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The multiplicity of solutions to a new class of superlinear fractional Schrödinger-Poisson systems

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Abstract. The purpose of this article is to establish the multiplicity of distributional solutions to a new class of fractional Schrödinger-Poisson systems of the following form

$$\begin{cases} -div^{\sigma}(\nabla^{\sigma} u) + V(x)u + K(x)\phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ -div^{\beta}(\nabla^{\beta}\phi) = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\sigma, \beta \in (0,1)$, $4\sigma + 2\beta > 3$, and $div^{\sigma}(\nabla^{\sigma})$ denotes the distributional Riesz fractional derivative. First, we introduce the latter operator and investigate their natural functional space. Then, we pose the given problem in that space. By using variational methods, based on the symmetric mountain pass theorem under certain assumptions imposed on g, K and V, we investigate the existence of infinitely many distributional solutions. Some new results are acquired which enrich the previous conclusions on this topic.

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1. Introduction

The nonlinear problems involving fractional elliptic operators have been extensively investigated by many authors in the previous years, it has become a very active and interesting research area, both form a pure mathematical standpoint and for the concrete applications. Indeed, these types of operators could serve as mathematical models for various problems in physics, biology, mechanics, obstacle problem and so on. For broad overviews of these topics, we refer the readers to [5, 15, 7, 19, 20, 2] and their references.

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Over the past few years, the search for a good concept to the nonlocal operator has led many researchers to develop a variety of different definitions of fractional derivatives operators. Particularly, the distributional Riesz fractional derivatives, these operators have achieved an immense attention, and many excellent results have emerged. On the other hand, the pioneering work of Shieh and Spector of 2015 [19] leads to a new class of fractional PDEs and new problems in the calculus of variations based on distributional Riesz fractional gradient, they proved that the fractional Laplacian coincides with the composition of fractional divergence and fractional gradient. Furthermore, they showed that the fractional gradient is an interesting object for the study of fractional partial differential equations problems due to its intriguing structure and properties analogous to the classical gradient, and also they introduced an appropriate functional space to study this new class of fractional PDE problems. Since then, the fractional gradient has been the subject of extensive research by an increasing number of authors, see e.g [4, 8, 15, 19, 20, 23]. Furthermore, this kind of operator satisfies three basic qualitative requirements as stated in Šilhavý paper [23], and has impressive applications in the theory of electromagnetic fields [1], multidimensional processes [10], and in fractal media [17].

In last years, non local systems involving fractional Laplacian operators have received the attention of several scholars, mainly the following fractional Schrödinger-Poisson system

$$(1.1)\qquad \begin{cases} (-\Delta)^{\sigma} u + V(x)u + K(x)\phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\beta}\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

which is used to study the solitary wave solutions for the fractional Schrödinger equations. Recently, there have been many papers devoted to the study on the existence of solutions to (1.1) by means of variational tools and critical point theory under different assumptions on V, K and g, see for instance [3] for the existence and multiplicity of distributional solutions for (1.1) when $g(x,u) = \lambda g(x,u)$. Furthermore, we call attention to the work of He and Lei [12], where it was proved the multiplicity of non-trivial solutions for (1.1) under some assumptions on V and K with $\sigma, \beta \in (0,1)$, $2\beta + 4\sigma > 3$, by exploiting the symmetric mountain pass theorem. We notice that when $K(x) \equiv 1$, Zhang [25] considered the following system

$$(1.2)\qquad \begin{cases} (-\Delta)^{\sigma} u + V(x)u + \phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\beta}\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\sigma, \beta \in (0,1)$, and established the multiplicity of solutions via the fountain theorem. In recent years, a similar system like (1.2) has been widely studied by many authors, for instance, Kim et al [13] for infinitely many solutions when $g(x,u) = \lambda g(x,u)$, Li [14] for non-trivial solutions when $V(x) = 1$, and Gao et al [11] for ground state non-trivial solutions when $g(x,u) = g(u)$.

In this viewpoint, the present paper aims to improve the previous results related

to (1.1) and also we would like to improve the hypotheses required and to extend the result in [12] to our case considering the presence of distributional Riesz fractional derivative instead of fractional Laplacian, and we give its consistency with fractional Laplacian derivattive.
Motivated by the above discussion, we mainly study in the Bessel potential space the following new class of fractional Schrödinger-Poisson systems

$$(1.3) \qquad \begin{cases} -div^{\sigma}(\nabla^{\sigma}u) + V(x)u + K(x)\phi u = g(x,u) & \text{in } \mathbb{R}^3, \\ -div^{\beta}(\nabla^{\beta}\phi) = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

with $\sigma, \beta \in (0,1)$, $4\sigma + 2\beta > 3$.
The starting concept of research pursued in [19] for the development of a general theory for fractional PDEs involving the distributional Riesz fractional derivatives, is the distributional Riesz fractional gradient ∇^{σ}. For $1 < p < \infty$ and $u \in L^p\left(\mathbb{R}^d\right)$ such that $I_{1-\sigma} * u$ is well defined, the fractional gradient ∇^{σ} can be characterized as (see [15, 19])

