

A note on continuity of convex functions

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Abstract. In this note we generalize a well known theorem which says that *a function which is convex on an open subset of finite dimensional space is continuous* (Theorem 3.2, Corollary 3.3 ,Corollary 3.4.) Also we give some properties of quasiconvex functions and their connections with reflexivity (Proposition 2.5 , Corollary 2.6).

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1. Introduction

Convex functions play a very important role in mathematics especially in optimization, functional analysis and probability theory. Convexity of functions implies many "good" properties, such as continuity on an open interval or existence of right and left derivatives. Many of these properties, such as Jensen inequality, can be used to prove others inequalities like AM-GM inequality or Hölder inequality. There is no need to convince anyone that the convexity of functions has many applications in mathematics, as well as continuity.

The main purpose of this paper is to generalize the fact that a function that is convex on an open and convex subset of a finite dimensional normed space is continuous (see Corollary 10.11 in [4]). This is a well known property and we obtain it as a corollary of Theorem 3.2 which shows that the essential condition for a convex function to be continuous is its boundness. Moreover the boundness of convex function on a ball in finite dimensional normed space is a consequence of two facts: the first one is that *every two norms defined on a finite dimensional vector space are equivalent* and the second says that *in any finite dimensional vector space there is a norm in which a closed unit ball has a finite number of extremal points*. This shows that, from the point of view of continuity, convex functions behave very similarly to the linear functionals.

Also in this short note we give some connections between reaching minimum by quasiconvex functions and reflexivity or semi reflexivity of the space. In Proposition 2.5 we generalize a fact that *every convex function defined on closed bounded and convex subset of reflexive Banach space attains its minimum* (see [1] Prop.1.2 , also [3]) to the case quasiconvex function and semi reflexive topological vector space. The consequences of this proposition are given in Corollary 2.6.

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2. Some basic definitions and propositions

We start with some basic definitions and properties which will be useful for our purpose.

Definition 2.1. Let $W \subset X$ be a convex subset of the vector space X . We say that the function $f : W \rightarrow \mathbb{R}$ is *convex* if

$$f(\gamma x + \delta y) \leq \gamma f(x) + \delta f(y)$$

for all $x, y \in W, \gamma, \delta \in \mathbb{R}, \gamma, \delta \geq 0, \gamma + \delta = 1$.

Definition 2.2. Let $W \subset X$ be a convex subset of the vector space X . We say that a function $f : W \rightarrow \mathbb{R}$ is *quasiconvex* if the set

$$C_f^a = f^{-1}((-\infty, a]) = \{x \in W : f(x) \leq a\}$$

is convex for all $a \in \mathbb{R}$.

Every convex function is quasiconvex function but the converse statement is not true because, for example, every monotone increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ is quasiconvex.

Notice that $f : W \rightarrow \mathbb{R}$ is quasiconvex if and only if

$$f(\gamma x + \delta y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in X, \gamma, \delta \in \mathbb{R}, \gamma, \delta \geq 0, \gamma + \delta = 1$.

Proposition 2.3. Let $W \subset X$ be a convex subset of vector space X . A function $f : W \rightarrow \mathbb{R}$ is quasiconvex if and only if the sets

$$O_f^a = f^{-1}((-\infty, a)) = \{x \in W : f(x) < a\}$$

are convex for all $a \in \mathbb{R}$.

Proof. If f is quasiconvex then by the equality

$$O_f^a = \bigcup_{n \in \mathbb{N}} C_f^{a - \frac{1}{n}}$$

we obtain that the sets O_f^a are convex as a union of monotone family of convex sets.

Conversely if all the sets O_f^a are convex then since

$$C_f^a = \bigcap_{n \in \mathbb{N}} O_f^{a + \frac{1}{n}}$$

the set C_f^a is convex. Hence f is quasiconvex. □

Definition 2.4. Let X be a topological space. A function $f : X \rightarrow \mathbb{R}$ is called *lower semicontinuous* if the sets $f^{-1}((a, \infty))$ are open for all $a \in \mathbb{R}$.

Observe that if $f : W \rightarrow \mathbb{R}$ is lower semicontinuous then the sets

$$C_f^a = \{x \in W : f(x) \leq a\} = W \setminus f^{-1}((a, \infty))$$

are closed for all $a \in \mathbb{R}$.

