Novi Sad J. Math. Vol. 55, No. 2, 2025, 1–12
https://doi.org/10.30755/NSJOM.13897

On weakly $(2, J)$-ideals of commutative rings

Adam Anebri[1,2],
Najib Mahdou[3] **and Youssef Zahir**[4]

Abstract. Let R be a commutative ring with nonzero identity. In this paper, we introduce the notion of weakly $(2, J)$-ideals as a generalization of $(2, J)$-ideals. A proper ideal I of R is called a weakly $(2, J)$-ideal if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$. Besides giving various examples and properties of weakly $(2, J)$-ideals, we investigate the relations between weakly $(2, J)$-ideals and other classical ideals such as $(2, J)$-ideals, weakly J-ideals, weakly $(2, n)$-ideals and weakly 2-absorbing primary ideals. Finally, we characterize weakly $(2, J)$-ideals of the trivial ring extensions and amalgamation of a ring along an ideal to construct non-trivial and original examples of weakly $(2, J)$-ideals.

AMS Mathematics Subject Classification (2010): 13A15; 13C13

Key words and phrases: weakly $(2, J)$-ideal; $(2, J)$-ideal; weakly J-ideal; weakly $(2, n)$-ideal; weakly 2-absorbing primary ideal

1. Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. For any ring R, by $\sqrt{0_R}$ and $Jac(R)$, we denote the nilradical and the Jacobson radical of R, respectively. In 2017, Tekir et al., [17] introduced the concept of n-ideals. A proper ideal I of R is said to be an n-ideal if whenever $ab \in I$ and $a \notin \sqrt{0_R}$, then $b \in I$. In [16], Tamekkante and Bouba introduced a generalization of the class of n-ideals called $(2, n)$-ideals. A proper ideal I of R is called a $(2, n)$-ideal if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{0_R}$ or $bc \in \sqrt{0_R}$. They proved that an ideal I of R is $(2, n)$-ideal if and only if I is a 2-absorbing primary ideal and $I \subseteq \sqrt{0_R}$ (see [16, Theorem 2.4]). In [5], Anebri et al. generalized the concept of $(2, n)$-ideals to ϕ-(n, N)-ideals. Let $\mathfrak{I}(R)$ be the set of

[1]Laboratory of Education, Sciences and Technics-LEST, Higher School of Education and Training Berrechid (ESEFB), Hassan First University, Avenue de l'Université , B.P:218, Berrechid 26100, Morocco,
e-mail: adam.anebri@uhp.ac.ma, ORCID iD: orcid.org/0000-0001-5958-8548
[2]Corresponding author
[3]Laboratory of Modelling and Mathematical Structures, Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco,
e-mail: mahdou@hotmail.com, ORCID iD: orcid.org/0000-0001-6353-1114
[4]Laboratory: Mathematics, Computing and Applications – Information Security (LabMiA-SI), Department of Mathematics, Faculty of Sciences, Mohammed V University in Rabat, Morocco,
e-mail: y.zahir@um5r.ac.ma, ORCID iD: orcid.org/0000-0002-5121-8822

all ideals of R and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a function. A proper ideal I of R is called a ϕ-(n, N)-ideal if $x_1 \cdots x_{n+1} \in I \setminus \phi(I)$ for some elements $x_1, \ldots, x_{n+1} \in R$, then $x_1 \cdots x_n \in I$ or the product of x_{n+1} with $(n-1)$ of $x_1, \ldots, x_n$ is in $\sqrt{0_R}$. So, a proper ideal I of R is weakly $(2, n)$-ideal if and only if I is ϕ_0-$(2, N)$-ideal, where $\phi_0(I) = 0$. In [14], Khashan and Bani-Ata introduced the notion of J-ideals as a generalization of n-ideals in commutative rings, as follows: a proper ideal I of a ring R is said to be a J-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin Jac(R)$, then $b \in I$. In [18], Yıldız et al. introduced $(2, J)$-ideals in a commutative ring. A proper ideal I of R is called a $(2, J)$-ideal if $abc \in I$ for some $a, b, c \in R$, then $ab \in I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$. Moreover, Khashan and Celikel defined weakly J-ideals [15]. A proper ideal I of R is called a weakly J-ideal if whenever $a, b \in R$ with $0 \neq ab \in I$ and $a \notin Jac(R)$, then $b \in I$.

Some of our results use the $R(+)M$ construction. Let R be a ring and M be an R-module. Then $R(+)M$, the *trivial (ring) extension of R by M*, is the ring whose additive structure is that of the external direct sum $R \oplus M$ and whose multiplication is defined by $(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + r_2m_1)$ for all $r_1, r_2 \in R$ and all $m_1, m_2 \in M$. The basic properties of trivial ring extensions are summarized in the books [12, 13]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (see for instance [1, 2, 3, 4, 10]).

In this paper, our aim is to introduce a generalization of the concept of $(2, J)$-ideals in commutative rings with a nonzero identity. For this, firstly with Definition 2.1, we introduce the notion of weakly $(2, J)$-ideal of R as follows: a proper ideal I of R is called a weakly $(2, J)$-ideal if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$. Moreover, we observe the relations between this new class of ideals and others that already exist. Namely, the weakly J-ideals, the weakly $(2, n)$-ideals and the weakly 2-absorbing primary ideals (see Remark 2.2, Example 2.3, Example 2.4 and Example 2.5). Furthermore, we prove that if I_1 or I_2 is a weakly J-ideal of R, then I_1I_2 and $I_1 \cap I_2$ are weakly $(2, J)$-ideals (see Corollary 2.8). Also, we give a new characterization of local rings in terms of weakly $(2, J)$-ideals. The third section deals with the transfer of weakly $(2, J)$-ideals to the pre-mentioned ring extensions in particular cases. At this point, we construct non-trivial examples of weakly $(2, J)$-ideals.