$$\left(\nabla^{\sigma}u\right)_j = \frac{\partial^{\sigma}u}{\partial x_j^{\sigma}} = \frac{\partial}{\partial x_j} I_{1-\sigma} * u, \ 0 < \sigma < 1, \ j = 1, ..., d,$$

where $\dfrac{\partial}{\partial x_j}$ is defined as follows

$$\langle \frac{\partial^{\sigma}u}{\partial x_j^{\sigma}}, w\rangle = -\langle I_{1-\sigma} * u, \frac{\partial w}{\partial x_j}\rangle = -\int\limits_{\mathbb{R}^d} (I_{1-\sigma} * u)\frac{\partial w}{\partial x_j}dx, \quad \forall w \in C_0^{\infty}(\mathbb{R}^d),$$

and I_{σ} denotes the Riesz potential of order σ, $0 < \sigma < 1$:

$$\left(I_{\sigma} * u\right)(x) = C_{d,\sigma} \int\limits_{\mathbb{R}^d} \frac{u\left(y\right)}{\left|x - y\right|^{d-\sigma}} dy, \quad \text{with} \quad C_{d,\sigma} := \pi^{-\frac{d}{2}} \frac{\Gamma(\frac{d-\sigma}{2})}{2^{\sigma}\Gamma(\frac{\sigma}{2})}.$$

Furthermore, the fractional divergence div^{σ} and the fractional gradient ∇^{σ} can be written in finite integral form for sufficiently smooth vector φ and function u ([8, 15, 20]), respectively by

$$\begin{aligned} div^{\sigma}\varphi(x) \quad &:= \quad C_{d,\sigma} \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^d} \frac{h.\varphi(x+h)}{|h|^{d+\sigma+1}} \chi_{\varepsilon}(0,h) dh \\ &= \quad C_{d,\sigma} \int\limits_{\mathbb{R}^d} [\varphi(x) - \varphi(y)]. \frac{x-y}{|x-y|^{d+\sigma+1}} dy, \end{aligned}$$

$$\begin{aligned} \nabla^{\sigma}u(x) \quad &:= \quad C_{d,\sigma} \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^d} \frac{hu(x+h)}{|h|^{d+\sigma+1}} \chi_{\varepsilon}(0,h) dh \\ &= \quad C_{d,\sigma} \int\limits_{\mathbb{R}^d} [u(x) - u(y)] \frac{x-y}{|x-y|^{d+\sigma+1}} dy, \end{aligned}$$

where for $\varepsilon > 0$, the characteristic function of $\{(x,h) : |h-x| > \varepsilon\}$ is $\chi_\varepsilon(x,h)$. It was observed in [19] that for $u \in C_0^\infty(\mathbb{R}^d)$, the composition of div^σ and ∇^σ coincides with the well known fractional Laplacian as follows:

$$\begin{aligned}(-\Delta)^\sigma u &= -\sum_{j=1}^{d} \frac{\partial^\sigma}{\partial x_j^\sigma}\frac{\partial^\sigma}{\partial x_j^\sigma} u \\ &= -div^\sigma(\nabla^\sigma u),\end{aligned} \tag{1.4}$$

where the fractional Laplacian can be given ([9]), for $\sigma \in (0,1)$ by

$$(-\Delta)^\sigma u(x) = \frac{1}{2} C_{d,\sigma}^2 \int_{\mathbb{R}^d} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2\sigma}} dy.$$

Moreover, for $u, w \in C_0^\infty(\mathbb{R}^d)$, equation (1.4) means that the following holds

$$\int_{\mathbb{R}^d} \nabla^\sigma u . \nabla^\sigma w dx = \int_{\mathbb{R}^d} (-\Delta)^\sigma u . w dx = \int_{\mathbb{R}^d} (-\Delta)^{\frac{\sigma}{2}} u . (-\Delta)^{\frac{\sigma}{2}} w dx,$$

which is useful for the weak formulation of PDEs involving fractional operator. For full exposition of this fractional operator detailing its nice properties, we refer the readers to [8, 15, 19, 20, 23].
Next, we require the assumptions on g, K and V:
(g_1) : $g \in C(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R})$ $\forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}$, and there exist constants $C_1 > 0$, and $p \in]2; 2_\sigma^*[$ such that

$$|g(x,u)| \leq C_1(|u| + |u|^{p-1}),$$

where $2_\sigma^* = \frac{6}{3-2\sigma}$ is the fractional critical Sobolev exponent.
(g_2) : $g(x,-u) = -g(x,u)$, $x \in \mathbb{R}^3$, $u \in \mathbb{R}$.
(g_3) : $\lim\limits_{|u|\to\infty} \frac{G(x,u)}{|u|^4} = \infty$ unifomly in $x \in \mathbb{R}^3$ and $G(x,u) \geq 0$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$ where $G(x,u) = \int_0^u g(x,t)dt$.
(g_4) : There exist $\lambda > 0$ and $L > 0$ such that

$$4G(x,u) \leq ug(x,u) + \lambda |u|^2 \quad \text{for a.e. } x \in \mathbb{R}^3 \text{ and } \forall |u| \geq L.$$

