi+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\},\ j = 0, 1, ..., k-1. \tag{4.6}$$

By Lemma 4.8 for $p = 1$ and (4.6), we have

$$T(r, f) = m(r, f) \leq 2\pi C\left(1 + \sum_{j=0}^{k-1} rM(r, A_j)\right)$$

$$\leq 2\pi C\left(1 + kr\exp^{[2]}\left\{\alpha^{-1}((\varpi+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right)$$

$$\leq \exp^{[2]}\left\{\alpha^{-1}((\varpi+2\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}. \tag{4.7}$$

Therefore, we get from (4.7) and Proposition 2.1 that

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f] \leq \max\{\sigma_{(\alpha,\beta,\gamma)}[A_j] : j = 0, 1, ..., k-1\}.$$

□

5. Proof of the Main Results

Proof of Theorem 3.1. Set $\Upsilon = \sup\{\sigma_{(\alpha(\log),\beta,\gamma)}[f] \mid L(f) = 0\}$ and $\digamma = \sup\{\sigma_{(\alpha,\beta,\gamma)}[A_j] \mid j = 0, ..., k-1\}$.

First we prove that $\digamma \leq \Upsilon$. If $\Upsilon = +\infty$, it is trivial. Hence we just consider the case of $\Upsilon < +\infty$. Let $f_1(z), f_2(z), ..., f_k(z)$ be a solution base of (1.1) with $\sigma_{(\alpha(\log),\beta,\gamma)}[f_j] < +\infty$, $j = 1, ..., k$. It is clear that $W = W(f_1, f_2, ..., f_k) \neq 0$ by the properties of the Wronski determinant.

It follows from Proposition 2.2, Proposition 2.4 and Lemma 4.3 that

$$\sigma_{(\alpha(\log),\beta,\gamma)}[W] < +\infty.$$

By the properties of the Wronski determinant ([23, p. 55]),

$$A_{k-s}(z) = -W_{k-s}((f_1, f_2, ..., f_k) \cdot W^{-1},\ s \in \{1, ..., k\},$$

where

$$W_j(f_1, f_2, ..., f_k) = \begin{vmatrix} f_1 & \cdots & f_k \\ \vdots & \vdots & \vdots \\ f_1^{(j-1)} & \cdots & f_k^{(j-1)} \\ f_1^{(j)} & \cdots & f_k^{(j)} \\ f_1^{(j+1)} & \cdots & f_k^{(j+1)} \\ \vdots & \vdots & \vdots \\ f_1^{(k-1)} & \cdots & f_k^{(k-1)} \end{vmatrix}.$$

In view of Proposition 2.2 we can conclude that $\sigma_{(\alpha(\log),\beta,\gamma)}[A_i] < +\infty$, $i = 0, 1, ..., k-1$. By Lemma 4.1 to f_i, $i = 1, ..., k$

$$m\left(r, \frac{f_i^{(l)}}{f_i}\right) = O\left(\exp\left\{\alpha^{-1}((\Upsilon+\varepsilon)\beta(\log\gamma(r)))\right\}\right),\ l = 1, 2, ..., k,\ r \notin F.$$

We now apply the standard order reduction procedure ([23, p. 53-57]). Let us denote

$$\nu_1(z) := \frac{d}{dz}\left(\frac{f(z)}{f_1(z)}\right),$$

$A_k = 1$ and $\nu_1^{(-1)} := \frac{f}{f_1}$, i.e., $(\nu_1^{(-1)})' := \nu_1$. Hence,

$$f^{(l)} = \sum_{m=0}^{l} \binom{l}{m} f_1^{(m)} \nu_1^{(l-1-m)},\ l = 0, ..., k. \tag{5.1}$$

Substituting (5.1) into (1.1) and using the fact that f_1 solves (1.1), we obtain

$$\nu_1^{(k-1)} + A_{1,k-2}(z)\nu_1^{(k-2)} + \cdots + A_{1,0}(z)\nu_1 = 0, \tag{5.2}$$

where

$$A_{1,j} = A_{j+1} + \sum_{m=1}^{k-j-1} \binom{j+1+m}{m} A_{j+1+m} \frac{f_1^{(m)}}{f_1},\ j = 0, ..., k-2.$$

By $\Upsilon < +\infty$ and Lemma 4.3, the meromorphic functions

$$\nu_{1,j}(z) = \frac{d}{dz}\left(\frac{f_{j+1}(z)}{f_1(z)}\right),\ j = 1, ..., k-1, \tag{5.3}$$

form a solution base to (5.2) of finite $(\alpha(\log), \beta, \gamma)$-order.