2. Main results

We shall begin with the following definition.

Definition 2.1. Let R be a ring. A proper ideal I of R is said to be a weakly $(2, J)$-ideal if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$.

We next collect some immediate classes of weakly $(2, J)$-ideals.

Remark 2.2. Let R be a ring. Then the following statements hold:

(1) Every $(2, J)$-ideal is a weakly $(2, J)$-ideal.

(2) Every weakly J-ideal is a weakly $(2, J)$-ideal.

(3) Every weakly $(2, n)$-ideal is a weakly $(2, J)$-ideal.

(4) The intersection of any family of weakly $(2, J)$-ideals of R is a weakly $(2, J)$-ideal.

The following examples show that the converses of Remark 2.2 (1), (2) and (3) are not true in general.

Example 2.3. Consider the ideal $I = (\bar{0})$ of the ring $R = \mathbb{Z}_{30}$. It is clear that I is a weakly $(2, J)$-ideal of R. However, I is not a $(2, J)$-ideal. Indeed, $\bar{2} \cdot \bar{3} \cdot \bar{5} \in I$, but $\bar{2} \cdot \bar{3} \notin I$ and $\bar{2} \cdot \bar{5} \notin Jac(R)$ and $\bar{3} \cdot \bar{5} \notin Jac(R)$.

Example 2.4. Let $R = \mathbb{Z}(+)\mathbb{Z}$ and $I = 0(+)4\mathbb{Z}$. One can see that I is a weakly $(2, J)$-ideal. But I is not a weakly J-ideal since $(0, 0) \neq (2, 2)(0, 2) \in I$ and neither $(2, 2) \in Jac(R)$ nor $(0, 2) \in I$.

Example 2.5. Let $R = k[[X]]$ be the ring of formal power series, where k is a field, and let $I = (X^3)$. The fact that $Jac(R) = (X)$ is the unique maximal ideal of R gives that I is a weakly $(2, J)$-ideal of R. On the other hand, $I \nsubseteq \sqrt{0_R}$, which implies that I is not a weakly $(2, n)$-ideal by [5, Proposition 2.3].

According to [6], a proper ideal I of R is called a weakly 2-absorbing primary ideal if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Proposition 2.6. *Let R be a ring and I be a proper ideal of R. The following statements hold:*

(1) *If I is a weakly $(2, J)$-ideal of R, then $I \subseteq Jac(R)$.*

(2) *Assume that I is a weakly 2-absorbing primary ideal. Then I is weakly $(2, J)$-ideal if and only if $I \subseteq Jac(R)$.*

Proof. (1) Note that the zero ideal is always a weakly $(2, J)$-ideal. We assume that $I \neq (0)$, so we can pick a nonzero element $x \in I$. Since $0 \neq 1 \cdot 1 \cdot x \in I$, then $x \in Jac(R)$ and hence $I \subseteq Jac(R)$.
(2) Follows from (1). □

Theorem 2.7. *Let R be a ring and $I \subseteq K$ be proper ideals of R. If K is a weakly J-ideal of R, then I is a weakly $(2, J)$-ideal.*

Proof. Assume that $0 \neq abc \in I$ for some $a, b, c \in R$ and $ab \notin I$. If $a \in Jac(R)$, we are done. Suppose that $a \notin Jac(R)$. Since K is a weakly J-ideal, we have $bc \in K$ and thus $bc \in Jac(R)$, as desired. □

Corollary 2.8. *Let R be a ring and I_1, I_2 be two proper ideals of R. If I_1 or I_2 is a weakly J-ideal of R, then $I_1 I_2$ and $I_1 \cap I_2$ are weakly $(2, J)$-ideals.*

Theorem 2.9. *Let R be a ring. Then the following assertions are equivalent:*

(1) *R is a local ring.*

(2) *Every proper principal ideal of R is a weakly $(2, J)$-ideal.*

(3) *Every proper ideal of R is a weakly $(2, J)$-ideal.*

Proof. (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) Let I be a proper ideal of R and let $a, b, c \in R$ such that $0 \neq abc \in I$. Since $0 \neq abc \in (abc)$ and (abc) is a weakly $(2, J)$-ideal, we have that either $ab \in (abc) \subseteq I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$.
(3) $\Rightarrow$ (1) If M is a maximal ideal of R, then M is a weakly $(2, J)$-ideal and thus $M \subseteq Jac(R)$, as needed. $\square$

Definition 2.10. Let I be a weakly $(2, J)$-ideal of a ring R. We say that (a, b, c) is a J-triple-zero of I if $abc = 0$, $ab \notin I$, $ac \notin Jac(R)$ and $bc \notin Jac(R)$.

Theorem 2.11. *Let I be a weakly $(2, J)$-ideal that is not a $(2, J)$-ideal, then $I^3 = 0$.*

We need the following lemma before proving Theorem 2.11.

Lemma 2.12. *Let I be a weakly $(2, J)$-ideal of R. If (a, b, c) is a J-triple-zero of I, then:*

(1) $abI = acI = bcI = 0$.

(2) $aI^2 = bI^2 = cI^2 = 0$.