(K) : $K(x) \in L^{\frac{6}{4\sigma+2\beta-3}}(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$, $K(x) \geq 0$ $\forall x \in \mathbb{R}^3$.
(V) : $V(x) \in C\left(\mathbb{R}^3, \mathbb{R}\right)$, $\inf_{x\in\mathbb{R}^3} V(x) > 0$, and for any $V_0 > 0$ meas $\left\{x \in \mathbb{R}^3 \ V(x) \leq V_0\right\} < +\infty$.
The main result of this paper is as follows:

Theorem 1.1. *Assume that system* (1.3) *satisfies* (g_1)-(g_4), (K) *and* (V). *Then,* (1.3) *has infinitely many distributional solutions.*

The remainder of the paper is structured as follows. In the next section, we state some preliminaries and we give the variational framework for studying system (1.3). In Section 3, we use the symmetric mountain pass theorem to prove Theorem 1.1.

Notation: Throughout the paper we fix of the following notations.

-For any $p \in [1, \infty)$, $L^p\left(\mathbb{R}^d\right)$ is the Lesbesgue space with the norm $\|u\|_p = \Big(\int_{\mathbb{R}^d} |u|^p dx\Big)^{\frac{1}{p}}$.

-The letters C or C_i, $i = 1, 2...$ denote some positive constants.

2. Preliminaries and variational settings

Firstly, we introduce the variational framework for studying system (1.3), and the complete introduction on fractional Sobolev space $W^{\sigma,2}(\mathbb{R}^d)$ and Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$ can be found respectively, in [9, 19].

In the sequel for $\sigma \in (0,1)$, the vector space of fractional differentiable functions $S^{\sigma,2}(\mathbb{R}^d)$ is defined for $u \in C_0^\infty(\mathbb{R}^d)$ as the closure of $C_0^\infty(\mathbb{R}^d)$ with the norm

$$\|u\|^2_{S^{\sigma,2}} = \|u\|_2^2 + \|\nabla^\sigma u\|_2^2. \tag{2.1}$$

By Theorem 1.7 in [19], it is exactly the Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$ defined for $\sigma \in \mathbb{R}_+$ by

$$L^{\sigma,2}(\mathbb{R}^d) = \{u : u = G_\sigma * f \quad \text{for some} \quad f \in L^2(\mathbb{R}^d)\},$$

where the Bessel potential G_σ is defined by (see [19, 21])

$$G_\sigma(x) := \frac{1}{(4\pi)^{\frac{\sigma}{2}}\Gamma(\frac{\sigma}{2})}\int_0^{+\infty} e^{\frac{-\pi|x|^2}{t}} e^{\frac{-t}{4\pi}} t^{\frac{\sigma-d}{2}-1} dt.$$

The norm of $L^{\sigma,2}(\mathbb{R}^d)$ is $\|u\|_{L^{\sigma,2}} = \|f\|_2$ if $u = G_\sigma * f$.

Now, we summarize the main properties of $L^{\sigma,2}(\mathbb{R}^d)$ (see [19]).

Theorem 2.1. *1. For $\sigma \in (0,1)$, we have*

$$H^\sigma(\mathbb{R}^d) = W^{\sigma,2}(\mathbb{R}^d) = L^{\sigma,2}(\mathbb{R}^d) = S^{\sigma,2}(\mathbb{R}^d).$$

2. If $\sigma \geq 0$ and $2 \leq q \leq 2^*_\sigma$, then $L^{\sigma,2}(\mathbb{R}^d)$ is continuously embedded in $L^q(\mathbb{R}^d)$.

Remark 2.2.

- *Though our working space involves $\|\nabla^\sigma u\|_2$, we will not separate the Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$ from the fractional Sobolev space $H^\sigma(\mathbb{R}^d)$ despite the fact that $L^{\sigma,2}(\mathbb{R}^d)$ is topologically compatible with $H^\sigma(\mathbb{R}^d)$, and the norm in the two spaces being equivalent given by* (2.1).
- *As stated in [19], the natural setting for solving system* (1.3) *involving the distributional Riesz fractional gradient, is the Bessel potential space $L^{\sigma,2}(\mathbb{R}^d)$.*