Next, we claim that

$$m(r, A_i) = O\left(\exp\left\{\alpha^{-1}((\Upsilon+\varepsilon)\beta(\log\gamma(r)))\right\}\right),\ i = 0, ..., k-1,\ r \notin F, \tag{5.4}$$

when

$$m(r, A_{1,j}) = O\left(\exp\left\{\alpha^{-1}((\Upsilon+\varepsilon)\beta(\log\gamma(r)))\right\}\right),\ j = 0, ..., k-2,\ r \notin F. \tag{5.5}$$

In fact, we prove it by induction on i following [23]. By (5.2), for $j = k-2$, we have $A_{1,k-2} = A_{k-1} + k\frac{f_1'}{f_1}$. By Lemma 4.1 and (5.5), we get that

$$\begin{aligned} m(r, A_{k-1}) &\leq m(r, A_{1,k-2}) + m\left(r, \frac{f_1'}{f_1}\right) + O(1) \\ &= O\left(\exp\left\{\alpha^{-1}((\Upsilon+\varepsilon)\beta(\log\gamma(r)))\right\}\right). \end{aligned}$$

We assume that

$$m(r, A_i) = O\left(\exp\left\{\alpha^{-1}((\Upsilon+\varepsilon)\beta(\log\gamma(r)))\right\}\right),\ i = k-1, ..., k-l. \tag{5.6}$$

Since

$$A_{1,k-(l+2)} = A_{k-(l+1)} + \sum_{m=1}^{l+1} \binom{m+k-l-1}{m} A_{m+k-l-1} \frac{f_1^{(m)}}{f_1},$$

by Lemma 4.1, (5.5) and (5.6), we obtain that

$$\begin{aligned} m(r, A_{k-(l+1)}) &\leq m(r, A_{1,k-(l+2)}) + m(r, A_{k-1}) + \cdots + m(r, A_{k-l}) \\ &\quad + m\left(r, \frac{f_1'}{f_1}\right) + \cdots + m\left(r, \frac{f_1^{(l+1)}}{f_1}\right) + O(1) \\ &= O\left(\exp\left\{\alpha^{-1}((\Upsilon+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right), \ r \notin F. \end{aligned} \tag{5.7}$$

We may now proceed as above to further reduce the order of (5.2). In each reduction step, according to (5.3), we obtain a solution base of meromorphic functions of finite $(\alpha(\log), \beta, \gamma)$-order according to (5.3), and the implication (5.5) to (5.4) remains valid. Hence, we finally obtain an equation of type $w' + B(z)w = 0$, and w is any solution of the equation with $\sigma_{(\alpha(\log),\beta,\gamma)}[w] < +\infty$. Then

$$m(r, B) = m\left(r, \frac{w'}{w}\right) = O\left(\exp\left\{\alpha^{-1}((\Upsilon+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right), \ r \notin F.$$

Observing the reasoning corresponding to (5.4) and (5.5) in the subsequent reduction steps, we see that

$$m(r, A_j) = O\left(\exp\left\{\alpha^{-1}((\Upsilon+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right), \ j = 0, ..., k-1, \ r \notin F.$$

It implies that

$$T(r, A_j) = O\left(\exp\left\{\alpha^{-1}((\Upsilon+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right), \ j = 0, ..., k-1, \ r \notin F.$$

By using Lemma 4.2 for $d = 2$, $\gamma(2r) \leq 2\gamma(r)$ and $\beta(x+O(1)) = (1+o(1))\beta(x)$ as $x \to +\infty$, for sufficiently large r, we obtain that

$$\begin{aligned} T(r, A_j) &= O\left(\exp\left\{\alpha^{-1}((\Upsilon+\varepsilon)\beta(\log\gamma(2r)))\right\}\right) \\ &\leq \exp\left\{\alpha^{-1}((\Upsilon+2\varepsilon)\beta\left(\log 2 + \log\gamma\left(r\right)\right))\right\} \\ &= \exp\left\{\alpha^{-1}((\Upsilon+2\varepsilon)\left(1+o\left(1\right)\right)\beta\left(\log\gamma\left(r\right)\right))\right\} \\ &\leq \exp\left\{\alpha^{-1}((\Upsilon+3\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}, \ j = 0, ..., k-1. \end{aligned}$$

Hence $\frac{\alpha(\log T(r,A_j))}{\beta(\log\gamma(r))} \leq \Upsilon + 3\varepsilon$. By the arbitrariness of $\varepsilon > 0$, we get that $\digamma \leq \Upsilon$.

