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Entire solutions for quadratic trinomial partial differential-difference functional equations in $\mathbb{C}^n$

Raju Biswas[1,2] **and Rajib Mandal**[3]

Abstract. In this paper, we solve four certain quadratic trinomial partial differential-difference equations for finite order entire functions in $\mathbb{C}^n$. Our results significantly improve some earlier results, mainly the results due to Haldar and Ahamed (Entire solutions of several quadratic binomial and trinomial partial differential-difference equations in $\mathbb{C}^2$ [Anal. Math. Physics 12(113) (2022)]). The earlier results were on $\mathbb{C}^2$ whereas our results are on $\mathbb{C}^n$. We exhibit some examples to fortify our results.

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1. Introduction, Definitions and Results

By a meromorphic (resp. entire) function we shall always mean meromorphic (resp. entire) function on $\mathbb{C}$. Value distribution theory of Nevanlinna for meromorphic functions on $\mathbb{C}$ has been extensively applied to resolve growth, value distribution (see, [16, 21, 44]), and solvability of meromorphic solutions of linear and nonlinear differential equations (see, [17, 26, 42, 43]).

A meromorphic mapping $f : \mathbb{C}^n \to \mathbb{P}^1$ into the Riemann sphere $\mathbb{P}^1$ with $f(\mathbb{C}^n) \neq \infty$ is called a meromorphic function on $\mathbb{C}^n$. In other words, f is a meromorphic function in $\mathbb{C}^n$ in the sense that f can be written as a quotient of two relatively prime holomorphic functions. In particular, the entire function of several complex variables are holomorphic throughout $\mathbb{C}^n$. Considering meromorphic function f on $\mathbb{C}^n$, we assume that the reader is familiar with the standard notations and results such as the proximity function $m(r, f)$, valence function $N(r, f)$, characteristic function $T(r, f)$, the first and second main theorems, lemma on the logarithmic derivatives etc. of Nevanlinna theory, see e.g. [18, 22, 31, 36]. The exceptional sets are needed throughout the Nevanlinna theory, as well as in this paper. Typically, it means considering the linear measure $m(E) := \int_E dt$ and the logarithmic measure $l(E) := \int_{E\cap[1,\infty)} \frac{dt}{t}$ for a set $E \subset [0, \infty)$. Such a set E is always called an exceptional set if it is of

[1]Department of Mathematics, Raiganj University, Raiganj, West Bengal-733134, India, e-mail: rajubiswasjanu02@gmail.com
[2]Corresponding author
[3]Department of Mathematics, Raiganj University, Raiganj, West Bengal-733134, India, e-mail: rajibmathresearch@gmail.com

finite linear measure or of finite logarithmic measure. Trivially, $l(E) \leq m(E)$. Hence E is of finite logarithmic measure whenever it is of finite linear measure. However, the set $E = \cup_{n=1}^{+\infty}[n, n+\frac{1}{n}]$ shows that the converse is not true. Given a meromorphic function f, recall that a meromorphic function α is said to be a small function of f, if $T(r,\alpha) = S(r,f)$, where $S(r,f)$ is used to denote any quantity that satisfies $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside of a set of r of finite linear measure.

We consider some notations from [18], i.e., let $\mathscr{M}(\mathbb{C}^n)$ (resp. $\mathscr{E}(\mathbb{C}^n)$) denotes the set of all meromorphic (resp. entire) functions in $\mathbb{C}^n$. We denote by $\mathscr{M}_T(\mathbb{C}^n)$, $\mathscr{M}^{<\infty}(\mathbb{C}^n)$ and $\mathscr{M}_T^{<\infty}(\mathbb{C}^n)$ (resp. $\mathscr{E}_T(\mathbb{C}^n)$, $\mathscr{E}^{<\infty}(\mathbb{C}^n)$, $\mathscr{E}_T^{<\infty}(\mathbb{C}^n)$) respectively the set of all transcendental meromorphic functions, finite order meromorphic functions and finite order transcendental meromorphic functions (resp. transcendental entire functions, finite order entire functions and finite order transcendental entire functions) in $\mathbb{C}^n$. We denote the order of $f \in \mathscr{M}(\mathbb{C}^n)$ by $\rho(f)$ such that

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

We define $z + t = (z_1 + t_1, z_2 + t_2, \ldots, z_n + t_n)$ for any $z = (z_1, z_2, \ldots, z_n)$ and $t = (t_1, t_2, \ldots, t_n)$.

Definition 1.1. Given a meromorphic function $f(z)$ in $\mathbb{C}^n$, $f(z+c)$ is called a shift of f and $\Delta(f) = f(z+c) - f(z)$ is called a difference operator of f, where $c(\neq 0) \in \mathbb{C}^n$.

A difference polynomial (resp. a partial differential-difference polynomial) in f is a finite sum of difference products of f and its shifts (resp. of products of f, partial derivatives of f and of their shifts) with all the coefficients of these monomials being small functions of f.

Given three meromorphic functions $f(z)$, $g(z)$ and $h(z)$, $f^n(z) + g^n(z) = h^n(z)$ is called a Fermat-type functional equation over some function field, where $n \in \mathbb{N}$. If the equation contains shifts or derivatives of the functions, then it is called a Fermat-type differential-difference equation.

We now consider the Fermat-type functional equation

$$f^n(z) + g^n(z) = 1, \text{ where } n \in \mathbb{N}. \tag{1.1}$$

The result due to Iyer [10] is the gateway to find out the non-constant solutions of the Fermat-type functional equation (1.1). Actually, Iyer [10] proved that, for $n = 2$, all the non-constant entire solutions of (1.1) on $\mathbb{C}$ are of the form $f(z) = \cos(\psi(z))$ and $g(z) = \sin(\psi(z))$, where $\psi(z)$ is an entire function. We summarize the classical results on the solutions of the functional equation (1.1) in the following:

Proposition 1.2. *(i)[12] The functional equation (1.1) with $n = 2$ has the non-constant entire solutions $f(z) = \cos(\eta(z))$ and $g(z) = \sin(\eta(z))$, where $\eta(z)$ is any entire function. No other solutions exist.*
(ii)[11, 12, 29] For $n \geq 3$, there are no non-constant entire solutions of (1.1) on $\mathbb{C}$.

Proposition 1.3. *(i)[12] The functional equation (1.1) with* $n = 2$ *has the non-constant meromorphic solutions* $f = \frac{2\omega}{1+\omega^2}$ *and* $g = \frac{1-\omega^2}{1+\omega^2}$*, where* ω *is an arbitrary meromorphic function on* $\mathbb{C}$*.*
(ii)[3, 11] The functional equation (1.1) with $n = 3$ *has the non-constant meromorphic solutions* $f = \frac{1}{2\wp(h)}\left(1 + \frac{\wp'(h)}{\sqrt{3}}\right)$, $g = \frac{1}{2\wp(h)}\left(1 - \frac{\wp'(h)}{\sqrt{3}}\right)$*, where* $\wp(z)$ *denotes the Weierstrass elliptic* $\wp$*-function with periods* ω_1 *and* ω_2 *is defined as*

$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{\mu,\nu;\mu^2+\nu^2\neq 0}\left\{\frac{1}{(z+\mu\omega_1+\nu\omega_2)^2} - \frac{1}{(\mu\omega_1+\nu\omega_2)^2}\right\},$$

which is even and satisfying, after appropriately choosing ω_1 *and* ω_2, $(\wp')^2 = 4\wp^3 - 1$*.*
(iii)[11, 12] For $n \geq 4$*, there are no non-constant meromorphic solutions of (1.1) on* $\mathbb{C}$*.*

The above conclusions were also extended to the case of several complex variables (see [33]). The solutions of the Fermat-type partial difference equations was originally investigated by Li [23] and Saleeby [32]. In 2008, Li [24] showed that meromorphic solutions f and g of the Fermat-type functional equation $f^2 + g^2 = 1$ in $\mathbb{C}^2$ must be constant if and only if $\frac{\partial f}{\partial z_2}$ and $\frac{\partial g}{\partial z_1}$ have the same zeros (counting multiplicities). This shows the relationship between existence of solutions and differential operator of solutions. Furthermore, any entire solutions of the partial differential equations $\left(\frac{\partial u}{\partial z_1}\right)^2 + \left(\frac{\partial u}{\partial z_2}\right)^2 = 1$ in $\mathbb{C}^2$ are necessarily linear [19]. In 2013, Saleeby [34] considered the quadratic trinomial equations of the form

$$f^2 + 2\alpha fg + g^2 = 1, \quad \text{where } \alpha \in \mathbb{C} \setminus \{\pm 1\} \tag{1.2}$$

and the associated partial differential equations $\left(\frac{\partial u}{\partial x}\right)^2 + 2\alpha\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^2 = 1$ and proved the following results.