The space $D^{\sigma,2}(\mathbb{R}^d)$ is defined by

$$D^{\sigma,2}(\mathbb{R}^d) = \left\{ u \in L^{2^*_\sigma}(\mathbb{R}^d) : \nabla^\sigma u \in L^2(\mathbb{R}^d) \right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_{D^{\sigma,2}} = (\int_{\mathbb{R}^d} |\nabla^\sigma u|^2 dx)^{\frac{1}{2}},$$

and the inner product

$$\langle u, w \rangle_{D^{\sigma,2}} = \int_{\mathbb{R}^d} (\nabla^\sigma u . \nabla^\sigma w) dx.$$

Now, we introduce the work space of (1.3) as

$$E = \left\{ u \in L^{\sigma,2}\left(\mathbb{R}^d\right) : \int_{\mathbb{R}^d} (|\nabla^\sigma u|^2 + V(x) u^2) dx < \infty \right\},$$

under the norm

$$\|u\|^2 = \int_{\mathbb{R}^d} (|\nabla^\sigma u|^2 + V(x) |u|^2) dx,$$

and the inner product

$$\langle u, w \rangle = \int_{\mathbb{R}^d} (\nabla^\sigma u . \nabla^\sigma w + V(x) u w) dx.$$

By assumption (V), clearly we have $\|u\| = \|u\|_{L^{\sigma,2}}$, and we have the following embedding property of E.

Lemma 2.3. *[22] E is compactly embedded in $L^q\left(\mathbb{R}^d\right)$ for $q \in [2, 2^*_\sigma)$, and continuously embedded in $L^q\left(\mathbb{R}^d\right)$ for $q \in [2, 2^*_\sigma]$.*

The previous definitions of $D^{\sigma,2}(\mathbb{R}^d)$ and E, coincide with any standard definitions found in the literature.
Now, we state the theorem which proves the solvability of linear fractional PDEs with variable coefficients.

Theorem 2.4. *[19] Let $\Omega \subset \mathbb{R}^d$ is an arbitrary bounded open set. Assume that $v \in L^{\sigma,2}(\mathbb{R}^d)$ and $h \in L^2(\Omega)$, such that $I_{1-\sigma} * v$ is well defined and $A : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ with coefficients bounded and measurable such that*

$$A(x)y.y \leq \Lambda |y|^2 \quad \text{and} \quad \Lambda_* |y|^2 \leq A(x)y.y$$

For all $x, y \in \mathbb{R}^d$ and $\Lambda_*, \Lambda > 0$. Then, there is a unique $u \in L^{\sigma,2}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} A(x) \nabla^\sigma u . \nabla^\sigma w dx = \int_{\mathbb{R}^d} g w dx,$$

for every $w \in L^{\sigma,2}(\mathbb{R}^d)$ and $u = v$ in $\mathbb{R}^d \backslash \Omega$.

In this work, we will use the above theorem only for the mapping A which maps any vector in $\mathbb{R}^d$ onto the identity matrix $\mathbb{I}$. Also, in this paper, we restrict the work space to dimension $d = 3$.

Lemma 2.5. *[9] For any $\sigma \in (0,1)$, $D^{\sigma,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*_\sigma}(\mathbb{R}^3)$, that is, there is $C_\sigma > 0$ such that :*

$$\left(\int_{\mathbb{R}^3} |u|^{2^*_\sigma} dx\right)^{\frac{2}{2^*_\sigma}} \leq C_\sigma \int_{\mathbb{R}^3} |\nabla^\sigma u|^2 dx, \ u \in D^{\sigma,2}\left(\mathbb{R}^3\right).$$

For any $u \in L^{\sigma,2}(\mathbb{R}^3)$, the linear operator $\mathcal{L}_u : D^{\beta,2}(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$\mathcal{L}_u(w) = \int_{\mathbb{R}^3} K(x) u^2 w dx,$$

is well defined on $D^{\beta,2}(\mathbb{R}^3)$ and continuous. Indeed, for $K \in L^{\frac{6}{4\sigma+2\beta-3}}(\mathbb{R}^3)$ using the Hölder inequality, (K), Lemma 2.3 and Lemma 2.5, we obtain

$$\begin{aligned} |\mathcal{L}_u(w)| &\leq \|K\|_{\frac{6}{4\sigma+2\beta-3}} \|u\|^2_{\frac{6}{3-2\sigma}} \|w\|_{2^*_\beta} \\ &\leq C\|u\|^2 \|w\|_{D^{\beta,2}}. \end{aligned} \tag{2.2}$$

While for $K \in L^\infty(\mathbb{R}^3)$, we derive that

$$\begin{aligned} |\mathcal{L}_u(w)| &\leq \|K\|_\infty \|u\|^2_{\frac{12}{3+2\beta}} \|w\|_{2^*_\beta} \\ &\leq C\|u\|^2 \|w\|_{D^{\beta,2}}. \end{aligned} \tag{2.3}$$

Therefore, set

$$\eta(u,w) = \int_{\mathbb{R}^3} \nabla^\beta u . \nabla^\beta w dx, \quad \forall u, w \in D^{\beta,2}(\mathbb{R}^3).$$