Proof. (1) Assume that $abI \neq 0$. Then $abx \neq 0$ for some element $x \in I$. Since I is a weakly $(2, J)$-ideal of R and $0 \neq ab(c + x) \in I$, we then have $a(c + x) \in Jac(R)$ or $b(c + x) \in Jac(R)$. This implies that $ac \in Jac(R)$ or $bc \in Jac(R)$, a contradiction. Thus $abI = 0$. Similarly, one can prove that $acI = bcI = 0$.
(2) Suppose that $aI^2 \neq 0$. So, there exist $x, y \in I$ such that $axy \neq 0$. From (1), we have $0 \neq a(b + x)(c + y) \in I$, which gives that either $a(b + x) \in I$ or $a(c + y) \in Jac(R)$ or $(b + x)(c + y) \in Jac(R)$. Hence $ab \in I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$, a contradiction. Consequently, $aI^2 = 0$. Similarly it can be easily verified that $bI^2 = cI^2 = 0$. $\square$

Proof of Theorem. Suppose that I is a weakly $(2, J)$-ideal that is not a $(2, J)$-ideal of R. So there exists a J-triple-zero (a, b, c) of I. If $I^3 \neq 0$, then $xyz \neq 0$ for some $x, y, z \in I$. By Lemma 2.12, $0 \neq (a + x)(b + y)(c + z) = xyz \in I$. This yields that $(a+x)(b+y) \in I$ or $(a+x)(c+z) \in Jac(R)$ or $(b+y)(c+z) \in Jac(R)$. Hence $ab \in I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$, the desired contradiction. Thus $I^3 = 0$.

Corollary 2.13. *Let R be a reduced ring and I be a nonzero proper ideal of R. Then I is a weakly $(2, J)$-ideal if and only if I is a $(2, J)$-ideal.*

Proof. It suffices to prove the "if" assertion. This, in turn, follows from the fact that $I \subseteq \sqrt{0}$ for every weakly $(2, J)$-ideal I of R that is not a $(2, J)$-ideal. This completes the proof. $\square$

The following example shows that a proper ideal I of R with the property $I^3 = 0$ need not be a weakly $(2, J)$-ideal of R.

Example 2.14. Let $R = \mathbb{Z}_{90}$ and $I = (\overline{30})$. It is clear that $I^3 = (\bar{0})$. Since $\bar{0} \neq \bar{2} \cdot \bar{3} \cdot \bar{5} \in I$, $\bar{2} \cdot \bar{3} \notin I$, $\bar{2} \cdot \bar{5} \notin Jac(R)$ and $\bar{3} \cdot \bar{5} \notin Jac(R)$. Hence I is not a weakly $(2, J)$-ideal of R.

Proposition 2.15. *Let R be a ring such that $\sqrt{0_R}$ is a J-ideal and let I be a proper ideal of R. Then I is a weakly $(2, J)$-ideal of R if and only if I is a $(2, J)$-ideal of R.*

Proof. Assume that I is a weakly $(2, J)$-ideal of R. Let $a, b, c \in R$ such that $abc \in I$. If $0 \neq abc$, we are done. We may assume that $abc = 0$. As $\sqrt{0_R}$ is a J-ideal of R, we have that either $a \in Jac(R)$ or $bc \in \sqrt{0_R}$ and thus $ac \in Jac(R)$ or $bc \in Jac(R)$. It follows that I is a $(2, J)$-ideal of R. The converse is obvious. □

Definition 2.16. Let I be a proper ideal of a ring R, and let I_1, I_2 and I_3 be ideals of R such that $0 \neq I_1I_2I_3 \subseteq I$. We say that I is free J-triple-zero with respect $I_1I_2I_3$ if (a, b, c) is not a J-triple-zero of I for every $a \in I_1$, $b \in I_2$ and $c \in I_3$.

Theorem 2.17. *Let R be a ring and I be a weakly $(2, J)$-ideal of R and suppose that $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R such that I is free J-triple-zero with respect to $I_1I_2I_3$. Then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq Jac(R)$ or $I_2I_3 \subseteq Jac(R)$.*

To prove this theorem, we need the following lemma.

Lemma 2.18. *Let I be a weakly $(2, J)$-ideal of a ring R, K be a proper ideal of R and let $a, b \in R$ such that $abK \subseteq I$ and (a, b, c) is not a J-triple-zero of I for every $c \in K$. Then either $ab \in I$ or $aK \subseteq Jac(R)$ or $bK \subseteq Jac(R)$.*

Proof. Assume that $ab \notin I$, $aK \nsubseteq Jac(R)$ and $bK \nsubseteq Jac(R)$. So, $ak_1 \notin Jac(R)$ and $bk_2 \notin Jac(R)$ for some $k_1, k_2 \in K$. As I is a weakly $(2, J)$-ideal and (a, b, k_1) is not a J-triple-zero of I, we have $bk_1 \in Jac(R)$. Similarly, we obtain $ak_2 \in Jac(R)$. Now, since $ab(k_1 + k_2) \in I$ and $(a, b, k_1 + k_2)$ is not a J-triple-zero of I, we conclude that either $a(k_1 + k_2) \in Jac(R)$ or $b(k_1 + k_2) \in Jac(R)$, and so either $ak_1 \in Jac(R)$ or $bk_2 \in Jac(R)$, a desired contradiction. □

Proof of Theorem. Suppose that $I_1I_2 \nsubseteq I$, $I_1I_3 \nsubseteq Jac(R)$ and $I_2I_3 \nsubseteq Jac(R)$. Take $j_1 \in I_1$ and $j_2 \in I_2$ such that $j_1I_3 \nsubseteq Jac(R)$ and $j_2I_3 \nsubseteq Jac(R)$. By Lemma 2.18, we have $j_1j_2 \in I$. On the other hand, the fact that $I_1I_2 \nsubseteq I$ implies that there are $a \in I_1$ and $b \in I_2$ such that $ab \notin I$. Also, by Lemma 2.18, we have that either $aI_3 \subseteq Jac(R)$ or $bI_3 \subseteq Jac(R)$.

Case 1: Suppose that $aI_3 \subseteq Jac(R)$ and $bI_3 \nsubseteq Jac(R)$. Since $j_1bI_3 \subseteq I$ and $j_1I_3 \nsubseteq Jac(R)$ and $bI_3 \nsubseteq Jac(R)$, we obtain that $j_1b \in I$. Moreover, by Lemma 2.18, we conclude that $(a + j_1)b \in I$ and hence $ab \in I$, a contradiction.