Theorem 1.4. *[34] The entire and meromorphic solutions of (1.2) have the following forms respectively*
(I) $f = \frac{1}{\sqrt{2}}\left(\frac{\cos(h)}{\sqrt{1+\alpha}} + \frac{\sin(h)}{\sqrt{1-\alpha}}\right)$, $g = \frac{1}{\sqrt{2}}\left(\frac{\cos(h)}{\sqrt{1+\alpha}} - \frac{\sin(h)}{\sqrt{1-\alpha}}\right)$*, where* h *is entire on* $\mathbb{C}^n$*;*
(II) $f = \frac{\alpha_1-\alpha_2\beta^2}{(\alpha_1-\alpha_2)\beta}$, $g = \frac{1-\beta^2}{(\alpha_1-\alpha_2)\beta}$*, where* $\alpha_1 = -\alpha+\sqrt{\alpha^2-1}$, $\alpha_1 = -\alpha-\sqrt{\alpha^2-1}$ *and* β *is a meromorphic function on* $\mathbb{C}^n$*.*

Theorem 1.5. *[34] The entire and meromorphic solutions of*

$$\left(\frac{\partial u}{\partial x}\right)^2 + 2\alpha\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^2 = 1, \quad (x,y) \in \mathbb{C}^2 \text{ and } \alpha \in \mathbb{C} \setminus \{\pm 1\}$$

have the form $u = ax + by + c$*, where* $a^2 + 2\alpha ab + b^2 = 1$*.*

From the proof of this result, we observe that under the transformation $f = \frac{1}{\sqrt{2}}(U+V)$ and $g = \frac{1}{\sqrt{2}}(U-V)$, the equation (1.2) transforms into $(1+$

$\alpha)U^2+(1-\alpha)V^2 = 1$, which is associated with an ellipse in $\mathbb{C}^2$. Also $f^2+2\alpha fg+g^2 = (f-\alpha_1 g)(f-\alpha_2 g)$, where $\alpha_1 = -\alpha+\sqrt{\alpha^2-1}$ and $\alpha_2 = -\alpha-\sqrt{\alpha^2-1}$.

Afterwards, Liu and Yang [27] have studied the existence of meromorphic solutions of trinomial quadratic functional equations and obtained the following results.

Theorem 1.6. *[27] If $\alpha \in \mathbb{C}\setminus\{0,\pm 1\}$, then equation $f(z)^2+2\alpha f(z)f'(z)+f'(z)^2 = 1$ has no transcendental meromorphic solutions.*

Theorem 1.7. *[27] If $\alpha \in \mathbb{C}\setminus\{0,\pm 1\}$, then the finite order transcendental entire solutions of the equation $f(z)^2+2\alpha f(z)f(z+c)+f(z+c)^2 = 1$ must be of order 1.*

In [27], Liu and Yang also showed that there exist finite and infinite order meromorphic solutions of $f(z)^2+2\alpha f(z)f(z+c)+f(z+c)^2 = 1$.

In 2021, Chen and Xu [8] considered the following Fermat-type partial differential functional equation

$$\left(a_1 f(z)+a_2\frac{\partial f}{\partial z_1}\right)^2+\left(a_3 f(z)+a_4\frac{\partial f}{\partial z_2}\right)^2 = 1 \tag{1.3}$$

in $\mathbb{C}^2$ and obtained the following result.

Theorem 1.8. *[8] Let $a_1,a_2,a_3,a_4 \in \mathbb{C}\setminus\{0\}$. Then any transcendental entire solution $f(z_1,z_2)$ with finite order of (1.3) must be of the following forms:*

(I) $f(z_1,z_2) = \pm\frac{1}{\sqrt{a_1^2+a_3^2}}+\eta e^{-\left(\frac{a_1}{a_2}z_1+\frac{a_3}{a_4}z_2\right)}$;

(II) $f(z_1,z_2) = \frac{(a_3+ia_1)e^{L(z)+B}}{2(\alpha_1 a_2 a_3-\alpha_2 a_1 a_4)}-\frac{(a_3-ia_1)e^{-L(z)-B}}{2(\alpha_1 a_2 a_3-\alpha_2 a_1 a_4)}+\eta e^{-\left(\frac{a_1}{a_2}z_1+\frac{a_3}{a_4}z_2\right)}$, where $L(z) = \alpha_1 z_1+\alpha_2 z_2$, $\alpha_1 = \frac{a_3}{a_2}i$, $\alpha_2 = -\frac{a_1}{a_4}i$ and $\eta, B \in \mathbb{C}$.

In 2022, Haldar and Ahamed [15] considered the Fermat-type partial differential-difference equation

$$\left(a_1\Delta f(z)+a_2\frac{\partial f}{\partial z_1}\right)^2+\left(a_3\Delta f(z)+a_4\frac{\partial f}{\partial z_1}\right)^2 = 1 \tag{1.4}$$

in $\mathbb{C}^2$ and obtained the following results.

Theorem 1.9. *[15] Let $c = (c_1,c_2) \in \mathbb{C}^2\setminus\{(0,0)\}$ and $a_1,a_2,a_3,a_4 \in \mathbb{C}$ such that $D = a_1a_4-a_2a_3 \neq 0$ and $a_2^2+a_4^2 \neq 0$. Then any transcendental entire solution $f(z_1,z_2)$ with finite order of (1.4) must be of the following forms:*

(I) If $c_2 \neq 0$, then $f(z_1,z_2) = \frac{1}{2\alpha_1 D}\left((a_3+ia_1)e^{L(z)+A}-(a_3-ia_1)e^{-L(z)-A}\right)+\Phi(z_2)$, where $\Phi(z_2)$ is a finite order entire function in z_2 with period c_2, $L(z) = \alpha_1 z_1+\alpha_2 z_2$ with $\alpha_1 = \frac{2iD}{a_2^2+a_4^2}, \alpha_2, A \in \mathbb{C}$ satisfying $e^{L(c)} = \frac{a_3+ia_1-(a_4+ia_2)\alpha_1}{a_3+ia_1} = \frac{a_3-ia_1}{a_3-ia_1+(a_4-ia_2)\alpha_1}$;

(II) If $c_2 = 0$, then $f(z_1,z_2) = \frac{1}{2\alpha_1 D}\left((a_3+ia_1)e^{L(z)+\Psi(z_2)+A}\right.$ $\left.-(a_3-ia_1)e^{-L(z)-\Psi(z_2)-A}\right)+\Phi(z_2)$, where $\Psi(z_2)$ is a polynomial in z_2, $\Phi(z_2)$ is a finite order entire function of z_2 only, $L(z) = \alpha_1 z_1+\alpha_2 z_2$ with $\alpha_1 = \frac{2iD}{a_2^2+a_4^2}, \alpha_2, A \in \mathbb{C}$ satisfying $e^{L(c)} = \frac{a_3+ia_1-(a_4+ia_2)\alpha_1}{a_3+ia_1} = \frac{a_3-ia_1}{a_3-ia_1+(a_4-ia_2)\alpha_1}$.

In [15], Halder and Ahamed considered the following trinomial partial differential-difference functional equations

$$\left(a_2\frac{\partial f(z)}{\partial z_1}\right)^2 + 2\alpha\left(a_2\frac{\partial f(z)}{\partial z_1}\right)\left(a_3 f(z+c) + a_4\frac{\partial^2 f(z)}{\partial z_1^2}\right) + \left(a_3 f(z+c) + a_4\frac{\partial^2 f(z)}{\partial z_1^2}\right)^2 = 1, \tag{1.5}$$

$$\left(a_2\frac{\partial f(z)}{\partial z_1}\right)^2 + 2\alpha\left(a_2\frac{\partial f(z)}{\partial z_1}\right)\left(a_3 \Delta f(z) + a_4\frac{\partial^2 f(z)}{\partial z_1^2}\right) + \left(a_3 \Delta f(z) + a_4\frac{\partial^2 f(z)}{\partial z_1^2}\right)^2 = 1, \tag{1.6}$$

$$(a_1 f(z+c))^2 + 2\alpha\,(a_1 f(z+c))\left(a_2\frac{\partial f}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1^2}\right) + \left(a_2\frac{\partial f}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1^2}\right)^2 = 1, \tag{1.7}$$

and

$$(a_1 f(z+c))^2 + 2\alpha\,(a_1 f(z+c))\left(a_2\frac{\partial f}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1\partial z_2}\right) + \left(a_2\frac{\partial f}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1\partial z_2}\right)^2 = 1 \tag{1.8}$$

in $\mathbb{C}^2$ and obtained the following results.