Thus, the Lax-Milgram theorem implies that there is a unique $\phi_u^\beta \in D^{\beta,2}\left(\mathbb{R}^3\right)$ such that $\mathcal{L}_u(w) = \eta(\phi_u, w)$ for every $w \in D^{\beta,2}\left(\mathbb{R}^3\right)$, that is,

$$\int_{\mathbb{R}^3} \nabla^\beta \phi_u^\beta . \nabla^\beta w dx = \int_{\mathbb{R}^3} K(x) u^2 w dx. \tag{2.4}$$

Therefore, ϕ_u^β is a weak solution of $-div^\beta\left(\nabla^\beta \phi_u^\beta\right) = K(x)u^2$. Moreover, for $x \in \mathbb{R}^3$, we have

$$\phi_u^\beta(x) = c_\beta \int_{\mathbb{R}^3} \frac{K(x)u^2(y)}{|x-y|^{3-2\beta}} dy, \tag{2.5}$$

which is called β-Riesz potential (see [21]), where

$$c_\beta = 2^{-2\beta} \frac{\Gamma\left(\frac{3-2\beta}{2}\right)}{\pi^{\frac{3}{2}}\Gamma(\beta)}.$$

Taking $w = \phi_u^\beta$ in (2.2)–(2.4), we get

$$\left\|\phi_u^\beta\right\|_{D^{\beta,2}} \leq C \|u\|^2 . \tag{2.6}$$

Substituting ϕ_u^β in (1.3), then (1.3) can be reduced to this fractional Schrödinger equation

$$-div^\sigma(\nabla^\sigma u) + V(x)u + K(x)\phi_u^\beta u = g(x,u), \quad x \in \mathbb{R}^3. \tag{2.7}$$

Whose solutions can be obtained by seeking critical values of the energy functional $\Phi : E \to \mathbb{R}$

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla^\sigma u|^2 + V(x)u^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\beta u^2 dx - \int_{\mathbb{R}^3} G(x,u)\,dx. \tag{2.8}$$

Moreover, if we take $w = \phi_u^\beta$ in (2.2)-(2.4) again, and by (2.6) we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} K(x)\phi_u^\beta u^2 dx &\leq C\|u\|^2 \left\|\phi_u^\beta\right\|_{D^{\beta,2}} \\ &\leq C\|u\|^4 \end{aligned} \tag{2.9}$$

Hence, Φ is well defined in E and $\Phi \in C^1(E,\mathbb{R})$, and for all $u, w \in E$,

$$< \Phi'(u), w >= \int_{\mathbb{R}^3} (\nabla^\sigma u.\nabla^\sigma w + V(x)uw + K(x)\phi_u^\beta uw - g(x,u)w)dx. \tag{2.10}$$

Definition 2.6. *1. $(u,\phi) \in E \times D^{\beta,2}(\mathbb{R}^3)$ is a distributional solution of (1.3) if u is a distributional solution of* (2.7).

2. u is a distributional solution of (2.7) *for any $w \in E$, if*

$$\int_{\mathbb{R}^3} \left(\nabla^\sigma u.\nabla^\sigma w + V(x)uw + K(x)\phi_u^\beta uw - g(x,u)w\right)dx = 0.$$

Now, we introduce the Cerami condition, which was established by Cerami [6]. Assume that $\Phi \in C^1(E,\mathbb{R})$.

Definition 2.7. *The functional Φ satisfies the Cerami condition ($(C)_c$ condition for short) at level $c \in \mathbb{R}$, if any $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c \quad \textit{and} \quad \|\Phi'(u_n)\|(1+\|u_n\|) \to 0 \quad \textit{as } n \to \infty.$$

possesses a convergent subsequence.

We choose $\{e_i\}$ an orthonormal basis of space E and define $X_i = \mathbb{R}e_i$,

$$Y_k = \oplus_{i=1}^{k} X_i \quad Z_k = \overline{\oplus_{i=k}^{\infty} X_i} \quad k \in \mathbb{Z}.$$

Evidently, we have $E = Y_k \oplus Z_k$.
To prove our main result, we need the following symmetric mountain-pass theorem [18].

Theorem 2.8. *Assume that $E = Y_k \oplus Z_k$ is a Banach space where Y is finite dimensional, let $\Phi \in C^1(E, \mathbb{R})$ be even and $\Phi(0) = 0$, satisfies the $(C)_c$ condition, if*
(i) there exist constants $\rho, \delta > 0$ satisfying $\Phi \big|_{\partial B_\rho \bigcap Z} \geq \delta$, where B_ρ is the open ball in E of radius ρ about 0 and ∂B_ρ is its boundary,
(ii) for any finite dimensional subspace $\tilde{E} \subset E$, there is $C = C(\tilde{E}) > 0$ such that $\Phi(u) \leq 0$ on $\tilde{E} \setminus B_C$,
then, Φ has an unbounded sequence of critical points.