We next prove the converse inequality under the assumption that $\digamma < +\infty$.

By Lemma 4.4, there exists a set $E \in \mathbb{R}_+$ of finite logarithmic measure, such that for all z satisfies $|f(z)| = M(r, f)$ and $|z| = r \notin E$,

$$f^{(i)}(z) = \left(\frac{\nu(r,f)}{z}\right)^i (1+o(1))f(z), \ i = 0, ..., k. \tag{5.8}$$

Now by substituting (5.8) into (1.1), we get that

$$\nu(r,f)^k + zA_{k-1}(z)\nu(r,f)^{k-1}(1+o(1))$$
$$+\cdots+z^{k-1}A_1(z)\nu(r,f)(1+o(1))+z^kA_0(z)(1+o(1))=0.$$

The Definition 2.1 of (α,β,γ)-order, yields that for any given $\varepsilon>0$ there exists $r_0>1$, such that for all $r\geq r_0$,

$$M(r,A_j)<\exp^{[2]}\left\{\alpha^{-1}((\digamma+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\},\ j=0,...,k-1.$$

By Lemma 4.5, we get that

$$\begin{aligned}
\nu(r,f) &\leq 1+\max_{0\leq j\leq k-1}\left|z^{k-j}A_j(z)(1+o(1))\right| \\
&\leq 1+\max_{0\leq j\leq k-1}\left(2r^{k-j}\exp^{[2]}\left\{\alpha^{-1}((\digamma+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right) \\
&\leq 1+2r^k\exp^{[2]}\left\{\alpha^{-1}((\digamma+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\} \\
&\leq \exp^{[2]}\left\{\alpha^{-1}((\digamma+2\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\},\ r\notin E.
\end{aligned}$$

It follows from [18, p. 36-37] that

$$\begin{aligned}
T(r,f) &\leq \log M(r,f)\leq\log\mu(r,f)+\log(\nu(2r,f)+2) \\
&\leq \nu(r,f)\log r+\log(2\nu(2r,f)) \\
&\leq \exp^{[2]}\left\{\alpha^{-1}((\digamma+2\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\log r \\
&\quad +\log(2\exp^{[2]}\left\{\alpha^{-1}((\digamma+2\varepsilon)\beta(\log\gamma(2r)))\right\}) \\
&\leq \exp^{[2]}\left\{\alpha^{-1}((\digamma+3\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}+\log 2 \\
&\quad +\exp\left\{\alpha^{-1}((\digamma+2\varepsilon)\beta(\log\gamma(2r)))\right\} \\
&\leq \exp^{[2]}\left\{\alpha^{-1}((\digamma+4\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}.
\end{aligned}$$

Since $\varepsilon>0$ is an arbitrary number, we obtain that $\Upsilon\leq\digamma$. Thus, the theorem is proved under the assumption that $\max\{\digamma,\Upsilon)]<+\infty$. If only one of $\digamma$ or Υ is finite, then by the previous proof we obtain a contradiction. Therefore, either, $\Upsilon=\digamma<+\infty$ or $\Upsilon=\digamma=+\infty$. Thus, Theorem 3.1 follows.

Proof of Theorem 3.2. By the assumption there exist two numbers λ_1 and λ such that $\sigma_{(\alpha,\beta,\gamma)}[A_m]\geq\lambda$ and $\sigma_{(\alpha,\beta,\gamma)}[A_l]\leq\lambda_1<\lambda$ for $l=m+1,...,k-1$.