Theorem 1.10. *[15] Let $c=(c_1,c_2)\in\mathbb{C}^2$ with $c_2\neq 0$ and $\alpha\neq\pm 1$. Then any transcendental entire solution of (1.5) must be of the form $f(z)=\frac{-i}{2\sqrt{2}a_2a_3}\left(B_1e^{i(L(z)-L(c)+B)}-B_2e^{-i(L(z)-L(c)+B)}\right)$, where $B_1=a_4\alpha_1+ia_2\mathcal{A}_2$, $B_2=a_4\alpha_1-ia_2\mathcal{A}_1$, $L(z)=\alpha_1z_1+\alpha_2z_2$ with $\alpha_1(\neq 0),\alpha_2,B\in\mathbb{C}$ satisfying $e^{2iL(c)}=\frac{\mathcal{A}_2B_1}{\mathcal{A}_1B_2}$ and $a_3^2\mathcal{A}_1\mathcal{A}_2=\alpha_1^2B_1B_2$, where $\mathcal{A}_1=\frac{1}{\sqrt{1+\alpha}}-\frac{i}{\sqrt{1-\alpha}}$ and $\mathcal{A}_2=\frac{1}{\sqrt{1+\alpha}}+\frac{i}{\sqrt{1-\alpha}}$.*

Theorem 1.11. *[15] Let $c=(c_1,c_2)\in\mathbb{C}^2$ with $c_2\neq 0$ and $\alpha\neq\pm 1$. Then any transcendental entire solution of (1.6) must be of the form $f(z)=\frac{-i}{2\sqrt{2}a_2\alpha_1}\left(\mathcal{A}_1e^{i(L(z)+B)}-\mathcal{A}_2e^{-i(L(z)+B)}\right)$, where $L(z)=\alpha_1z_1+\alpha_2z_2$ with $\alpha_1(\neq 0),\alpha_2,B\in\mathbb{C}$ satisfying $e^{2iL(c)}=\frac{\mathcal{A}_2\left(a_4\mathcal{A}_1\alpha_1^2+ia_2\mathcal{A}_2\alpha_1-a_3\mathcal{A}_1\right)}{\mathcal{A}_1\left(a_4\mathcal{A}_2\alpha_1^2-ia_2\mathcal{A}_1\alpha_1+a_3\mathcal{A}_2\right)}$ and $a_3^2\mathcal{A}_1\mathcal{A}_2=\left(a_4\mathcal{A}_1\alpha_1^2+ia_2\mathcal{A}_2\alpha_1-a_3\mathcal{A}_1\right)\left(a_4\mathcal{A}_2\alpha_1^2-ia_2\mathcal{A}_1\alpha_1+a_3\mathcal{A}_2\right)$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are given in Theorem 1.10.*

Theorem 1.12. *[15] Let $c=(c_1,c_2)\in\mathbb{C}^2$ with $c_2\neq 0$ and $\alpha\neq\pm 1$. Then any transcendental entire solution of (1.7) must be of the form $f(z)=\frac{1}{\sqrt{2}}\left(\frac{\cos(L(z)-L(c)+B)}{\sqrt{1+\alpha}}+\frac{\sin(L(z)-L(c)+B)}{\sqrt{1-\alpha}}\right)$, where $L(z)=\alpha_1z_1+\alpha_2z_2$ with $\alpha_1(\neq 0),\alpha_2,B\in\mathbb{C}$ satisfying $e^{2iL(c)}=\frac{\mathcal{A}_1^2(a_3\alpha_1-ia_2)}{\mathcal{A}_2^2(a_3\alpha_1+ia_2)}$ and $a_1^2=\alpha_1^2\left(a_3^2\alpha_1^2+a_2^2\right)$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are given in Theorem 1.10.*

Theorem 1.13. *[15] Let $c=(c_1,c_2)\in\mathbb{C}^2$ with $c_2\neq 0$ and $\alpha\neq\pm 1$. Then any transcendental entire solution of (1.8) must be of the form*

$f(z) = \frac{1}{\sqrt{2}}\left(\frac{\cos(L(z)-L(c)+B)}{\sqrt{1+\alpha}} + \frac{\sin(L(z)-L(c)+B)}{\sqrt{1-\alpha}}\right)$, where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$ with $\alpha_1(\neq 0), \alpha_2, B \in \mathbb{C}$ satisfying $e^{2iL(c)} = \frac{\mathcal{A}_1^2(a_3\alpha_2 - ia_2)}{\mathcal{A}_2^2(a_3\alpha_2 + ia_2)}$ and $a_1^2 = \alpha_1^2\left(a_3^2\alpha_2^2 + a_2^2\right)$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are given in Theorem 1.10.

If $I = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_+^n$ is a multi-index with length $|I| = \sum_{j=1}^n i_j$, then any polynomial $\mathcal{P}(z)$ in $\mathbb{C}^n$ of degree d can be expressed as $\mathcal{P}(z) = \sum_{|I|=0}^d a_I z_1^{i_1} \cdots \partial z_n^{i_n}$, where $a_I \in \mathbb{C}$ such that a_I are not all zero at a time. Now the results in [8, 15] motivate us to improve and generalize further. This can be done by the following ways:
(I) replacing "1" on the R.H.S. in (1.3)-(1.8) by the term "$e^{g(z)}$", where $g(z)$ is a polynomial;
(II) considering more general forms of the differential-difference parts on the L.H.S. in (1.3)-(1.8) and
(III) also considering the generalized forms of (1.3)-(1.8) in $\mathbb{C}^n$ $(n \geq 2)$.
Note that, one can replace 1 on the R.H.S. in the functional equations (1.3)-(1.8) by any finite order entire function, like, $F(z)$ such that $F(0) \neq 0$ and attempt to generalize the results in[8, 15]. Actually, for this kind of functions, we can easily apply Lemma 2.4, given in the lemma section. In this paper, we have considered $e^{g(z)}$ on the R.H.S. of equations (1.3)-(1.8) only. Thus keeping all the above facts in mind, in this paper we consider the following quadratic trinomial partial differential-difference functional equations in $\mathbb{C}^n$:

$$\left(a_1\frac{\partial f(z)}{\partial z_1}\right)^2 + 2\alpha\left(a_1\frac{\partial f(z)}{\partial z_1}\right)\left(a_2 f(z) + a_3 f(z+c) + a_4\frac{\partial^2 f(z)}{\partial z_1^2}\right) + \left(a_2 f(z) + a_3 f(z+c) + a_4\frac{\partial^2 f(z)}{\partial z_1^2}\right)^2 = e^{g(z)}, \tag{1.9}$$

$$a_1^2 f^2(z+c) + 2\alpha a_1 f(z+c)\left(a_2\frac{\partial f(z)}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1^2}\right) + \left(a_2\frac{\partial f(z)}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1^2}\right)^2 = e^{g(z)} \text{ and} \tag{1.10}$$

$$a_1^2 f^2(z+c) + 2\alpha a_1 f(z+c)\left(a_2\frac{\partial f(z)}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1\partial z_2}\right) + \left(a_2\frac{\partial f(z)}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1\partial z_2}\right)^2 = e^{g(z)}, \tag{1.11}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C} \setminus \{0\}$, $\alpha \in \mathbb{C} \setminus \{0, \pm 1\}$ and $g(z)$ is a polynomial in $\mathbb{C}^n$. Furthermore, Theorems 1.8 and 1.9 motivate us to investigate the solutions of the following quadratic trinomial partial differential-difference functional equation in $\mathbb{C}^n$:

$$\left(a_1\Delta f(z) + a_2\frac{\partial f}{\partial z_1}\right)^2 + 2\alpha\left(a_1\Delta f(z) + a_2\frac{\partial f}{\partial z_1}\right)\left(a_3\Delta f(z) + a_4\frac{\partial f}{\partial z_2}\right) + \left(a_3\Delta f(z) + a_4\frac{\partial f}{\partial z_2}\right)^2 = e^{g(z)}, \tag{1.12}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C} \setminus \{0\}$, $\alpha \in \mathbb{C} \setminus \{0, \pm 1\}$ and $g(z)$ is a polynomial in $\mathbb{C}^n$.

Throughout the paper, we denote

$$(1.13)\qquad A_1=\frac{1}{2\sqrt{1+\alpha}}+\frac{1}{2i\sqrt{1-\alpha}},\ A_2=\frac{1}{2\sqrt{1+\alpha}}-\frac{1}{2i\sqrt{1-\alpha}}\ \text{and}\ \omega_1=\sum_{j=2}^{n}z_j.$$

Then $A_1A_2=\frac{1}{2(1-\alpha^2)}$, $A_1^2+A_2^2=\frac{\alpha}{\alpha^2-1}$ and $A_1^2-A_2^2=\frac{1}{i\sqrt{1-\alpha^2}}$. For entire solutions of equation (1.9), we obtain the following result.