Case 2: If $aI_3 \nsubseteq Jac(R)$ and $bI_3 \subseteq Jac(R)$, then we have a contradiction by a similar argument.

Case 3: Assume that $aI_3 \subseteq Jac(R)$ and $bI_3 \subseteq Jac(R)$. Since $j_1(b + j_2)I_3 \subseteq I$

and neither $j_1I_3 \subseteq Jac(R)$ nor $(b+j_2)I_3 \subseteq Jac(R)$, we conclude that $j_1(b+j_2) \in I$ and hence $j_1b \in I$. Similarly, we can see that $aj_2 \in I$. Moreover, since $(a+j_1)(b+j_2)I_3 \subseteq I$ and neither $(a+j_1)I_3 \subseteq Jac(R)$ nor $(b+j_2)I_3 \subseteq Jac(R)$, we get $(a+j_1)(b+j_2) \in I$ and thus $ab \in I$, which gives a contradiction. It follows that either $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq Jac(R)$ or $I_2I_3 \subseteq Jac(R)$.

3. Extension of weakly $(2, J)$-ideals

Proposition 3.1. *Let I and K be two proper ideals of a ring R with $K \subseteq I$. Then the following statements hold:*

(1) *If I is a weakly $(2, J)$-ideal of R, then I/K is a weakly $(2, J)$-ideal of R/K.*

(2) *If I/K is a weakly $(2, J)$-ideal of R/K and K is a weakly $(2, J)$-ideal of R, then I is a weakly $(2, J)$-ideal of R.*

Proof. (1) Assume that $K \neq (a+K)(b+K)(c+K) \in I/K$ for some $(a+K), (b+K), (c+K) \in R/K$. So, $0 \neq abc \in I$. As I is a weakly $(2, J)$-ideal, we then have $ab \in I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$. This yields that $ab+K \in I/K$ or $ac + K \in Jac(R)/K \subseteq Jac(R/K)$ or $bc + K \in Jac(R/K)$. Thus I/K is a weakly $(2, J)$-ideal.
(2) Let $a, b, c \in R$ such that $0 \neq abc \in I$. If $abc \in K$, then by hypothesis $ab \in K \subseteq I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$. If $abc \notin K$, we must have $K \neq (a+K)(b+K)(c+K) \in I/K$. By hypothesis, we obtain that $ab+K \in I/K$ or $ac+K \in Jac(R/K)$ or $bc+K \in Jac(R/K)$. Since $K \subseteq Jac(R)$, then $ab \in I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$, as required. □

Theorem 3.2. *Let R be a Noetherian ring such that (0) is a $(2, J)$-ideal. Then I is a $(2, J)$-ideal of R if and only if $I \subseteq Jac(R)$ and I/I^n is a weakly $(2, J)$-ideal of R/I^n for every positive integer n.*

Proof. It suffices to prove the converse. Suppose that $abc \in I$ for some $a, b, c \in R$. If $abc \notin I^n$ for some $n \geq 2$, then $I^n \neq (a+I^n)(b+I^n)(c+I^n) \in I/I^n$. As I/I^n is a weakly $(2, J)$-ideal of R/I^n, we conclude that $(a+I^n)(b+I^n) \in I/I^n$ or $(a+I^n)(c+I^n) \in Jac(R/I^n)$ or $(b+I^n)(c+I^n) \in Jac(R/I^n)$. Which implies that $ab \in I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$ since $I^n \subseteq Jac(R)$. Now, we assume that $abc \in I^n$ for all n. Since R is Noetherian and $I \subseteq Jac(R)$, the Krull intersection theorem implies that $abc \in \cap_{n \geq 1} I^n = 0$. Therefore, $ab = 0$ or $ac \in Jac(R)$ or $bc \in Jac(R)$. This completes the proof. □

Proposition 3.3. *Let R be a ring, I and J be two weakly $(2, J)$-ideals of R. Then $I + J$ is a weakly $(2, J)$-ideal of R.*

Proof. Since $I \subseteq Jac(R)$ and J is a proper ideal, we then have $I+J$ is a proper ideal of R. On the other hand, $I \cap J$ and $I/(I \cap J)$ are weakly $(2, J)$-ideals of R and $R/(I \cap J)$, respectively. From the isomorphism $(I+J)/J \cong I/(I \cap J)$ and Proposition 3.1(2), we conclude that $I + J$ is a weakly $(2, J)$-ideal, as needed. □

Proposition 3.4. *Let $R = R_1 \times R_2$ be a decomposable ring. Let I_1 be a nonzero proper ideal of R_1 and I_2 be a nonzero ideal of R_2. The following assertions are equivalent:*

(1) *$I_1 \times I_2$ is a weakly $(2, J)$-ideal of R.*

(2) *I_1 is a J-ideal of R_1 and I_2 is a J-ideal of R_2.*

(3) *$I_1 \times I_2$ is a $(2, J)$-ideal of R.*

Proof. (1) $\Rightarrow$ (2) Since $I_1 \times I_2$ is a weakly $(2, J)$-ideal, then I_1 and I_2 are proper ideals of R_1 and R_2, respectively. Now, we will show that I_2 is a J-ideal of R_2. Suppose that $ab \in I_2$ for some $a, b \in R_2$ and take $0 \neq i \in I_1$. So, $(0, 0) \neq (i, 1)(1, a)(1, b) \in I_1 \times I_2$. By assymption, we have $(i, 1)(1, a) \in I_1 \times I_2$ or $(i, 1)(1, b) \in Jac(R_1 \times R_2)$, which gives $a \in I_2$ or $b \in Jac(R_2)$. Similarly, we prove that I_1 is a J-ideal of R_1.
(2) $\Rightarrow$ (3) By [18, Theorem 3].
(3) $\Rightarrow$ (1) Clear. □