Let $f_1,...,f_{m+1}$ be linearly independent solutions of (1.1) such that

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f_i]<\lambda,\quad i=1,...,m+1.$$

If $m=k-1$, then all $f_1,...,f_k$ are of $\sigma_{(\alpha(\log),\beta,\gamma)}[f_i]<\lambda$, this contradicts with Theorem 3.1. Hence, $m<k-1$. Applying the order reduction procedure as in the proof of Theorem 3.1, and using the notation ν_0 instead of f with

$A_{0,0}, ..., A_{0,k-1}$ instead of $A_0, ..., A_{k-1}$, and on the basis of general reduction step, we obtain an equation of type

$$\nu_j^{(k-j)} + A_{j,k-j-1}(z)\nu_j^{(k-j-1)} + \cdots + A_{j,0}(z)\nu_j = 0,\ j = 1, ..., k-1, \tag{5.9}$$

where

$$A_{j,l} = A_{j-1,l+1} + \sum_{n=1}^{k-l-j} \binom{l+1+n}{n} A_{j-1,l+1+n} \frac{\nu_{j-1,1}^{(n)}}{\nu_{j-1,1}}, \tag{5.10}$$

and the functions

$$\nu_{j,l}(z) = \frac{d}{dz}\left(\frac{\nu_{j-1,l+1}(z)}{\nu_{j-1,1}(z)}\right),\ l = 1, ..., k-j,$$

$\nu_0(z) = f(z)$, $\nu_j(z) = \frac{d}{dz}\left(\frac{\nu_{j-1}(z)}{\nu_{0,j-1}(z)}\right)$, determine at each reduction step a solution base of (5.9) in terms of the preceding solution base. We may express (1.1) and the $m-$th reduction steps by the following table. The rows correspond to (5.9) for $\nu_0(z), ..., \nu_m(z)$, i.e., the first row corresponds to (1.1), and columns from k to 0 give the coefficients of these equations, while the last column lists those solutions with $\sigma_{(\alpha(\log),\beta,\gamma)}[f] < \lambda$.

	k	$k-1$	$\cdot$	$\mathbf{m}$	$m-1$	$\cdot$	0	$\sigma_{(\alpha(\log),\beta,\gamma)}[f]<\lambda$
v_0	1	$A_{0,k-1}$	$\cdot$	$\mathbf{A}_{0,m}$	$A_{0,m-1}$	$\cdot$	$A_{0,0}$	$v_{0,1},...,v_{0,m+1}$
v_1		1	$\cdot$	$A_{1,m}$	$\mathbf{A}_{1,m-1}$	$\cdot$	$A_{1,0}$	$v_{1,1},...,v_{1,m}$
$\cdot$				$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$				$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$				$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
v_{m-1}				$A_{m-1,m}$	$A_{m-1,m-1}$	$\cdot$	$A_{m-1,0}$	$v_{m-1,1},v_{m-1,2}$
v_m				$A_{m,m}$	$A_{m,m-1}$	$\cdot$	$\mathbf{A}_{m,0}$	$v_{m,1}$

By Lemma 4.1 and (5.10), we see that in the second row, corresponding to the first reduction step,

$$m(r, A_{1,l}) = O\left(\exp\left\{\alpha^{-1}((\lambda_1+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),\ l = m, ..., k-2,\ r \notin F,$$

while $\lambda_1 + \varepsilon < \lambda$ and

$$m(r, A_{1,m-1}) \neq O\left(\exp\left\{\alpha^{-1}((\lambda_1+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),\ r \notin F.$$

Similarly in each reduction step of (5.10) implies that

$$m(r, A_{j,l}) = O\left(\exp\left\{\alpha^{-1}((\lambda_1+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),\ r \notin F, \tag{5.11}$$

when $l = m+1-j, ..., k-(j+1)$, i.e., for all coefficients to the left from the bold face coefficient $A_{j,m-j}$, while for $j = 1, ..., m$,

$$m(r, A_{j,m-j}) \neq O\left(\exp\left\{\alpha^{-1}((\lambda_1+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),\ r \notin F.$$

In particular,

$$m(r, A_{m,0}) \neq O\left(\exp\left\{\alpha^{-1}((\lambda_1+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),\ r \notin F.$$

Applying Lemma 4.6 to the coefficient $A_{m,0}$ with the constant λ, and obtain that

$$T(r, A_{m,0}) > \exp\left\{\alpha^{-1}((\lambda+\varepsilon)\,\beta\left(\log\gamma\left(r\right)\right))\right\},\ r \to +\infty,\ r \in I. \tag{5.12}$$

On the other hand, after the $m-$th reduction step, by (5.10) we have

$$A_{m,0} = -\frac{\nu_{m,1}^{(k-m)}}{\nu_{m,1}} - A_{m,k-m-1}\frac{\nu_{m,1}^{(k-m-1)}}{\nu_{m,1}} - \cdots - A_{m,1}\frac{\nu'_{m,1}}{\nu_{m,1}}.$$