Theorem 1.14. *Let $c=(c_1,c_2,\dots,c_n)\in\mathbb{C}^n\setminus\mathbb{O}$ with each c_i is non-zero for $n\geq 3$, $\tau=\sum_{j=1}^n c_j\neq 0$, $\tau_1=\sum_{j=2}^n c_j\neq 0$ and $a_1,a_2,a_3,a_4\in\mathbb{C}\setminus\{0\}$. Let $y_1=(z_2,z_3,\dots,z_n)$, $s_1=(c_2,c_3,\dots,c_n)$ and $f(z)\in\mathscr{E}_T^{<\infty}(\mathbb{C}^n)$ satisfies (1.9). Then the following circumstances arise.*

(I) If $g\in\mathbb{C}$ and $e^g\equiv K\in\mathbb{C}\setminus\{0\}$, then $f(z)$ has one of the following forms:

(I_1)

$$f(z)=\begin{cases} G_1(y_1)+\frac{\phi_4}{a_3\tau}\omega+k_0, & \text{if } a_2+a_3=0,\\ e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}G_2(y_1)+\frac{\phi_4}{a_2+a_3}, & \text{if } a_2+a_3\neq 0,\end{cases}$$

where $k_0,\phi_3,\phi_4\in\mathbb{C}$ with $\phi_3^2+2\alpha\phi_3\phi_4+\phi_4^2=K$, $\omega=\sum_{j=1}^n z_j$, $G_1(y_1)$ (resp. $G_2(y_1)$) is a finite order transcendental entire (resp. a finite order entire) periodic function with period $s_1\in\mathbb{C}^{n-1}\setminus\mathbb{O}$ and $\phi_3=\begin{cases}\frac{a_1\phi_4}{a_3\tau}, & \text{if } a_2+a_3=0,\\ 0, & \text{if } a_2+a_3\neq 0;\end{cases}$

(I_2)

$$f(z)=\frac{\left(A_1K_1e^{\sum_{j=2}^n b_jz_j+b_{n+1}}+A_2K_2e^{-\sum_{j=2}^n b_jz_j-b_{n+1}}\right)z_1}{\sqrt{2}a_1}+e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\phi(y_1)$$
$$-\frac{(a_1A_2+a_2c_1A_1)K_1\omega_1e^{\sum_{j=2}^n b_jz_j+b_{n+1}}}{\sqrt{2}a_1a_2\tau_1}-\frac{(a_1A_1+a_2c_1A_2)K_2\omega_1\frac{e^{-\sum_{j=2}^n b_jz_j}}{e^{b_{n+1}}}}{\sqrt{2}a_1a_2\tau_1},$$

where $b_j,K_1,K_2\in\mathbb{C}$ for $2\leq j\leq n+1$, $e^{\sum_{j=2}^n b_jc_j}=-\frac{a_2}{a_3}=e^{-\sum_{j=2}^n b_jc_j}$ and $\phi(y_1)$ is a finite order entire function with period s_1;

(I_3) $f(z)=\frac{\left(A_1K_1e^{\sum_{j=1}^n b_jz_j+b_{n+1}}-A_2K_2e^{-\sum_{j=1}^n b_jz_j-b_{n+1}}\right)}{\sqrt{2}a_1b_1}+e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\psi(y_1)$,
where $b_j,K_1,K_2\in\mathbb{C}$ for $1\leq j\leq n+1$ with $b_1\neq 0$, $K_1K_2=K$, $e^{\sum_{j=1}^n b_jc_j}\equiv\frac{a_1}{a_3}\left(\frac{A_2}{A_1}b_1-\frac{a_4}{a_1}b_1^2-\frac{a_2}{a_1}\right)$, $e^{-\sum_{j=1}^n b_jc_j}\equiv-\frac{a_1}{a_3}\left(\frac{A_1}{A_2}b_1+\frac{a_4}{a_1}b_1^2+\frac{a_2}{a_1}\right)$ and $\psi(y_1)$ is a finite order entire periodic function with period s_1;

(II) If $g(z)$ is a non-constant polynomial in $\mathbb{C}^n$, then $g(z)$ reduces to a linear polynomial and $f(z)$ has one of the following forms:

(II_1)

$$f(z)=\frac{K_3}{a_1}z_1e^{\frac12\sum_{j=2}^n b_jz_j+\frac12 b_{n+1}}+e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\Psi_2(y_1)$$
$$-\left(\frac{K_4}{a_2\tau_1}+\frac{K_3c_1}{a_1\tau_1}\right)\omega_1e^{\frac12\sum_{j=2}^n b_jz_j+\frac12 b_{n+1}},$$

where $b_j, K_1, K_2, K_3(\neq 0), K_4, \xi(\neq 0) \in \mathbb{C}$ *for* $2 \leq j \leq n+1$ *with* $\omega_1 = \sum_{j=2}^{n} z_j$, $e^{\frac{1}{2}\sum_{j=2}^{n} b_j c_j} \equiv -\frac{a_2}{a_3}$, $K_1K_2 = 1$, $K_3 = \frac{A_1K_1\xi + A_2K_2\xi^{-1}}{\sqrt{2}}$, $K_4 = \frac{A_2K_1\xi + A_1K_2\xi^{-1}}{\sqrt{2}}$ *and* $\Psi_2(y_1)$ *is a finite order entire periodic function of period* s_1;

(II₂)

$$f(z) = \frac{2K_3}{a_1b_1} e^{\frac{1}{2}\sum_{j=1}^{n} b_j z_j + \frac{1}{2}b_{n+1}} + e^{\frac{\sum_{j=2}^{n} z_j}{\sum_{j=2}^{n} c_j}\log\left(-\frac{a_2}{a_3}\right)}\Psi_3(y_1),$$

where $b_j, K_1, K_2, K_3(\neq 0), K_4, \xi(\neq 0) \in \mathbb{C}$ *for* $1 \leq j \leq n+1$ *with* $b_1 \neq 0$, $K_1K_2 = 1$, $\frac{a_1}{a_3K_3}\left(\frac{K_4b_1}{2} - \frac{a_4K_3}{4a_1}b_1^2 - \frac{a_2K_3}{a_1}\right) \equiv e^{\frac{1}{2}\sum_{j=1}^{n} b_j c_j}$, $K_3 = \frac{A_1K_1\xi + A_2K_2\xi^{-1}}{\sqrt{2}}$, $K_4 = \frac{A_2K_1\xi + A_1K_2\xi^{-1}}{\sqrt{2}}$ *and* $\Psi_3(y_1)$ *is a finite order entire periodic function with period* s_1. *Also* $g(z) = \sum_{j=1}^{n} b_j z_j + b_{n+1}$;

(II₃)

$$f(z) = \frac{A_1K_1e^{\sum_{j=2}^{n} b_j z_j + b_{n+1}}}{\sqrt{2}a_1} z_1 + \frac{A_2K_2e^{\sum_{j=2}^{n} d_j z_j + d_{n+1}}}{\sqrt{2}a_1} z_1 + e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\Phi_3(y_1)$$
$$-\frac{(a_1A_2 + a_2c_1A_1)K_1\omega_1}{\sqrt{2}a_1a_2\tau_1} e^{\sum_{j=2}^{n} b_j z_j + b_{n+1}} - \frac{(a_1A_1 + a_2c_1A_2)K_2\omega_1}{\sqrt{2}a_1a_2\tau_1} \frac{e^{\sum_{j=2}^{n} d_j z_j}}{e^{-d_{n+1}}},$$

where $b_j, d_j, K_1, K_2 \in \mathbb{C}$ *for* $2 \leq j \leq n+1$ *with* $K_1K_2 = 1$, $e^{\sum_{j=2}^{n} b_j c_j} \equiv -\frac{a_2}{a_3}$, $e^{\sum_{j=2}^{n} d_j c_j} \equiv -\frac{a_2}{a_3}$ *and* $\Phi_3(y_1)$ *is a finite order entire periodic function with period* s_1. *Also* $g(z) = \sum_{j=2}^{n}(b_j + d_j)z_j + b_{n+1} + d_{n+1}$;

(II₄)

$$f(z) = \frac{A_1K_1z_1e^{\sum_{j=2}^{n} b_j z_j + b_{n+1}}}{\sqrt{2}a_1} + \frac{A_2K_2e^{\sum_{j=1}^{n} d_j z_j + d_{n+1}}}{\sqrt{2}a_1d_1} + e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\Psi_5(y_1)$$
$$-\frac{(a_1A_2 + a_2c_1A_1)K_1\omega_1e^{\sum_{j=2}^{n} b_j z_j + b_{n+1}}}{\sqrt{2}a_1a_2\tau_1},$$

where $d_1(\neq 0), b_j, d_j, K_1, K_2 \in \mathbb{C}$ *for* $2 \leq j \leq n+1$ *with* $K_1K_2 = 1$, $e^{\sum_{j=2}^{n} b_j c_j} \equiv -\frac{a_2}{a_3}$, $e^{\sum_{j=1}^{n} d_j c_j} \equiv \frac{a_1}{a_3}\left(\frac{A_1}{A_2}d_1 - \frac{a_4}{a_1}d_1^2 - \frac{a_2}{a_1}\right)$ *and* $\Psi_5(y_1)$ *is a finite order entire periodic function with period* s_1. *Also* $g(z) = d_1z_1 + \sum_{j=2}^{n}(b_j + d_j)z_j + b_{n+1} + d_{n+1}$;