Proposition 3.5. *Let R_1, R_2, R_3 be rings. If I is a weakly $(2, J)$-ideal of $R = R_1 \times R_2 \times R_3$, then $I = (0, 0, 0)$.*

Proof. Assume that $I = I_1 \times I_2 \times I_3 \neq (0, 0, 0)$. Then there is a nonzero element $(a, b, c) \in I$. Hence $(0, 0, 0) \neq (a, 1, 1)(1, b, 1)(1, 1, c) \in I$, and so $(a, 1, 1)(1, b, 1) \in I$. This implies that $I_3 = R_3$, which contradicts the fact that $I \subseteq Jac(R)$. Thus $I = (0, 0, 0)$. □

Remark 3.6. By Proposition 3.5, we can see that the zero ideal of a decomposable ring $R = R_1 \times R_2 \times R_3$ is a weakly $(2, J)$-ideal that is not a $(2, J)$-ideal.

Proposition 3.7. *Let R be a ring and S be a multiplicatively closed subset of R. If I is a weakly $(2, J)$-ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a weakly $(2, J)$-ideal of $S^{-1}R$.*

Proof. Assume that $0 \neq \frac{a}{s}\frac{b}{t}\frac{c}{u} \in S^{-1}I$ for some $\frac{a}{s}, \frac{b}{t}, \frac{c}{u} \in S^{-1}R$. So, there exists $v \in S$ such that $0 \neq vabc \in I$. As I is a weakly $(2, J)$-ideal, we get $vab \in I$ or $vac \in Jac(R)$ or $bc \in Jac(R)$. Thus $\frac{a}{s}\frac{b}{t} \in S^{-1}I$ or $\frac{a}{s}\frac{c}{u} \in S^{-1}Jac(R) \subseteq Jac(S^{-1}R)$ or $\frac{b}{t}\frac{c}{u} \in Jac(S^{-1}R)$, which implies that $S^{-1}I$ is a weakly $(2, J)$-ideal of $S^{-1}R$. □

Theorem 3.8. *Let R be a ring, M be an R-module and I be a proper ideal of R. Then:*

(1) *$I(+)M$ is a $(2, J)$-ideal of $R(+)M$ if and only if I is a $(2, J)$-ideal of R.*

(2) *$I(+)M$ is a weakly $(2, J)$-ideal of $R(+)M$ if and only if I is a weakly $(2, J)$-ideal of R and for every J-triple-zero (a, b, c) of I, we have $ab \in Ann(M), ac \in Ann(M)$ and $bc \in Ann(M)$.*

Proof. (1) Follows from [18, Proposition 6] because $I \cong \frac{I(+)M}{0(+)M}$.
(2) Suppose that $I(+)M$ is a weakly $(2, J)$-ideal of $R(+)M$ and let $a, b, c \in R$ such that $0 \neq abc \in I$. So, $(0, 0) \neq (a, 0)(b, 0)(c, 0) \in I(+)M$. By the hypothesis, we have $(a, 0)(b, 0) \in I(+)M$ or $(a, 0)(c, 0) \in Jac(R(+)M)$ or $(b, 0)(c, 0) \in Jac(R(+)M)$, which implies that $ab \in I$ or $ac \in Jac(R)$ or $bc \in Jac(R)$ since $Jac(R(+)M) = Jac(R)(+)M$. Thus, I is a weakly $(2, J)$-ideal. On the other hand, take a J-triple-zero (a, b, c) of I. We first prove that $ab \in Ann(M)$. Suppose not. Then there is $m \in M$ such that $abm \neq 0$. Moreover, the fact that $(0, 0) \neq (a, 0)(b, 0)(c, m) = (0, abm) \in I(+)M$ implies that $(a, 0)(b, 0) \in I(+)M$ or $(a, 0)(c, m) \in Jac(R(+)M)$ or $(b, 0)(c, m) \in Jac(R(+)M)$, a contradiction. Hence $ab \in Ann(M)$. Similarly, one can verify that $ac, bc \in Ann(M)$. For the converse, suppose that $(0, 0) \neq (a, m_1)(b, m_2)(c, m_3) \in I(+)M$ and $(a, m_1)(c, m_3) \notin Jac(R(+)M)$ and $(b, m_2)(c, m_3) \notin Jac(R(+)M)$. Then, two cases are possible:
Case 1: If $0 \neq abc$, then $ab \in I$ and hence $(a, m_1)(b, m_2) \in I(+)M$.
Case 2: If $0 = abc$. Suppose that $ab \notin I$, then (a, b, c) is a J-triple-zero of I. This implies that $(a, m_1)(b, m_2)(c, m_3) = (0, 0)$, a contradiction. So, $ab \in I$ and thus $(a, m_1)(b, m_2) \in I(+)M$. This completes the proof. □

Remark 3.9. Let R be a ring, M be an R-module and N be an R-submodule of M. In general, an ideal of the form $I(+)N$ need not be a weakly $(2, J)$-ideal of $R(+)M$, where I is a weakly $(2, J)$-ideal of R. Indeed, we take $R = \mathbb{Z}_{30}$, $I = (\bar{0})$ and $N = (\overline{15})$. So, I is clearly a weakly $(2, J)$-ideal of R. However $I(+)N$ is not a weakly $(2, J)$-ideal of $R(+)R$ since $(\bar{0}, \bar{0}) \neq (\bar{2}, \bar{1})(\bar{3}, \bar{0})(\bar{5}, \bar{0}) \in I(+)N$, $(\bar{2}, \bar{1})(\bar{3}, \bar{0}) \notin I(+)N$, $(\bar{2}, \bar{1})(\bar{5}, \bar{0}) \notin Jac(R(+)R)$ and $(\bar{3}, \bar{0})(\bar{5}, \bar{0}) \notin Jac(R(+)R)$.