3. Proof of Main Result

Lemma 3.1. Φ *satisfies $(C)_c$ condition for every $c \in \mathbb{R}$ on E, if (g_1)-(g_4) and (V) hold.*

Proof. For $c \in \mathbb{R}$, let $\{u_n\}$ be the $(\mathrm{C})_c$ sequence in E, that is

$$c = \Phi(u_n) + o_n(1) \ \text{ and } \ \langle \Phi'(u_n), u_n \rangle = o_n(1), \tag{3.1}$$

where $o_n(1) \to 0$ as $n \to \infty$. First, we argue by contradiction to verify that $\{u_n\}$ is bounded in E. Assume that $\|u_n\| \to \infty$ as $n \to \infty$. Consider a sequence $\{v_n\} \subset E$ such that $v_n = \frac{u_n}{\|u_n\|}$. Then it is obvious that $\|v_n\| = 1$. Hence, there is a subsequence $\{v_n\}$ (we still write it as v_n) such that $v_n \rightharpoonup v$ in E as $n \to \infty$. We have by Lemma 2.3 as $n \to \infty$ that,

$$v_n \to v \quad \text{a.e. in } \mathbb{R}^3 \ \text{ and } \ v_n \to v \quad \text{in } L^p\left(\mathbb{R}^3\right) \ \text{ for } \ 2 \leq p < 2_\sigma^*. \tag{3.2}$$

There are two possible cases.
Case $v(x) \neq 0$ in $\mathbb{R}^3$. Set $A_0 = \left\{x \in \mathbb{R}^3 : v(x) \neq 0\right\}$. Thus meas$(A_0) > 0$. Since $\|u_n\| \to \infty$ as $n \to \infty$ and by (3.1), we get

$$\begin{aligned} c &= \Phi(u_n) + o_n(1) \\ &= \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^{\beta} u_n^2 dx - \int_{\mathbb{R}^3} G(x, u_n)dx + o_n(1). \end{aligned} \tag{3.3}$$

Since $\|u_n\| \to \infty$ as $n \to \infty$, we derive

$$\int_{\mathbb{R}^3} G(x, u_n)dx \geq \frac{1}{2}\|u_n\|^2 - c + o_n(1) \to \infty \ \text{ as } n \ \to \infty. \tag{3.4}$$

Combining (2.9) and (3.3), we obtain for n large enough that the following holds

$$\int_{\mathbb{R}^3} G(x,u_n)dx + c - o_n(1) = \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^{\beta} u_n^2 dx \leq \frac{3}{4}\|u_n\|^4. \tag{3.5}$$

Moreover, the assumption (g_3) implies that there is $u_0 > 1$ such that $G(x,u) > |u|^4$ for all $x \in \mathbb{R}^3$ and $|u| > u_0$. By means of (g_1), we derive that there is $C > 0$ such that $|G(x,u)| < C$ for all $(x,u) \in \mathbb{R}^3 \times [-u_0, u_0]$. Hence, there is $C_0 \in \mathbb{R}$ such that $G(x,u) \geq C_0$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$. Thus

$$\frac{G(x,u_n) - C_0}{\|u_n\|^4} \geq 0. \tag{3.6}$$

The assumption (g_3) implies that

$$\lim_{n\to\infty} \frac{G(x,u_n)}{\|u_n\|^4} = \lim_{n\to\infty} \frac{G(x,u_n)}{|u_n|^4}|v_n|^4 = \infty, \tag{3.7}$$

for all $x \in A_0$. Thus, we see that $\text{meas}(A_0) = 0$. Indeed, suppose that $\text{meas}(A_0) \neq 0$, combining with (3.4)-(3.7) and the Fatou lemma and taking into that A_0 have a finite measure, we infer that

$$\begin{aligned}
\frac{3}{4} &= \liminf_{n\to\infty} \frac{\frac{3}{4}\int_{\mathbb{R}^3} G(x,u_n)dx}{\int_{\mathbb{R}^3} G(x,u_n)dx + c - o_n(1)} \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{\frac{3}{4}G(x,u_n)}{\frac{3}{4}\|u_n\|^4}dx \\
&\geq \liminf_{n\to\infty} \int_{A_0} \frac{G(x,u_n)}{\|u_n\|^4}dx - \limsup_{n\to\infty}\int_{A_0} \frac{C_0}{\|u_n\|^4}dx \\
&\geq \liminf_{n\to\infty} \int_{A_0} \frac{G(x,u_n) - C_0}{\|u_n\|^4}dx. \\
&\geq \int_{A_0} \liminf_{n\to\infty} \frac{G(x,u_n)}{\|u_n\|^4}dx - \int_{A_0} \limsup_{n\to\infty} \frac{C_0}{\|u_n\|^4}dx = \infty, \qquad (3.8)
\end{aligned}$$

which is a contradiction. This means that $v(x) = 0$ for a.e. $x \in \mathbb{R}^3$.