(II₅)

$$f(z) = \frac{A_1K_1e^{\sum_{j=1}^{n} b_j z_j + b_{n+1}}}{\sqrt{2}a_1b_1} + \frac{A_2K_2z_1e^{\sum_{j=2}^{n} d_j z_j + d_{n+1}}}{\sqrt{2}a_1} + e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\Psi_6(y_1)$$
$$-\frac{(a_1A_1 + a_2c_1A_2)K_2\omega_1e^{\sum_{j=2}^{n} d_j z_j + d_{n+1}}}{\sqrt{2}a_1a_2\tau_1},$$

where $b_1(\neq 0), b_j, d_j, K_1, K_2 \in \mathbb{C}$ *for* $2 \leq j \leq n+1$ *with* $K_1K_2 = 1$, $e^{\sum_{j=1}^{n} b_j c_j} \equiv \frac{a_1}{a_3}\left(\frac{A_2}{A_1}b_1 - \frac{a_4}{a_1}b_1^2 - \frac{a_2}{a_1}\right)$, $e^{\sum_{j=2}^{n} d_j c_j} \equiv -\frac{a_2}{a_3}$ *and* $\Psi_6(y_1)$ *is a finite order entire periodic function with period* s_1. *Also* $g(z) = b_1z_1 + \sum_{j=2}^{n}(b_j + d_j)z_j + b_{n+1} + d_{n+1}$;

(II$_6$)

$$f(z)=\frac{A_1K_1e^{\sum_{j=1}^{n}b_jz_j+b_{n+1}}}{\sqrt{2}a_1b_1}+\frac{A_2K_2e^{\sum_{j=1}^{n}d_jz_j+d_{n+1}}}{\sqrt{2}a_1d_1}+e^{\frac{\sum_{j=2}^{n}z_j}{\sum_{j=2}^{n}c_j}\log\left(-\frac{a_2}{a_3}\right)}\Psi_8(y_1),$$

where $b_j, d_j, K_1, K_2\in\mathbb{C}$ $(1\le j\le n+1)$ with $b_1\neq 0$, $d_1\neq 0$, $K_1K_2=1$, $\frac{a_1}{a_3}\left(\frac{A_2}{A_1}b_1-\frac{a_4}{a_1}b_1^2-\frac{a_2}{a_1}\right)\equiv e^{\sum_{j=1}^{n}b_jc_j}$, $\frac{a_1}{a_3}\left(\frac{A_1}{A_2}d_1-\frac{a_4}{a_1}d_1^2-\frac{a_2}{a_1}\right)\equiv e^{\sum_{j=1}^{n}d_jc_j}$ and $\Psi_8(y_1)$ is a finite order entire periodic function with period s_1. Also $g(z)=\sum_{j=1}^{n}(b_j+d_j)z_j+b_{n+1}+d_{n+1}$.

Note that Theorem 1.14 is the generalization of Theorems 1.10 and 1.11. The following examples related to Theorem 1.14 are reasonable.

Example 1.15. Let $c=(c_1,c_2,c_3,c_4)\in\mathbb{C}^4$ such that $\sum_{j=2}^{4}c_j\neq 0$, $\alpha=3$, $\xi=1$, $K_1=2$ and $K_2=1/2$. Then $K_3=(5\sqrt{2}-6)/16$ and $K_4=(5\sqrt{2}+6)/16$. Clearly, $f(z)=K_3e^{z_1+\frac{3}{2}z_2+\frac{1}{2}z_3+\frac{5}{2}z_4+\frac{7}{2}}+e^{\frac{z_2+z_3+z_4}{c_2+c_3+c_4}\log\left(-\frac{2}{3}\right)}G(z_2,z_3,z_4)$ satisfies (1.9) with $a_j=j$ $(1\le j\le 4)$ and $g(z)=2z_1+3z_2+z_3+5z_4+7$, where G is a finite order entire function satisfying $G(z_2+c_2,z_3+c_3,z_4+c_4)=G(z_2,z_3,z_4)$ and $e^{c_1+\frac{3}{2}c_2+\frac{1}{2}c_3+\frac{5}{2}c_4}=(1+30\sqrt{2})/21$.

Example 1.16. Let $c=(c_1,c_2,c_3,c_4)\in\mathbb{C}^4$ such that $c_2+c_3+c_4\neq 0$. Let $\alpha=-\frac{1}{2}$, $K_1=2$ and $K_2=\frac{1}{2}$. Then $A_1=\sqrt{\frac{2}{3}}\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)$ and $A_2=\sqrt{\frac{2}{3}}\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)$. Clearly, $f(z)=\frac{(\sqrt{3}-i)e^{z_1+2z_2+3z_3+4z_4+7}}{2\sqrt{3}}+e^{\frac{z_2+z_3+z_4}{c_2+c_3+c_4}\log\left(-\frac{3}{4}\right)}G(z_2,z_3,z_4)$ $+\frac{(\sqrt{3}+i)e^{2z_1+z_2+4z_3+5z_4+8}}{16\sqrt{3}}$ satisfies (1.9) with $a_j=j+1$ $(1\le j\le 4)$ and $g(z)=3z_1+3z_2+7z_3+9z_4+15$, where G is a finite order entire function satisfying $G(z_2+c_2,z_3+c_3,z_4+c_4)=G(z_2,z_3,z_4)$, $e^{c_1+2c_2+3c_3+4c_4}=-7/4+i\sqrt{3}/4$ and $e^{2c_1+c_2+4c_3+5c_4}=-21/4-i\sqrt{3}/2$.

For entire solutions of equation (1.12), we obtain the following result.

Theorem 1.17. *Let $c=(c_1,c_2,\ldots,c_n)\in\mathbb{C}^n\setminus\mathbb{O}$ with each c_i is non-zero for $n\geq 3$ and $a_1,a_2,a_3,a_4\in\mathbb{C}\setminus\{0\}$ with $a_1^2+a_3^2+2\alpha a_1a_3\neq 0$, i.e., $(a_3A_2-a_1A_1)(a_1A_2-a_3A_1)\neq 0$. Let $y=(a_1a_4z_1+a_2a_3z_2,z_3,\cdots,z_n)$, $s=(a_1a_4c_1+a_2a_3c_2,c_3,\cdots,c_n)$ and $f(z)\in\mathscr{E}_T^{<\infty}(\mathbb{C}^n)$ satisfies (1.12). Then the following circumstances arise.*

(I) If $g\in\mathbb{C}$ and $e^g\equiv K\in\mathbb{C}\setminus\{0\}$, then $f(z)$ has one of the following forms:

(I$_1$) $f(z)=\frac{a_3\phi_1-a_1\phi_2}{a_2a_3}z_1+G(y)$, where $\phi_1,\phi_2\in\mathbb{C}$ with $\phi_1^2+2\alpha\phi_1\phi_2+\phi_2^2=K$, $G(y)$ is a finite order transcendental entire function satisfying $a_2a_3\frac{\partial G(y)}{\partial z_1}=a_1a_4\frac{\partial G(y)}{\partial z_2}$ and

$$\begin{aligned}G(y+s)-G(y)&=\frac{\phi_1}{a_1}-\frac{a_3\phi_1-a_1\phi_2}{a_2a_3}\left(\frac{a_2}{a_1}+c_1\right)-\frac{a_2}{a_1}\frac{\partial G(y)}{\partial z_1}\\&=\frac{\phi_2}{a_3}-\frac{a_3\phi_1-a_1\phi_2}{a_2a_3}c_1-\frac{a_4}{a_3}\frac{\partial G(y)}{\partial z_2};\end{aligned}$$

(I_2)

$$f(z)=\frac{(a_3A_1-a_1A_2)K_1e^{\sum_{j=1}^{n}b_jz_j+b_{n+1}}-(a_3A_2-a_1A_1)K_2e^{-\sum_{j=1}^{n}b_jz_j-b_{n+1}}}{\sqrt{2}(a_2a_3b_1-a_1a_4b_2)}+\Psi\left(y\right),$$

where $b_j(1\leq j\leq n+1), K_1, K_2\in\mathbb{C}$ *with* $K_1K_2=K$, $a_2a_3b_1-a_1a_4b_2\neq 0$, $\Psi(y)$ *is a finite order entire function with*

$$a_2a_3\frac{\partial\Psi(y)}{\partial z_1}=a_1a_4\frac{\partial\Psi(y)}{\partial z_2}\ \text{and}\ \Psi(y+s)-\Psi(y)=-\frac{a_2}{a_1}\frac{\partial\Psi(y)}{\partial z_1}=-\frac{a_4}{a_3}\frac{\partial\Psi(y)}{\partial z_2}.$$

Also $e^{\sum_{j=1}^{n}b_jc_j}-1\equiv\frac{a_4b_2A_1-a_2b_1A_2}{a_1A_2-a_3A_1}$ *and* $e^{-\sum_{j=1}^{n}b_jc_j}-1\equiv\frac{a_2b_1A_1-a_4b_2A_2}{a_1A_1-a_3A_2}$.