Let R and S be two rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. In this setting, we can consider the following subring of $R \times S$:

$$R \bowtie^f J = \{(r, f(r) + j) \mid r \in R \text{ and } j \in J\}$$

called the amalgamation of R with S along J with respect to f. This construction has been first indroduced and studied D'Anna, Finocchiaro, and Fontana in [7, 8]. In particular, if I is an ideal of R and $id_R : R \to R$ is the identity homomorphism on R, then $R \bowtie J = R \bowtie^{id_R} J = \{(r, r + j) \mid r \in R \text{ and } j \in J\}$ is the amalgamated duplication of R along J (introduced and studied by D'Anna and Fontana in [9]). See for instance [11].
For all ideals I of R and ideals K of S, set:

$$I \bowtie^f J = \{(r, f(r) + j) \mid r \in I \text{ and } j \in J\},$$

$$\overline{K}^f = \{(r, f(r) + j) \mid r \in R,\ j \in J \text{ and } f(r) + j \in K\}.$$

Lemma 3.10. *[8, Proposition 2.6] Let R and S be a pair of rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. Then, the set of maximal ideals of $R \bowtie^f J$ is $Max(R \bowtie^f J) = \{M \bowtie^f J \mid M \in Max(R)\} \cup \{\overline{Q}^f \mid Q \in Max(S) \setminus V(J)\}$, where $V(J)$ denotes the set of all prime ideals containing J.*

Theorem 3.11. *Let R and S be a pair of rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. Let I be a proper ideal of R. Then the following statements hold:*

(1) *If $I \bowtie^f J$ is a $(2, J)$-ideal of $R \bowtie^f J$, then I is a $(2, J)$-ideal of R. Moreover, the converse is true if $J \subseteq Jac(S)$.*

(2) *If $I \bowtie^f J$ is a weakly $(2, J)$-ideal of $R \bowtie^f J$, then I is a weakly $(2, J)$-ideal of R and for a J-triple-zero (a, b, c) of I, we have $(f(a) + i)(f(b) + j)(f(c) + k) = 0$ for every $i, j, k \in J$. The converse is true if $J \subseteq Jac(S)$.*

Proof. (1) It suffices to apply [18, Proposition 6] since $I \cong \frac{I \bowtie^f J}{0 \bowtie^f J}$.
(2) By a similar argument, we can see that if $I \bowtie^f J$ is a weakly $(2, J)$-ideal of $R \bowtie^f J$, then I is a weakly $(2, J)$-ideal of R. Now, let $a, b, c \in R$ such that (a, b, c) is a J-triple-zero of I and let $i, j, k \in J$. Suppose that $(f(a) + i)(f(b) + j)(f(c) + k) \neq 0$. So, $(0, 0) \neq (a, f(a) + i)(b, f(b) + j)(c, f(c) + k) \in I \bowtie^f J$ and hence $(a, f(a) + i)(b, f(b) + j) \in I \bowtie^f J$ or $(a, f(a) + i)(c, f(c) + k) \in Jac(R \bowtie^f J)$ or $(b, f(b)+j)(c, f(c)+k) \in Jac(R \bowtie^f J)$, a desired contradiction. Now, we will prove the converse under additional hypothesis that $J \subseteq Jac(S)$. Assume that $(0, 0) \neq (a, f(a) + i)(b, f(b) + j)(c, f(c) + k) \in I \bowtie^f J$ for some $(a, f(a)+i), (b, f(b)+j), (c, f(c)+k) \in R \bowtie^f J$ with $(a, f(a)+i)(c, f(c)+k) \notin Jac(R \bowtie^f J)$ and $(b, f(b)+i)(c, f(c)+k) \notin Jac(R \bowtie^f J)$. So, by hypothesis, we have $abc \in I$ and $ac \notin Jac(R)$ and $bc \notin Jac(R)$. Then two cases are possible:
Case 1: If $0 \neq abc$ then $ab \in I$ and thus $(a, f(a) + i)(b, f(b) + j) \in I \bowtie^f J$.
Case 2: Assume that $0 = abc$. If $ab \notin I$, then (a, b, c) is a J-triple-zero of I, which implies that $(a, f(a)+i)(b, f(b)+j)(c, f(c)+k) = (0, 0)$ a contradiction. It follows that $ab \in I$ and thus $I \bowtie^f J$ is a weakly $(2, J)$-ideal, as required. □

Corollary 3.12. *Let R be a ring and let I and J be proper ideals of R.*

(1) *If $I \bowtie J$ is a $(2, J)$-ideal of $R \bowtie J$, then I is a $(2, J)$-ideal of R. The converse is true if $J \subseteq Jac(R)$.*

(2) *If $I \bowtie J$ is a weakly $(2, J)$-ideal of $R \bowtie J$, then I is a $(2, J)$-ideal of R and for a J-triple-zero (a, b, c) of I, we have $(a + i)(b + j)(c + k) = 0$ for every $i, j, k \in J$. The converse is true if $J \subseteq Jac(R)$.*

In the following example, we prove that the condition $J \subseteq Jac(S)$ can not be discarded in the proof of the converse of Theorem 3.11.