Case $v(x)=0$. From (g_4), one has

$$\begin{aligned} c+1 &\geq \Phi\left(u_n\right)-\frac{1}{4}\left\langle\Phi'\left(u_n\right), u_n\right\rangle \\ &= \left(\frac{1}{2}-\frac{1}{4}\right)\left\|u_n\right\|^2+\int\limits_{\mathbb{R}^3}\left(\frac{g\left(x, u_n\right) u_n}{4}-G\left(x, u_n\right)\right) dx \\ &\geq \frac{1}{4}\left\|u_n\right\|^2-\frac{\lambda}{4} \int\limits_{\mathbb{R}^3}\left|u_n\right|^2 dx \\ &\geq \frac{1}{4}\left\|u_n\right\|^2-\frac{\lambda}{4}\left\|v_n\right\|_2^2\left\|u_n\right\|^2, \end{aligned}$$

which implies

$$\frac{c+1}{\left\|u_n\right\|^2} \geq \frac{1}{4}-\frac{\lambda}{4}\left\|v_n\right\|_2^2 .$$

Since $\|u_n\| \to \infty$ as $n \to \infty$, then

$$\left\|v_n\right\|_2^2 \geq \frac{1}{\lambda},$$

which shows that $v(x) \neq 0$, then we arrive at contradiction in any cases. Hence, $\{u_n\}$ is bounded in E.
Next, we verify that u_n has a convergent subsequence. Up to a subsequence, we have that $u_n \rightharpoonup u$ in E, from Lemma 2.3 we conclude $u_n \to u$ in $L^p\left(\mathbb{R}^3\right)$, for all $2 \leq p < 2_\sigma^*$. Clearly, we have

$$\left\langle\Phi'\left(u_n\right)-\Phi'\left(u\right), u_n-u\right\rangle \to 0 \text{ and } \left\|u_n-u\right\|_2^2 \to 0 \text{ as } n \to \infty. \tag{3.9}$$

For $K \in L^\infty(\mathbb{R}^3)$, from (K) and the Hölder inequality we infer

$$\left|\int\limits_{\mathbb{R}^3} K(x) \phi_{u_n}^\beta u_n\left(u_n-u\right) dx\right| \leq \|K\|_\infty\left\|\phi_{u_n}^\beta\right\|_{\frac{6}{3-2\beta}}\left\|u_n\right\|_{\frac{12}{3+2\beta}}\left\|u_n-u\right\|_{\frac{12}{3+2\beta}} \to 0$$

as $n \to \infty$. Thus, we deduce that

$$\begin{aligned} rl&\left|\int\limits_{\mathbb{R}^3} K(x)\left(\phi_{u_n}^\beta u_n-\phi_u^\beta u\right)\left(u_n-u\right) dx\right| \\ &\leq \left|\int\limits_{\mathbb{R}^3} K(x) \phi_{u_n}^\beta u_n\left(u_n-u\right) dx\right| \\ &+ \left|\int\limits_{\mathbb{R}^3} K(x) \phi_u^\beta u\left(u_n-u\right) dx\right| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.10}$$

For $K \in L^{\frac{6}{4\sigma+2\beta-3}}(\mathbb{R}^3)$, we can prove similarly as in [16] (see Proposition 2.4) and [24] (see Lemma 2.1), that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^{\beta}u_n\,(u_n-u)\,dx=\int_{\mathbb{R}^3} K(x)\phi_u^{\beta}u\,(u_n-u)\,dx.$$

According to (g_1) and the Hölder inequality, we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}(g(x,u_n)-g(x,u))(u_n-u)\,dx\right|\\&\leq C_1\left(\|u_n\|_2+\|u\|_2\right)\|u_n-u\|_2+C_1\left(\|u_n\|_p^{p-1}+\|u\|_p^{p-1}\right)\|u_n-u\|_p\\&\leq C\left(\|u_n\|+\|u\|\right)\|u_n-u\|_2+C\left(\|u_n\|^{p-1}+\|u\|^{p-1}\right)\|u_n-u\|_p\to 0\end{aligned}$$

as $n\to\infty$. Thus

$$\begin{aligned}\|u_n-u\|^2 &= \langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle\\&\quad-\int_{\mathbb{R}^3}K(x)\left(\phi_{u_n}^{\beta}u_n-\phi_u^{\beta}u\right)(u_n-u)\,dx\\&\quad+\int_{\mathbb{R}^3}(g(x,u_n)-g(x,u))(u_n-u)dx\to 0 \text{ as } n\to\infty.\end{aligned}$$