That implies by (5.11) and Lemma 4.1, that

$$m(r, A_{m,0}) = O\left(\exp\left\{\alpha^{-1}((\lambda_1+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),\ r \notin F.$$

Since $\sigma_{(\alpha,\beta,\gamma)}[\nu_{m,1}] < \lambda_1$, in view of Proposition 2.2, Proposition 2.4 and Lemma 4.3, we obtain that

$$N(r, A_{m,0}) = O\left(\exp\left\{\alpha^{-1}((\lambda_1+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),\ r \notin F.$$

Therefore,

$$T(r, A_{m,0}) = O\left(\exp\left\{\alpha^{-1}((\lambda_1+\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}\right),\ r \notin F.$$

By using Lemma 4.2 for $d=2$, $\gamma(2r) \leq 2\gamma(r)$ and $\beta(x+O(1)) = (1+o(1))\beta(x)$ as $x \to +\infty$, for all sufficiently large r, we get that

$$\begin{aligned} T(r, A_{m,0}) &= O\left(\exp\left\{\alpha^{-1}((\lambda_1+\varepsilon)\beta(\log\gamma(2r)))\right\}\right) \\ &\leq O\left(\exp\left\{\alpha^{-1}((\lambda_1+\varepsilon)\left(1+o\left(1\right)\right)\beta(\log\gamma(r)))\right\}\right) \\ &\leq \exp\left\{\alpha^{-1}((\lambda_1+2\varepsilon)\beta\left(\log\gamma\left(r\right)\right))\right\}. \end{aligned} \tag{5.13}$$

Since $\varepsilon > 0$ satisfies $\lambda_1 + \varepsilon < \lambda$, by (5.12) and (5.13), we obtain a contradiction and this proves the assertion. Hence, there exist at most m linearly independent solutions of (1.1) with $\sigma_{(\alpha(\log),\beta,\gamma)}[f] < \lambda$.

Proof of Theorem 3.3. Let f be a nontrivial solution of (1.1). We denote $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_1$ and $\sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma_0$. The inequality $\sigma_0 \leq \sigma_1$ follows from Theorem 3.2 when $m=0$ and $\lambda = \sigma_0$.

To prove the conserve inequality, by Lemma 4.8 for $p=1$ and the definition of (α,β,γ)-order for any given $\varepsilon > 0$ and sufficiently large r

$$\begin{aligned} m(r,f) &\leq C\left(\sum_{j=0}^{k-1}\int_0^{2\pi}\int_0^r \left|A_j(se^{i\theta})\right|^{\frac{1}{k-j}} dsd\theta + 1\right) \\ &\leq C\left(k\max_{0\leq j\leq k-1}\int_0^{2\pi}\int_0^r \left|A_j(se^{i\theta})\right|^{\frac{1}{k-j}} dsd\theta + 1\right) \\ &\leq C_1 \max_{0\leq j\leq k-1}\int_0^r \left(\exp^{[2]}\left\{(\alpha^{-1}((\sigma_0+\varepsilon)\beta(\log\gamma(s)))\right\}\right)^{\frac{1}{k-j}} ds \\ &\leq C_1 \int_0^r \exp^{[2]}\left\{\alpha^{-1}((\sigma_0+\varepsilon)\beta(\log\gamma(s)))\right\} ds \\ &\leq C_1\, r\ \exp^{[2]}\left\{\alpha^{-1}((\sigma_0+\varepsilon)\beta(\log\gamma(r)))\right\} \\ &\leq \exp^{[2]}\left\{\alpha^{-1}((\sigma_0+2\varepsilon)\beta(\log\gamma(r)))\right\}, \end{aligned}$$

where $C_1 > 0$ is some constant. Therefore

$$\frac{\alpha(\log^{[2]} T(r,f))}{\beta(\log\gamma(r))} \leq \sigma_0 + 2\varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we get that $\sigma_1 \leq \sigma_0$, and then the proof is complete.