(II) If $g(z)$ *is a non-constant polynomial in* $\mathbb{C}^n$, *then* $g(z)$ *reduces to a linear polynomial and* $f(z)$ *has one of the following forms:*

(II_1) $f(z)=\frac{2(a_3K_3-a_1K_4)e^{\frac{1}{2}\sum_{j=1}^{n}b_jz_j+\frac{1}{2}b_{n+1}}}{(a_2a_3b_1-a_1a_4b_2)}+h_2\left(y\right)$, *where* $b_j(1\leq j\leq n+1)$, $K_1, K_2, K_3, K_4, \xi(\neq 0)\in\mathbb{C}$ *with* $K_1K_2=1$, $K_3=\frac{A_1K_1\xi+A_2K_2\xi^{-1}}{\sqrt{2}}$, $K_4=\frac{A_2K_1\xi+A_1K_2\xi^{-1}}{\sqrt{2}}$, $a_3K_3-a_1K_4\neq 0$, $a_2a_3b_1-a_1a_4b_2\neq 0$ *and* $h_2(y)$ *is a finite order entire function satisfying* $a_2a_3\frac{\partial h_2(y)}{\partial z_1}=a_1a_4\frac{\partial h_2(y)}{\partial z_2}$ *and*

$$h_2(y+s)-h_2(y)=\frac{a_4b_2K_3-a_2b_1K_4}{a_2a_3b_1-a_1a_4b_2}e^{\frac{1}{2}\sum_{j=1}^{n}b_jz_j+\frac{1}{2}b_{n+1}}-\frac{a_2}{a_1}\frac{\partial h_2(y)}{\partial z_1}.$$

Also $g(z)=\sum_{j=1}^{n}b_jz_j+b_{n+1}$ *and* $e^{\frac{1}{2}\sum_{j=1}^{n}b_jc_j}-1\equiv\frac{a_4K_3b_2-a_2K_4b_1}{a_1K_4-a_3K_3}$;

(II_2)

$$f(z)=\frac{(a_3A_1-a_1A_2)K_1e^{\sum_{j=1}^{n}b_jz_j+b_{n+1}}}{\sqrt{2}(a_2a_3b_1-a_1a_4b_2)}+\frac{(a_3A_2-a_1A_1)K_2e^{\sum_{j=1}^{n}d_jz_j+d_{n+1}}}{\sqrt{2}(a_2a_3d_1-a_1a_4d_2)}+\chi\left(y\right),$$

where $b_j, d_j, K_1, K_2\in\mathbb{C}$ *for* $1\leq j\leq n+1$ *with* $K_1K_2=1$, $a_2a_3b_1-a_1a_4b_2\neq 0$, $a_2a_3d_1-a_1a_4d_2\neq 0$ *and* $\chi(y)$ *is a finite order entire function with*

$$a_2a_3\frac{\partial\chi}{\partial z_1}=a_1a_4\frac{\partial\chi(y)}{\partial z_2}\ \text{and}\ \chi(y+s)-\chi(y)=-\frac{a_2}{a_1}\frac{\partial\chi(y)}{\partial z_1}=-\frac{a_4}{a_3}\frac{\partial\chi(y)}{\partial z_2}.$$

Also $g(z)=\sum_{j=1}^{n}\left(b_j+d_j\right)z_j+b_{n+1}+d_{n+1}$, $e^{\sum_{j=1}^{n}b_jc_j}-1\equiv\frac{a_4b_2A_1-a_2b_1A_2}{a_1A_2-a_3A_1}$ *and* $e^{\sum_{j=1}^{n}d_jc_j}-1\equiv\frac{a_4d_2A_2-a_2d_1A_1}{a_1A_1-a_3A_2}$.

The following example shows that the forms of the solutions in Theorem 1.17 are precise.

Example 1.18. Let $\alpha=-\frac{1}{2}$, $K_1=1$ and $K_2=1$. Then $A_1=\sqrt{\frac{2}{3}}\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)$ and $A_2=\sqrt{\frac{2}{3}}\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)$. Let $a_1=a_2=a_4=1$, $a_3=-1$ and $c=(c_1,c_2,c_3)\in\mathbb{C}^3$ be such that $e^{2c_1+3c_2+4c_3}-1=\frac{1}{2}-\frac{5}{2\sqrt{3}}i$ and $e^{5c_1+7c_2+2c_3}-1=1+2\sqrt{3}i$. Clearly $f(z)=\frac{1}{5}e^{2z_1+3z_2+4z_3+5}+\frac{1}{12}e^{5z_1+7z_2+2z_3+3}$ satisfies (1.12) with $g(z)=7z_1+10z_2+6z_3+8$.

For entire solutions of equation (1.10), we obtain the following result.

Theorem 1.19. *Let* $c=(c_1,c_2,\ldots,c_n)\in\mathbb{C}^n\setminus\mathbb{O}$ *and* $a_1,a_2,a_3\in\mathbb{C}\setminus\{0\}$. *Let* $f(z)\in\mathscr{E}_T^{<\infty}(\mathbb{C}^n)$ *satisfy (1.10). Then the following circumstances arise.*

(I) If $g\in\mathbb{C}$ *and* $e^g\equiv K\in\mathbb{C}\setminus\{0\}$, *then* $f(z)$ *has one of the forms:* $f(z)=\frac{A_1K_1e^{\sum_{j=1}^n b_j(z_j-c_j)+b_{n+1}}+A_2K_2e^{-\sum_{j=1}^n b_j(z_j-c_j)-b_{n+1}}}{\sqrt{2}a_1}$, *where* $b_j(1\le j\le n+1), K_1,K_2\in\mathbb{C}$ *with* $b_1\neq 0$, $K_1K_2=K$, $e^{\sum_{j=1}^n b_jc_j}\equiv\frac{(a_3b_1^2+a_2b_1)A_1}{a_1A_2}=\frac{a_1A_1}{(a_3b_1^2-a_2b_1)A_2}$, *when each* c_i*'s is non-zero for* $n\ge 3$, *otherwise* $f(z)$ *is of the form* $f(z)=\frac{1}{\sqrt{2}a_1}\left(A_1K_1e^{P(z-c)}+A_2K_2e^{-P(z-c)}\right)$, *where* $P(z)$ *is a polynomial satisfying* $e^{P(z+c)-P(z)}=\frac{A_1}{A_2}\left(\frac{a_3}{a_1}\left\{\frac{\partial^2P(z)}{\partial z_1^2}+\left(\frac{\partial P(z)}{\partial z_1}\right)^2\right\}+\frac{a_2}{a_1}\frac{\partial P(z)}{\partial z_1}\right)$ *and* $e^{P(z)-P(z+c)}=\frac{A_2}{A_1}\left(\frac{a_3}{a_1}\left\{\left(\frac{\partial P(z)}{\partial z_1}\right)^2-\frac{\partial^2P(z)}{\partial z_1^2}\right\}-\frac{a_2}{a_1}\frac{\partial P(z)}{\partial z_1}\right)$;

(II) If $g(z)$ *is a non-constant polynomial in* $\mathbb{C}^n$, *then* $f(z)$ *has one of the following forms:*

(II$_1$*)* $f(z)=\frac{K_3}{a_1}e^{\frac{g(z-c)}{2}}$, *where* $K_1,K_2,K_3,\xi\in\mathbb{C}\setminus\{0\}$ *such that* $K_1K_2=1$, $K_3=\frac{A_1K_1\xi+A_2K_2\xi^{-1}}{\sqrt{2}}$, $A_2K_1\xi+A_1K_2\xi^{-1}=0$ *and* $g(z)$ *is a non-constant polynomial in* $\mathbb{C}^n$ *with* $2a_2\frac{\partial g(z)}{\partial z_1}+a_3\left(\frac{\partial g(z)}{\partial z_1}\right)^2+2a_3\frac{\partial^2g(z)}{\partial z_1^2}\equiv 0$;