Therefore, $\{u_n\}$ converges strongly in E. □

Lemma 3.2. *Suppose that (g_1),(g_3) and (V) are satisfied. Then, for any finite dimensional subspace $\tilde{E}\subset E$, we have*

$$\Phi(u)\to-\infty,\qquad \|u\|\to\infty,\quad u\in\tilde{E}.$$

Proof. We argue indirectly, suppose that there is $C>0$ for some $\{u_n\}\subset\tilde{E}$ and all $n\in\mathbb{N}$, such that $\Phi(u_n)\geq -C$ with $\|u_n\|\to\infty$. Set $v_n=\frac{u_n}{\|u_n\|}$, then $\|v_n\|=1$. Passing to a subsequence, we may suppose that $v_n\rightharpoonup v$ in E. Since $\dim(\tilde{E})<\infty$, then $v_n\to v\in\tilde{E}$ in E, $v_n(x)\to v(x)$ a.e. on $\mathbb{R}^3$ and so $\|v\|=1$. Set $A_0=\left\{x\in\mathbb{R}^3: v(x)\neq 0\right\}$. Thus meas$(A_0)>0$, and we have $|u_n(x)|\to\infty$ for a.e. $x\in A_0$. From (2.9) we get

$$\begin{aligned}&\lim_{n\to\infty}\frac{4\int_{\mathbb{R}^3}G(x,u_n)}{\|u_n\|^4}dx\\ \text{(3.11)}\qquad &=\lim_{n\to\infty}\int_{A_0}\frac{2\|u_n\|^2+\int_{\mathbb{R}^3}K(x)\phi_{u_n}^{\beta}u_n^2dx-4\Phi(u_n)}{\|u_n\|^4}dx\leq C,\end{aligned}$$

for $x \in A_0$. Thus, since we have $|u_n(x)| \to \infty$, by similar arguments in (3.6)-(3.8) and from (g_1) and (g_3) we infer that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{4G(x,u_n)}{\|u_n\|^4} dx \geq \lim_{n\to\infty} \int_{A_0} \frac{4G(x,u_n)}{\|u_n\|^4} dx = \infty, \tag{3.12}$$

which is a contradiction with (3.11). □

Corollary 3.3. *Assume that (g_1),(g_3) and (V) are satisfied. Then, there is a constant $C = C(\tilde{E}) > 0$ for every finite dimensional subspace $\tilde{E} \subset E$ such that*

$$\Phi(u) \leq 0 \quad \text{for all } \ u \in \tilde{E} \quad \text{with } \|u\| \geq C.$$

Lemma 3.4. *For $2 \leq p < 2^*_\sigma$, we have that*

$$\Gamma_k := \sup_{u\in Z_k, \|u\|=1} \|u\|_p \to 0 \quad \text{as } \ k \to \infty$$

Proof. Clearly we have $0 < \Gamma_{k+1} \leq \Gamma_k$, hence $\Gamma_k \to \Gamma \geq 0$ as $k \to \infty$. For any $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_p \geq \frac{\Gamma_k}{2}$. Since the embedding from E into $L^p(\mathbb{R}^3)$ is compact, and by definition of $Z_k, u_k \to 0$ in E, we obtain that $u_k \to 0$ in $L^p(\mathbb{R}^3)$. This proves that $\Gamma = 0$. □

We can choose by Lemma 3.4 an integer $m \geq 1$ such that

$$\|u\|_2^2 \leq \frac{1}{2C_1} \|u\|^2, \quad \|u\|_p^p \leq \frac{p}{4C_1} \|u\|^p \quad \forall u \in Z_m. \tag{3.13}$$

Lemma 3.5. *Let (g_1) and (V) hold. Then, there exist constants $\rho, \delta > 0$ satisfying $\Phi \big|_{\partial B_\rho \bigcap Z_m} \geq \delta > 0$.*

Proof. For $u \in Z_m$, by (g_1) and (3.13), choosing $\rho := \|u\| = \frac{1}{2}$, we derive that

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla^\sigma u|^2 + V(x)u^2\right) dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^\beta u^2 dx - \int_{\mathbb{R}^3} G(x,u)\, dx \\
&\geq \frac{1}{2}\|u\|^2 - \frac{C_1}{2}\|u\|_2^2 - \frac{C_1}{p}\|u\|_p^p \\
&\geq \frac{1}{4}(\|u\|^2 - \|u\|^p) \\
&= \frac{2^{p-2}-1}{2^{p+2}} := \delta > 0.
\end{aligned}$$

This is the desired result. □

Proof of Theorem 1.1 Let $Y = Y_m$ and $Z = Z_m$. Due to (g_2), it is obvious that $\Phi(u)$ is even. Based on Lemma 3.1, Lemma 3.5 and Corollary 3.3, $\Phi(u)$ satisfies all conditions of Theorem 2.8. Consequently, the system (1.3) possesses infinitely many distributional solutions.

4. Conclusion

In this paper, we study a new class of fractional Schrödinger-Poisson systems. By applying the symmetric mountain pass theorem, infinitely many distributional solutions were obtained. From our perspective, this paper is an improvement for the study of fractional Schrödinger-Poisson systems. In addition, we give to readers an agile exposition of the distributional Riesz fractional derivatives and some of their elementary properties.

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