Example 3.13. Let $R := \mathbb{Z}(+)\mathbb{Z}_4$ and $J := 2\mathbb{Z}(+)\mathbb{Z}_4$. It is clear that $I := 0(+)\mathbb{Z}_4$ is a $(2, J)$-ideal (and so is a weakly $(2, J)$-ideal) of R. However, $I \bowtie J$ is not a weakly $(2, J)$-ideal (and so is not a $(2, J)$-ideal) of $R \bowtie J$ since $I \bowtie J \nsubseteq Jac(R \bowtie J)$ (because $((0, \bar{1}), (2, \bar{1})) \in (I \bowtie J) \setminus Jac(R \bowtie J)$).

Theorem 3.14. *Let $R \bowtie^f J$ be the amalgamation of R and S along J with respect to f, where f is an epimorphism. Let K be an ideal of S. Then:*

(1) *If $\overline{K}^f$ is a $(2, J)$-ideal of $R \bowtie^f J$, then K is a $(2, J)$-ideal of S. The converse holds provided that $J \subseteq Jac(S)$ and $f^{-1}(Jac(S)) \subseteq Jac(R)$.*

(2) *If $\overline{K}^f$ is a weakly $(2, J)$-ideal of $R \bowtie^f J$, then K is a weakly $(2, J)$-ideal of S and when $(f(a) + i), (f(b) + j), (f(c) + k))$ is a J-triple-zero of K with $a, b, c \in R$ and $i, j, k \in J$, then $abc = 0$. The converse is true if $J \subseteq Jac(S)$ and $f^{-1}(Jac(S)) \subseteq Jac(R)$.*

Proof. (1) Let $x, y, z \in S$, say, $x = f(a), y = f(b), z = f(c)$ for some $a, b, c \in R$. Assume that $xyz \in K$ and $xz \notin Jac(S)$ and $yz \notin Jac(S)$. So, $(a, f(a))(b, f(b))(c, f(c)) \in \overline{K}^f$. By hypothesis, we get $(a, f(a))(b, f(b)) \in \overline{K}^f$ or $(a, f(a))(c, f(c)) \in Jac(R \bowtie^f J)$ or $(b, f(b))(c, f(c)) \in Jac(R \bowtie^f J)$. Suppose that $(a, f(a))(c, f(c)) \in Jac(R \bowtie^f J)$, then $ac \in Jac(R)$. Since f is an epimorphism, we then have $xz = f(a)f(c) \in Jac(S)$, a contradiction. Hence, $(a, f(a))(c, f(c)) \notin Jac(R \bowtie^f J)$. Similarly, we obtain that $(b, f(b))(c, f(c)) \notin Jac(R \bowtie^f J)$. Moreover, $(a, f(a))(b, f(b)) \in \overline{K}^f$ and so $xy \in K$. Thus K is a $(2, J)$-ideal of S. For the converse, suppose that $J \subseteq Jac(S)$ and $f^{-1}(Jac(S)) \subseteq Jac(R)$. Take $(a, f(a) + i), (b, f(b) + j), (c, f(c) + k) \in R \bowtie^f J$ such that $(a, f(a)+i)(b, f(b)+j)(c, f(c)+k) \in \overline{K}^f$. So, $(f(a)+i)(f(b)+j)(f(c)+k) \in K$. As K is a $(2, J)$-ideal, we have $(f(a)+i)(f(b)+j) \in K$ or $(f(a)+i)(f(c)+k) \in Jac(S)$ or $(f(b) + j)(f(c) + k) \in Jac(S)$. If $(f(a) + i)(f(b) + j) \in K$ then $(a, f(a) + i)(b, f(b) + j) \in \overline{K}^f$. Now, we assume that $(f(a) + i)(f(c) + k) \in Jac(S)$. So, by hypothesis, $f(ac) \in Jac(S)$ and hence $ac \in Jac(R)$, which implies that $(a, f(a)+i)(c, f(c)+k) \in Jac(R \bowtie^f J)$. Thus $\overline{K}^f$ is a $(2, J)$-ideal of $R \bowtie^f J$.

(2) If $\overline{K}^f$ is a weakly $(2, J)$-ideal of $R \bowtie^f J$, then K is a weakly $(2, J)$-ideal of S via similar arguments as in the proof of (1). Moreover, let $a, b, c \in R$ and $i, j, k \in J$ such that $(f(a) + i), (f(b) + j), (f(c) + k))$ is a J-triple-zero of K. If $abc \neq 0$ then $(0, 0) \neq (a, f(a) + i)(b, f(b) + j)(c, f(c) + k) \in \overline{K}^f$. Since $\overline{K}^f$ is a weakly $(2, J)$-ideal, $(a, f(a) + i)(b, f(b) + j) \in \overline{K}^f$ or $(a, f(a) + i)(c, f(c) + k) \in Jac(R \bowtie^f J)$ or $(b, f(b) + j)(c, f(c) + k) \in Jac(R \bowtie^f J)$. If $(a, f(a) + i)(b, f(b) + j) \in \overline{K}^f$ then $(f(a) + i)(f(b) + j) \in K$, a contradiction. Assume that $(a, f(a)+i)(c, f(c)+k) \in Jac(R \bowtie^f J)$, so $ac \in Jac(R)$ and hence $f(ac) \in Jac(S)$ because f is an epimorphism. Also, we have $(f(a) + i)(f(c) + k) \in Q$ for all $Q \in Max(S) \setminus V(J)$. Moreover, note that $(f(a)+i)(f(c)+k) \in Q$, where $Q \in Max(S) \cap V(J)$. It follows that $(f(a)+i)(f(c)+k) \in Jac(S)$, a contradiction. Similarly, we prove that if $(b, f(b) + j)(c, f(c) + k) \in Jac(R \bowtie^f J)$ then $(f(b)+j)(f(c)+k) \in Jac(S)$, we also have a contradiction. Thus $abc = 0$. For the converse, suppose that $(0, 0) \neq (a, f(a)+i)(b, f(b)+j)(c, f(c)+k) \in \overline{K}^f$ for some $(a, f(a) + i), (b, f(b) + j), (c, f(c) + k) \in R \bowtie^f J$. Then two cases are possible:

Case 1: If $0 \neq (f(a) + i)(f(b) + j)(f(c) + k)$, then $(f(a) + i)(f(b) + j) \in K$ or $(f(a) + i)(f(c) + k) \in Jac(S)$ or $(f(b) + j)(f(c) + k)$, which gives the result by assumptions.