Proposition 2.5. *Let X be a semi-reflexive locally convex topological vector space and $W \subset X$ be a closed bounded and convex set. Then every quasiconvex lower semicontinuous function $f : W \rightarrow \mathbb{R}$ attains its infimum on W .*

Proof. Since W is a closed bounded convex set and the space X is semi-reflexive, it is compact in the topology $\sigma(X, X^*)$ (where the symbol $\sigma(X, X^*)$ denotes the weakest linear topology defined on X which makes continuous all linear and continuous functionals in the given topology on X).

First we prove that f is bounded below on W . Suppose, to the contrary, that $\inf_{x \in W} f(x) = -\infty$. The sets C_f^{-n} are nonempty convex closed and bounded for $n \in \mathbb{N}$ hence by the semi-reflexivity of X they are compact in the weak topology $\sigma(X, X^*)$. Hence

$$\bigcap_{n \in \mathbb{N}} C_f^{-n} \neq \emptyset$$

but this is impossible since f takes only finite values.

Therefore $m = \inf_{x \in W} f(x) < \infty$.

An analogous argument shows that the set $C = \bigcap_{n \in \mathbb{N}} C_f^{m+\frac{1}{n}}$ is not empty and therefore for some $v \in C$ we have

$$f(v) = \inf_{x \in W} f(x).$$

□

Corollary 2.6. *A Banach space X is reflexive if and only if every quasiconvex and lower semicontinuous function $f : B \rightarrow \mathbb{R}$, reaches its infimum, where B denotes a closed unit ball in X .*

Proof. If X is a reflexive Banach space then it is a semi-reflexive locally convex space and by Proposition 2.5 we obtain that every quasiconvex lower semicontinuous function attains its infimum on closed unit ball.

Assume that every quasiconvex lower semicontinuous function attains its infimum on unit ball. Since every continuous linear functional f defined on X is a quasiconvex and lower semicontinuous function defined on X , thus there exists $x_0 \in B$ such that $f(x_0) = \inf_{x \in B} f(x)$. Hence $f(-x_0) = -\inf_{x \in B} f(x) = \sup_{x \in B} f(x) = \|f\|$ and by the James Theorem (see [2], characterization of reflexivity of Banach spaces) we obtain that the space X is reflexive. □

3. Main results

Now we are going to prove our main result (Theorem 3.2) which shows some connections between boundness and continuity of a convex function. We start with the following lemma.

Lemma 3.1. *Let $f : W \rightarrow \mathbb{R}$ be a convex function defined on a convex subset W of normed space X . Then*

a) *If there is a $r > 0$ and $x_0 \in W$ such that*

$$\inf_{x \in B_W(x_0, r)} f(x) > -\infty$$

where $B_W(x_0, r) = \{x \in W : \|x - x_0\| < r\}$ then

$$\inf_{x \in G} f(x) > -\infty$$

for any bounded $G \subset W$.

b) *If there is a $r > 0$ and $x_0 \in W$ such that $x_0 - B_W(x_0, r) = B_W(x_0, r) - x_0$ then*

$$|f(z) - f(x_0)| \leq \mu \left| \inf_{x \in B_W(x_0, r)} f(x) - f(x_0) \right|$$

for all $z \in B_W(x_0, \mu r)$ and $0 < \mu < 1$.

c) *If there is a $r > 0$ and $x_0 \in W$ such that $x_0 - B_W(x_0, r) = B_W(x_0, r) - x_0$ then*

$$|f(z) - f(x_0)| \leq \mu \left| \sup_{x \in B_W(x_0, r)} f(x) - f(x_0) \right|$$

for all $z \in B_W(x_0, \mu r)$ and $0 < \mu < 1$.

Proof.

a) We can assume that $x_0 = 0 \in W$, since otherwise we can consider the function $f(x+x_0)$ which is also a convex function. Moreover we can also assume that $f(0) = 0$.