Proof of Theorem 3.4. Suppose that f is a nontrivial solution of (1.1). From (1.1), we can write that

$$(5.14)\qquad |A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right|.$$

If

$$\max\{\sigma_{(\alpha,\beta,\gamma)}[A_j], j=1,...,k-1\} < \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma_0 < +\infty,$$

then by Theorem 3.3, we obtain that

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha,\beta,\gamma)}[A_0].$$

Suppose that

$$\max\{\sigma_{(\alpha,\beta,\gamma)}[A_j], j=1,...,k-1\} = \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma_0 < +\infty$$

and

$$\begin{aligned} &\max\{\tau_{(\alpha,\beta,\gamma),M}[A_j] : \sigma_{(\alpha,\beta,\gamma)}[A_j] = \sigma_{(\alpha,\beta,\gamma)}[A_0] > 0\} \\ &< \tau_{(\alpha,\beta,\gamma),M}[A_0] = \tau_M < +\infty. \end{aligned}$$

First we prove that

$$\sigma_1 = \sigma_{(\alpha(\log),\beta,\gamma)}[f] \geq \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma_0.$$

Suppose the contrary $\sigma_1 = \sigma_{(\alpha(\log),\beta,\gamma)}[f] < \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma_0$. By assumption there exists a set $K \subseteq \{1, 2, ..., k-1\}$ such that

$$\sigma_{(\alpha,\beta,\gamma)}[A_j] = \sigma_{(\alpha,\beta,\gamma)}[A_0] = \sigma_0,\ \tau_{(\alpha,\beta,\gamma),M}[A_j] < \tau_{(\alpha,\beta,\gamma),M}[A_0] = \tau_M,\ j \in K$$

and

$$\sigma_{(\alpha,\beta,\gamma)}[A_j] < \sigma_{(\alpha,\beta,\gamma)}[A_0],\ j \in \{1, ..., k-1\} \setminus K.$$

Thus, we choose λ_1 and λ_2 satisfying

$$\max\{\tau_{(\alpha,\beta,\gamma),M}[A_j] : j \in K\} < \lambda_1 < \lambda_2 < \tau_{(\alpha,\beta,\gamma),M}[A_0] = \tau_M.$$

For sufficiently large r,

$$|A_j(z)| \le \exp^{[2]}\left\{\alpha^{-1}(\log(\lambda_1(\exp(\beta(\log\gamma(r))))^{\sigma_0}))\right\},\ j \in K \tag{5.15}$$

and

$$|A_j(z)| \le \exp^{[2]}\left\{\alpha^{-1}(\log(\lambda_1(\exp(\beta(\log\gamma(r))))^{\xi}))\right\}$$

$$\le \exp^{[2]}\left\{\alpha^{-1}(\log(\lambda_1(\exp(\beta(\log\gamma(r))))^{\sigma_0}))\right\},\ j \in \{1, ..., k-1\} \setminus K, \tag{5.16}$$

where $0 < \xi < \sigma_0$. By Lemma 4.7, there exists a set $I \subset (1, +\infty)$ with infinite logarithmic measure, such that for all $r \in I$,

$$|A_0(z)| > \exp^{[2]}\left\{\alpha^{-1}(\log(\lambda_2(\exp(\beta(\log\gamma(r))))^{\sigma_0}))\right\}. \tag{5.17}$$

By Lemma 4.9, there exist a constant $B > 0$ and a set $E \subset (1, +\infty)$ having finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \cup [0, 1]$,

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le B[T(2r, f)]^{k+1},\ j = 1, 2, ..., k.$$

By Proposition 2.1, for any given

$$\varepsilon \in \left(0, \max\left\{\frac{\lambda_2 - \lambda_1}{2}, \sigma_0 - \sigma_1\right\}\right)$$

and sufficiently large $|z| = r \notin E \cup [0, 1]$, we get that

$$\begin{aligned}\left|\frac{f^{(j)}(z)}{f(z)}\right| &\le B[T(2r, f)]^{k+1} \\ &\le B\left(\exp^{[2]}\left\{\alpha^{-1}((\sigma_1 + \varepsilon)\beta(\log\gamma(2r)))\right\}\right)^{k+1},\ j = 1, ..., k.\end{aligned} \tag{5.18}$$

Hence substituting (5.15), (5.16), (5.17) and (5.18) into (5.14), for sufficiently large $|z| = r \in I \setminus (E \cup [0, 1])$,

$$\begin{aligned}&\exp^{[2]}\left\{\alpha^{-1}(\log(\lambda_2(\exp(\beta(\log\gamma(r))))^{\sigma_0}))\right\} \\ \le\ &kB\exp^{[2]}\left\{\alpha^{-1}(\log(\lambda_1(\exp(\beta(\log\gamma(r))))^{\sigma_0}))\right\} \\ &\times\left(\exp^{[2]}\left\{\alpha^{-1}((\sigma_1 + \varepsilon)\beta(\log\gamma(2r)))\right\}\right)^{k+1}\end{aligned}$$

$$\leq \exp^{[2]}\left\{\alpha^{-1}(\log((\lambda_1+2\varepsilon)(\exp(\beta(\log\gamma(r))))^{\sigma_0}))\right\}. \tag{5.19}$$