Case 2: Assume that $0 = (f(a) + i)(f(b) + j)(f(c) + k)$. If $(f(a) + i), (f(b) + j), (f(c) + k))$ is a J-triple-zero of K, then $abc = 0$ and hence $(a, f(a) + i)(b, f(b) + j)(c, f(c) + k) = (0, 0)$, a desired contradiction. Finally, we conclude that $\overline{K}^f$ is a weakly $(2,J)$-ideal. This completes the proof. □

Corollary 3.15. *Let R be a ring and let I and J be ideals of R.*

(1) *If $\overline{I} := \{(a, a+i) \mid a \in R, j \in J$ and $a+j \in I\}$ is a $(2,J)$ of $R \bowtie J$, then I is a $(2,J)$ of R. The converse is true if $J \subseteq Jac(R)$.*

(2) *If $\overline{I}$ is a weakly $(2,J)$ of $R \bowtie J$, then I is a weakly $(2,J)$ of R and when $((a+i),(b+j),(c+k))$ is a J-triple-zero of I with $a,b,c \in R$ and $i,j,k \in J$, then $abc = 0$. The converse holds provided that $J \subseteq Jac(R)$.*

The following example shows that the converse of Theorem 3.14(1) fails if one deletes the hypothesis that $J \subseteq Jac(S)$.

Example 3.16. Let $R = \mathbb{Z}(+)\mathbb{Z}_6$, $S = \mathbb{Z}$ and $J = 2\mathbb{Z}$. Consider the canonical epimorphism $f : R \to S$ defined by $f(r,m) = r$. Observe that $K = (0)$ is a $(2,J)$-ideal of S. However, $\overline{K}^f$ is not a $(2,J)$-ideal of $R \bowtie^f J$ because $((2,\bar{0}),0) \in \overline{K}^f \setminus Jac(R \bowtie^f J)$.

Acknowledgement

The authors would like to express their sincere thanks to the referee, whose comments and corrections greatly improved this paper.

References

[1] Ahmed, M. T., Mahdou, N., and Zahir, Y. Commutative rings with invertible-radical factorization. *J. Algebra Appl. 21*, 8 (2022), Paper No. 2250153, 16.

[2] Anderson, D. D., and Winders, M. Idealization of a module. *J. Commut. Algebra 1*, 1 (2009), 3–56.

[3] Anebri, A., Khalfi, A. E., and Mahdou, N. On $(1,r)$-ideals of commutative rings. *J. Algebra Appl. 23*, 2 (2024), Paper No. 2450023, 12.

[4] Anebri, A., Mahdou, N., and Tekir, U. Commutative rings and modules that are r-Noetherian. *Bull. Korean Math. Soc. 58*, 5 (2021), 1221–1233.

[5] Anebri, A., Mahdou, N., Tekir, U., and Yi ldiz, E. On ϕ-(n,N)-ideals of commutative rings. *Algebra Colloq. 30*, 3 (2023), 481–492.

[6] Badawi, A., Tekir, U., and Yetkin, E. On weakly 2-absorbing primary ideals of commutative rings. *J. Korean Math. Soc. 52*, 1 (2015), 97–111.

[7] D'Anna, M., Finocchiaro, C. A., and Fontana, M. Amalgamated algebras along an ideal. In *Commutative algebra and its applications.* Walter de Gruyter, Berlin, 2009, pp. 155–172.

[8] D'Anna, M., Finocchiaro, C. A., and Fontana, M. Properties of chains of prime ideals in an amalgamated algebra along an ideal. *J. Pure Appl. Algebra 214*, 9 (2010), 1633–1641.

[9] D'Anna, M., and Fontana, M. An amalgamated duplication of a ring along an ideal: the basic properties. *J. Algebra Appl. 6*, 3 (2007), 443–459.

[10] Dumitrescu, T., Mahdou, N., and Zahir, Y. Radical factorization for trivial extensions and amalgamated duplication rings. *J. Algebra Appl. 20*, 2 (2021), Paper No. 2150025, 10.

[11] El Khalfi, A., Kim, H., and Mahdou, N. Amalgamation extension in commutative ring theory: a survey. *Moroc. J. Algebra Geom. Appl. 1*, 1 (2022), 139–182.

[12] Glaz, S. *Commutative coherent rings*, vol. 1371 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.

[13] Huckaba, J. A. *Commutative rings with zero divisors*, vol. 117 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1988.

[14] Khashan, H. A., and Bani-Ata, A. B. J-ideals of commutative rings. *Int. Electron. J. Algebra 29* (2021), 148–164.

[15] Khashan, H. A., and Celikel, E. Y. Weakly J-ideals of commutative rings. *Filomat 36*, 2 (2022), 485–495.

[16] Tamekkante, M., and Bouba, E. M. $(2, n)$-ideals of commutative rings. *J. Algebra Appl. 18*, 6 (2019), 1950103, 12.

[17] Tekir, U., Koc, S., and Oral, K. H. n-ideals of commutative rings. *Filomat 31*, 10 (2017), 2933–2941.

[18] Yi ldiz, E., Tekir, U., and Koç, S. $(2, j)$-ideals in commutative rings. *C. R. Acad. Bulgare Sci. 73*, 9 (2020), 1201–1209.

Received by the editors October 24, 2021
First published online April 5, 2024