To prove the thesis it is sufficient to show for all $R > r$ we have

$$\inf_{x \in B_W(x_0, R)} f(x) > -\infty.$$

Observe that convexity of f implies that for $\|x\| > r$ we have

$$\begin{aligned} f\left(\frac{rx}{2\|x\|}\right) &= f\left(\left(1 - \frac{r}{2\|x\|}\right) \cdot 0 + \frac{r}{2\|x\|} \cdot x\right) \leq \left(1 - \frac{r}{2\|x\|}\right) f(0) + \frac{r}{2\|x\|} f(x) \\ &= \frac{r}{2\|x\|} f(x) \end{aligned}$$

Hence

$$f(x) \geq \frac{2\|x\|}{r} f\left(\frac{rx}{2\|x\|}\right) \geq -\frac{2R}{r} \left| \inf_{x \in B_W(x_0, r)} f(x) \right| > -\infty.$$

b) Let $0 < \mu < 1$ and $x \in B_W(x_0, r)$, then from convexity of f we have

$$f(\mu x + (1 - \mu)x_0) \leq f(x_0) + \mu(f(x) - f(x_0)).$$

Similarly,

$$(1 + \mu)f(x_0) \leq f(\mu x + (1 - \mu)x_0) + \mu f(2x_0 - x).$$

Combining these two inequalities we obtain

$$\mu(f(x_0) - f(2x_0 - x)) \geq f(x_0 + \mu(x - x_0)) - f(x_0) \geq -\mu(f(x_0) - f(x)).$$

Now since $2x_0 - x \in B_W(x_0, r)$ we have that

$$\mu(f(x_0) - \inf_{x \in B_W(x_0, r)} f(x)) \geq f(x_0) - f(x_0 + \mu(x - x_0)) \geq -\mu(f(x_0) - \inf_{x \in B_W(x_0, r)} f(x)),$$

but this shows that

$$|f(z) - f(x_0)| \leq \mu \left| \inf_{x \in B_W(x_0, r)} f(x) - f(x_0) \right|$$

for all $z \in B_W(x_0, \mu r)$.

c) The proof of this statement analogous to the proof of b) and we omit it. \square

Theorem 3.2. *Let $W \subset X$ be a convex subset of a normed space X . If $f : W \rightarrow \mathbb{R}$ is a convex function and for some $x_0 \in W$ we have*

a) $x_0 - B_W(x_0, r) = B_W(x_0, r) - x_0,$

b) f is bounded from below on $B_W(x_0, r),$

or

a)' $x_0 - B_W(x_0, r) = B_W(x_0, r) - x_0,$

b)' f is bounded from above on $B_W(x_0, r)$

then f is continuous at x_0 .

Proof. Assume that the conditions a) and b) are satisfied (the proof in the case when we assume a)' and b)' is analogous). Let

$$m = \inf_{x \in B_W(x_0, r)} f(x),$$

then by Lemmma 3 b) we get

$$|f(z) - f(x_0)| \leq \mu|m - f(x_0)|$$

for $z \in B_W(x_0, \mu r)$ but this implies continuity of f at x_0 . \square

Corollary 3.3. *If f is a function defined on an open and convex subset of a normed space, which is bounded and convex, then it is continuous.*

Corollary 3.4. *Let W be a convex and open subset of a finite dimensional normed space. If $f : W \rightarrow \mathbb{R}$ is convex, then it is continuous.*

Proof. Let $x_0 \in W$. Since in finite dimensional vector space any two norms are equivalent, we can choose any norm. Thus let $\|\cdot\|_F$ be a norm defined on X such that the unit ball $B = B(0, 1) = \{x \in X : \|x\|_F \leq 1\}$ has only a finite numbers of extremal points. Then obviously every closed ball $B(x_0, r)$ has only a finite number of extremal points. Therefore by convexity of f we have

$$\sup_{x \in B(x_0, r)} f(x) \leq \sup\{f(t_i) : 1 \leq i \leq n\}$$

where t_1, \dots, t_n are extremal points of $B(x_0, r)$. Hence f is bounded above and by Theorem 3.2 it is continuous at x_0 . \square

Remark 3.5. The above theorem is not true when X is infinite dimensional since in any infinite dimensional normed space there exist linear and not continuous functionals.

Remark 3.6. A function f is *concave* or *quasiconcave* if the function $-f$ is convex or quasiconvex respectively. It is clear that all these properties can be expressed in notion of concave functions and *upper semicontinuity*.

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