Obviously, $I\setminus(E\cup[0,1])$ is of infinite logarithmic measure. By (5.19), there exists a sequence of points $\{|z_n|\}=\{r_n\}\subset I\setminus(E\cup[0,1])$ tending to $+\infty$ such that

$$\begin{aligned}&\exp^{[2]}\left\{\alpha^{-1}(\log(\lambda_2(\exp(\beta(\log\gamma(r_n))))^{\sigma_0}))\right\}\\ \leq\ &\exp^{[2]}\left\{\alpha^{-1}(\log((\lambda_1+2\varepsilon)(\exp(\beta(\log\gamma(r_n))))^{\sigma_0}))\right\}.\end{aligned}$$

From above we get that $\lambda_1\geq\lambda_2$. This contradiction implies

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f]\geq\sigma_{(\alpha,\beta,\gamma)}[A_0].$$

On the other hand, by Lemma 4.10, we get that

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f]\leq\max\{\sigma_{(\alpha,\beta,\gamma)}[A_j]:j=0,1,...,k-1\}=\sigma_{(\alpha,\beta,\gamma)}[A_0].$$

Hence every nontrivial solution f of (1.1) satisfies $\sigma_{(\alpha(\log),\beta,\gamma)}[f]=\sigma_{(\alpha,\beta,\gamma)}[A_0]$.

Acknowledgements

The first author is supported by University of Mostaganem (UMAB) (PRFU Project Code C00L03UN270120220007).

References

[1] Belaïdi, B. Estimation of the hyper-order of entire solutions of complex linear ordinary differential equations whose coefficients are entire functions. *Electron. J. Qual. Theory Differ. Equ.* (2002), no. 5, 8.

[2] Belaïdi, B. Growth and oscillation of solutions to linear differential equations with entire coefficients having the same order. *Electron. J. Differential Equations* (2009), No. 70, 10.

[3] Belaïdi, B. Growth of ρ_φ-order solutions of linear differential equations with entire coefficients. *PanAmer. Math. J. 27*, 4 (2017), 26–42.

[4] Belaïdi, B. Fast growing solutions to linear differential equations with entire coefficients having the same ρ_φ-order. *J. Math. Appl. 42* (2019), 63–77.

[5] Belaïdi, B., and Biswas, T. Study of complex oscillation of solutions of a second order linear differential equation with entire coefficients of (α,β,γ)-order. *WSEAS Trans. Math 21* (2022), 361–370.

[6] Belaïdi, B., and Biswas, T. Growth properties of solutions of complex differential equations with entire coefficients of finite (α,β,γ)-order. *Electron. J. Differential Equations* (2023), Paper No. 27, 14.

[7] Belaïdi, B., and Hamouda, S. Orders of solutions of an n-th order linear differential equation with entire coefficients. *Electron. J. Differential Equations* (2001), No. 61, 5.

[8] Bernal, L. G. On growth k-order of solutions of a complex homogeneous linear differential equation. *Proc. Amer. Math. Soc. 101*, 2 (1987), 317–322.

[9] Bouabdelli, R., and Belaïdi, B. Growth and complex oscillation of linear differential equations with meromorphic coefficients of [p, q]-φ order. *Int. J. Anal. Appl. 6*, 2 (2014), 178–194.

[10] Chen, Z.-X., and Yang, C.-C. Quantitative estimations on the zeros and growths of entire solutions of linear differential equations. *Complex Variables Theory Appl. 42*, 2 (2000), 119–133.

[11] Chyzhykov, I., and Semochko, N. Fast growing entire solutions of linear differential equations. *Mat. Visn. Nauk. Tov. Im. Shevchenka 13* (2016), 68–83.

[12] Gundersen, G. G. Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates. *J. London Math. Soc. (2) 37*, 1 (1988), 88–104.

[13] Gundersen, G. G. Finite order solutions of second order linear differential equations. *Trans. Amer. Math. Soc. 305*, 1 (1988), 415–429.