(II$_2$*)* $f(z)=\frac{K_3}{a_1}e^{\frac{1}{2}\sum_{j=1}^n b_j(z_j-c_j)+\frac{1}{2}b_{n+1}}$, *where* $b_j,K_1,K_2,K_3(\neq 0),K_4(\neq 0),\xi(\neq 0)\in\mathbb{C}$ *for* $1\le j\le n+1$ *with* $b_1\neq 0$, $K_1K_2=1$, $K_3=\frac{A_1K_1\xi+A_2K_2\xi^{-1}}{\sqrt{2}}$, $K_4=\frac{A_2K_1\xi+A_1K_2\xi^{-1}}{\sqrt{2}}$, $e^{\frac{1}{2}\sum_{j=1}^n b_jc_j}\equiv\frac{K_3}{K_4}\left(\frac{a_2}{2a_1}b_1+\frac{a_3}{4a_1}b_1^2\right)$ *and* $g(z)=\sum_{j=1}^n b_jz_j+b_{n+1}$, *when each* c_i*'s is non-zero for* $n\ge 3$, *otherwise* $f(z)$ *is of the form of (II*$_1$*), where* $g(z)$ *is a polynomial satisfying* $\left(\frac{a_2}{2a_1}\frac{\partial g(z)}{\partial z_1}+\frac{a_3}{4a_1}\left\{\left(\frac{\partial g(z)}{\partial z_1}\right)^2+2\frac{\partial^2g(z)}{\partial z_1^2}\right\}\right)\equiv\frac{K_4}{K_3}e^{\frac{g(z+c)-g(z)}{2}}$;

(II$_3$*)* $f(z)=\frac{A_1K_1e^{\sum_{j=1}^n b_j(z_j-c_j)+b_{n+1}}+A_2K_2e^{\sum_{j=1}^n d_j(z_j-c_j)+d_{n+1}}}{\sqrt{2}a_1}$, *where* $b_j,d_j,K_1,K_2\in\mathbb{C}$ *for* $1\le j\le n+1$ *with* $b_1\neq 0$, $d_1\neq 0$, $K_1K_2=1$, $e^{\sum_{j=1}^n b_jc_j}\equiv\frac{(a_3b_1^2+a_2b_1)A_1}{a_1A_2}$, $e^{\sum_{j=1}^n d_jc_j}\equiv\frac{(a_3d_1^2+a_2d_1)A_2}{a_1A_1}$ *and* $g(z)=\sum_{j=1}^n(b_j+d_j)z_j+b_{n+1}+d_{n+1}$, *when each* c_i*'s is non-zero for* $n\ge 3$, *otherwise* $f(z)$ *is of the form* $f(z)=\frac{1}{\sqrt{2}a_1}\left(A_1K_1e^{\gamma_1(z-c)}+A_2K_2e^{\gamma_2(z-c)}\right)$, *where* $\gamma_k(z)$ $(k=1,2)$ *are polynomials satisfying* $e^{\gamma_k(z+c)-\gamma_k(z)}=\alpha_k\left(\frac{a_3}{a_1}\left\{\frac{\partial^2\gamma_k}{\partial z_1^2}+\left(\frac{\partial\gamma_k}{\partial z_1}\right)^2\right\}+\frac{a_2}{a_1}\frac{\partial\gamma_k}{\partial z_1}\right)$ *with* $\alpha_1=A_1/A_2=\alpha_2^{-1}$.

Remark 1.20. In particular, if we choose $g(z)\equiv 0$ in Theorem 1.19, then $K=1$ and (1.10) becomes (1.7). If $K_1=K_2=1$ and $n=2$, then from (I) of Theorem 1.19, we have

$$(1.14)\, f(z)=\frac{\cos\left(\sum_{j=1}^2 d_j(z_j-c_j)+d_3\right)}{\sqrt{2(1+\alpha)}a_1}+\frac{\sin\left(\sum_{j=1}^2 d_j(z_j-c_j)+d_3\right)}{\sqrt{2(1-\alpha)}a_1},$$

where $b_j = id_j \in \mathbb{C}$ for $1 \leq j \leq 3$ such that $d_1 \neq 0$, $a_1^2 - d_1^2\left(a_3^2 d_1^2 + a_2^2\right) = 0$ and $e^{2i\sum_{j=1}^{n} d_j c_j} = \frac{(a_3 d_1 - i a_2)A_1^2}{(a_3 d_1 + i a_2)A_2^2}$. In this sense, Theorem 1.19 is a significant improvement of Theorem 1.12.

The following example related to Theorem 1.19 is reasonable.

Example 1.21. Let $c = (c_1, c_2, c_3, c_4) \in \mathbb{C}^4 \setminus \mathbb{O}$, $\alpha = 3$, $\xi = 1$, $K_1 = 2$ and $K_2 = \frac{1}{2}$. Then $K_3 = \frac{5\sqrt{2}-6}{16}$ and $K_4 = \frac{5\sqrt{2}+6}{16}$. Clearly, $f(z) = \frac{(5\sqrt{2}-6)}{32} e^{\frac{1}{2}\{3(z_1-c_1)+2(z_2-c_2)+5(z_3-c_3)+7(z_4-c_4)+1\}}$ satisfies the equation (1.10) with $g(z) = 3z_1 + 2z_2 + 5z_3 + 7z_4 + 1$, $a_1 = 2, a_2 = 7, a_3 = 3, b_1 = 3, b_2 = 2, b_3 = 5, b_4 = 7, b_5 = 1$ and $e^{\frac{1}{2}(3c_1+2c_2+5c_3+7c_4)} = 69(43 - 30\sqrt{2})/56$.

For entire solutions of equation (1.11), we obtain the following result.

Theorem 1.22. *Let $c = (c_1, c_2, \ldots, c_n) \in \mathbb{C}^n \setminus \mathbb{O}$ and $a_1, a_2, a_3 \in \mathbb{C} \setminus \{0\}$. Let $f(z) \in \mathscr{E}_T^{<\infty}(\mathbb{C}^n)$ satisfy (1.11). Then the following circumstances arise.*

(I) If $g \in \mathbb{C}$ and $e^g \equiv K \in \mathbb{C} \setminus \{0\}$, then $f(z)$ has one of the forms: $f(z) = \frac{A_1 K_1 e^{\sum_{j=1}^{n} b_j(z_j-c_j)+b_{n+1}} + A_2 K_2 e^{-\sum_{j=1}^{n} b_j(z_j-c_j)-b_{n+1}}}{\sqrt{2}a_1}$, where $b_j (1 \leq j \leq n+1), K_1, K_2 \in \mathbb{C}$ with $b_1 \neq 0$, $K_1 K_2 = K$, $e^{\sum_{j=1}^{n} b_j c_j} \equiv \frac{(a_3 b_1 b_2 + a_2 b_1)A_1}{a_1 A_2} = \frac{a_1 A_1}{(a_3 b_1 b_2 - a_2 b_1)A_2}$, when each c_i is non-zero for $n \geq 3$, otherwise $f(z)$ is of the form $f(z) = \frac{1}{\sqrt{2}a_1}\left(A_1 K_1 e^{P(z-c)} + A_2 K_2 e^{-P(z-c)}\right)$, where $P(z)$ is a polynomial satisfying $e^{P(z+c)-P(z)} = \frac{A_1}{A_2}\left(\frac{a_3}{a_1}\left\{\frac{\partial^2 P(z)}{\partial z_1 \partial z_2} + \frac{\partial P(z)}{\partial z_1}\frac{\partial P(z)}{\partial z_2}\right\} + \frac{a_2}{a_1}\frac{\partial P(z)}{\partial z_1}\right)$ and $e^{P(z)-P(z+c)} = \frac{A_2}{A_1}\left(\frac{a_3}{a_1}\left\{\frac{\partial P(z)}{\partial z_1}\frac{\partial P(z)}{\partial z_2} - \frac{\partial^2 P(z)}{\partial z_1 \partial z_2}\right\} - \frac{a_2}{a_1}\frac{\partial P(z)}{\partial z_1}\right)$;

(II) If $g(z)$ is a non-constant polynomial in $\mathbb{C}^n$, then $f(z)$ has one of the following forms:

(II_1) $f(z) = \frac{K_3}{a_1} e^{\frac{g(z-c)}{2}}$, where $K_1, K_2, K_3, \xi \in \mathbb{C} \setminus \{0\}$ such that $K_1 K_2 = 1$, $K_3 = \frac{A_1 K_1 \xi + A_2 K_2 \xi^{-1}}{\sqrt{2}}$, $A_2 K_1 \xi + A_1 K_2 \xi^{-1} = 0$ and $g(z)$ is a non-constant polynomial in $\mathbb{C}^n$ with $2a_2\frac{\partial g(z)}{\partial z_1} + a_3\frac{\partial g(z)}{\partial z_1}\frac{\partial g(z)}{\partial z_2} + 2a_3\frac{\partial^2 g(z)}{\partial z_1 \partial z_2} \equiv 0$;