[14] Hayman, W. K. *Meromorphic functions.* Oxford Mathematical Monographs. Clarendon Press, Oxford, 1964.

[15] Hayman, W. K. The local growth of power series: a survey of the Wiman-Valiron method. *Canad. Math. Bull. 17*, 3 (1974), 317–358.

[16] Heittokangas, J., Korhonen, R., and Rättyä, J. Growth estimates for solutions of linear complex differential equations. *Ann. Acad. Sci. Fenn. Math. 29*, 1 (2004), 233–246.

[17] Heittokangas, J., Wang, J., Wen, Z. T., and Yu, H. Meromorphic functions of finite ϕ-order and linear q-difference equations. *J. Difference Equ. Appl. 27*, 9 (2021), 1280–1309.

[18] Jank, G., and Volkmann, L. *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen.* UTB für Wissenschaft: Grosse Reihe. [UTB for Science: Large Series]. Birkhäuser Verlag, Basel, 1985.

[19] Juneja, O. P., Kapoor, G. P., and Bajpai, S. K. On the (p, q)-order and lower (p, q)-order of an entire function. *J. Reine Angew. Math. 282* (1976), 53–67.

[20] Juneja, O. P., Kapoor, G. P., and Bajpai, S. K. On the (p, q)-type and lower (p, q)-type of an entire function. *J. Reine Angew. Math. 290* (1977), 180–190.

[21] Kara, M. A., and Belaïdi, B. Growth of φ-order solutions of linear differential equations with meromorphic coefficients on the complex plane. *Ural Math. J. 6*, 1 (2020), 95–113.

[22] Kinnunen, L. Linear differential equations with solutions of finite iterated order. *Southeast Asian Bull. Math. 22*, 4 (1998), 385–405.

[23] Laine, I. *Nevanlinna theory and complex differential equations*, vol. 15 of *De Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 1993.

[24] Li, L.-M., and Cao, T.-B. Solutions for linear differential equations with meromorphic coefficients of (p, q)-order in the plane. *Electron. J. Differential Equations* (2012), No. 195, 15.

[25] Liu, J., Tu, J., and Shi, L.-Z. Linear differential equations with entire coefficients of $[p, q]$-order in the complex plane. *J. Math. Anal. Appl. 372*, 1 (2010), 55–67.

[26] Liu, S. G., Tu, J., and Zhang, H. The growth and zeros of linear differential equations with entire coefficients of [p, q]- φ order. *J. Comput. Anal. Appl. 27*, 4 (2019), 681–689.

[27] Long, J., Qin, H., and Tao, L. On $[p, q]_{,\varphi}$-order and complex differential equations. *J. Nonlinear Math. Phys. 30*, 3 (2023), 932–955.

[28] Mulyava, O. M., Sheremeta, M. M., and Trukhan, Y. S. Properties of solutions of a heterogeneous differential equation of the second order. *Carpathian Math. Publ. 11*, 2 (2019), 379–398.

[29] Sato, D. On the rate of growth of entire functions of fast growth. *Bull. Amer. Math. Soc. 69* (1963), 411–414.

[30] Shen, X., Tu, J., and Xu, H. Y. Complex oscillation of a second-order linear differential equation with entire coefficients of $[p, q]$-φ order. *Adv. Difference Equ.* (2014), 2014:200, 14.

[31] Tu, J., and Chen, Z.-X. Growth of solutions of complex differential equations with meromorphic coefficients of finite iterated order. *Southeast Asian Bull. Math. 33*, 1 (2009), 153–164.

[32] Tu, J., and Yi, C.-F. On the growth of solutions of a class of higher order linear differential equations with coefficients having the same order. *J. Math. Anal. Appl. 340*, 1 (2008), 487–497.

[33] Valiron, G. *Lectures on the general theory of integral functions, translated by E. F. Collingwood, New York*. Chelsea Publishing Company, 1949.

[34] Xu, H.-Y., and Tu, J. Oscillation of meromorphic solutions to linear differential equations with coefficients of $[p, q]$-order. *Electron. J. Differential Equations* (2014), No. 73, 14.

[35] Yang, C.-C., and Yi, H.-X. *Uniqueness theory of meromorphic functions*, vol. 557 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 2003.

[36] Yang, L. *Value distribution theory.* Springer-Verlag, Berlin; Science Press Beijing, Beijing, 1993. Translated and revised from the 1982 Chinese original.

Received by the editors August 11, 2023
First published online March 23, 2024