(II_2) $f(z) = \frac{K_3}{a_1} e^{\frac{1}{2}\sum_{j=1}^{n} b_j(z_j-c_j)+\frac{1}{2}b_{n+1}}$, where $b_j, K_1, K_2, K_3(\neq 0), K_4(\neq 0), \xi(\neq 0) \in \mathbb{C}$ for $1 \leq j \leq n+1$ with $b_1 \neq 0$, $K_1 K_2 = 1$, $K_3 = \frac{A_1 K_1 \xi + A_2 K_2 \xi^{-1}}{\sqrt{2}}$, $K_4 = \frac{A_2 K_1 \xi + A_1 K_2 \xi^{-1}}{\sqrt{2}}$ and $e^{\frac{1}{2}\sum_{j=1}^{n} b_j c_j} \equiv \frac{K_3}{K_4}\left(\frac{a_2}{2a_1}b_1 + \frac{a_3}{4a_1}b_1 b_2\right)$ and $g(z) = \sum_{j=1}^{n} b_j z_j + b_{n+1}$, when each c_i is non-zero for $n \geq 3$, otherwise $f(z)$ is of the form of (II_1), where $g(z)$ is a non-constant polynomial satisfying $\frac{K_3}{K_4}\left(\frac{a_2}{2a_1}\frac{\partial g(z)}{\partial z_1} + \frac{a_3}{4a_1}\left\{\frac{\partial g(z)}{\partial z_1}\frac{\partial g(z)}{\partial z_2} + 2\frac{\partial^2 g(z)}{\partial z_1 \partial z_2}\right\}\right) \equiv e^{\frac{g(z+c)-g(z)}{2}}$;

(II_3) $f(z) = \frac{A_1 K_1 e^{\sum_{j=1}^{n} b_j(z_j-c_j)+b_{n+1}} + A_2 K_2 e^{\sum_{j=1}^{n} d_j(z_j-c_j)+d_{n+1}}}{\sqrt{2}a_1}$, where $b_j, d_j, K_1, K_2 \in \mathbb{C}$ for $1 \leq j \leq n+1$ with $b_1 \neq 0$, $d_1 \neq 0$, $K_1 K_2 = 1$, $e^{\sum_{j=1}^{n} b_j c_j} \equiv \frac{(a_3 b_1 b_2 + a_2 b_1)A_1}{a_1 A_2}$, $e^{\sum_{j=1}^{n} d_j c_j} \equiv \frac{(a_3 d_1 d_2 + a_2 d_1)A_2}{a_1 A_1}$ and $g(z) = \sum_{j=1}^{n}(b_j + d_j)z_j + b_{n+1} + d_{n+1}$, when each c_i is non-zero for $n \geq 3$, otherwise $f(z)$ is of the form

$f(z) = \frac{1}{\sqrt{2}a_1}\left(A_1K_1e^{\gamma_1(z-c)} + A_2K_2e^{\gamma_2(z-c)}\right)$, *where* $\gamma_k(z)$ $(k = 1, 2)$ *are polynomials satisfying* $e^{\gamma_k(z+c)-\gamma_k(z)} = \alpha_k\left(\frac{a_3}{a_1}\left\{\frac{\partial^2\gamma_k}{\partial z_1\partial z_2} + \frac{\partial\gamma_k}{\partial z_1}\frac{\partial\gamma_k}{\partial z_2}\right\} + \frac{a_2}{a_1}\frac{\partial\gamma_k}{\partial z_1}\right)$ *with* $\alpha_1 = A_1/A_2 = \alpha_2^{-1}$.

Remark 1.23. In particular, if we choose $g(z) \equiv 0$ in Theorem 1.22, then $K = 1$ and (1.11) becomes (1.8). If $K_1 = K_2 = 1$ and $n = 2$, then from (I) of Theorem 1.22, we have $f(z)$ is in form of (1.14) with $b_j = id_j \in \mathbb{C}$ for $1 \le j \le 3$ such that $d_1 \ne 0$, $a_1^2 - d_1^2\left(a_3^2d_2^2 + a_2^2\right) = 0$ and $e^{2i\sum_{j=1}^n d_jc_j} = \frac{(a_3d_2-ia_2)A_1^2}{(a_3d_2+ia_2)A_2^2}$. In this sense, Theorem 1.22 is a significant improvement of Theorem 1.13.

The main tool of this paper is to consider both difference and differential operator in the functional equations by making use of Nevanlinna theory and the recent result on difference logarithmic derivative lemma of several complex variables [7, 20]. For recent development of the topic, we refer to the articles [1, 2, 4, 5, 9, 14] and references therein.

2. Some Lemmas

In the above section, we have studied some recent significant contributions related to transcendental entire solutions of some binomial and trinomial partial differential and differential-difference functional equations in several complex variables [8, 13, 25, 28, 37, 38, 40, 41, 39]. Note that, all the researchers used the following lemma to prove their results.

Lemma 2.1. *[30] If g and h are entire functions on $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*

(i) the internal function h is a polynomial and the external function g is of finite order; or else

(ii) the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.

Remark 2.2. Clearly the Lemma 2.1 holds on $\mathbb{C}$ but researchers used Lemma 2.1 to prove the results on $\mathbb{C}^n$. We think Lemma 2.1 may be true on $\mathbb{C}^n$ but it should be proved on $\mathbb{C}^n$ before it is used. To avoid this kind of ambiguity, we use here some other supplementary results in $\mathbb{C}^n$.

The following are relevant lemmas of this paper and are used in the sequel.

Lemma 2.3. [*[18], Lemma 3.1*] *Let $f_j \not\equiv 0$ $(j = 1, 2, 3)$ be meromorphic functions on $\mathbb{C}^n$ such that f_1 is not constant, and $f_1 + f_2 + f_3 \equiv 1$, and such that*

$$\sum_{j=1}^{3}\left\{N_2(r, 0; f_j) + 2\overline{N}(r, f_j)\right\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)) \textit{ holds,}$$

where $\lambda < 1$ is a positive number. Then either $f_2 \equiv 1$ or $f_3 \equiv 1$.

Lemma 2.4. *[22, 31, 36] For an entire function F on $\mathbb{C}^n$, $F(0) \neq 0$ and put $\rho(n_F) = \rho < \infty$. Then there exist a canonical function f_F and a function $g_F \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, f_F is the canonical product of Weierstrass.*

Lemma 2.5. [*[18], Lemma 3.58*] *If g is a transcendental entire function on $\mathbb{C}^m$ and if f is a meromorphic function of positive order on $\mathbb{C}$, then $f \circ g$ is of infinite order.*

Lemma 2.6. [*[18], Lemma 3.59*] *Let P be a non-constant entire function in $\mathbb{C}^m$. Then, $order(e^P) = \begin{cases} \deg(P) & : \text{if } P \text{ is a polynomial,} \\ +\infty & : \text{otherwise.} \end{cases}$*

Lemma 2.7. [*[18], Theorem 1.106*] *Suppose that $a_0(z), a_1(z), \ldots, a_n(z)$ ($n \geq 1$) are meromorphic functions on $\mathbb{C}^m$ and $g_0(z), g_1(z), \ldots, g_n(z)$ are entire functions on $\mathbb{C}^m$ such that $g_j(z) - g_k(z)$ are not constants for $0 \leq j < k \leq n$. If the following conditions $\sum_{j=0}^{n} a_j(z)e^{g_j(z)} \equiv 0$ and $T(r, a_j) = o(T(r))$, $j = 0, 1, \ldots, n$ hold, where $T(r) = \min_{0 \leq j < k \leq n} T(r, e^{g_j - g_k})$, then $a_j(z) \equiv 0$ ($j = 0, 1, 2, \ldots, n$).*

Lemma 2.8. [*[6], Lemma 3.2*] *Let f be a non-constant meromorphic function on $\mathbb{C}^n$. Then for any $I \in \mathbb{Z}_+^n$, $T(r, \partial^I f) = O(T(r, f))$ for all r except possibly a set of finite Lebesgue measure, and where $I = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_+^n$ denotes a multiple index with $|I| = i_1 + i_2 + \cdots + i_n$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $\partial^I f = \frac{\partial^{|I|} f}{\partial z_1^{i_1} \cdots \partial z_n^{i_n}}$.*

3. Proofs of main theorems

Proof of Theorem 1.14. Let $f \in \mathscr{E}_T^{<\infty}(\mathbb{C}^n)$ satisfy (1.9), where $g(z)$ is a polynomial in $\mathbb{C}^n$. Now the following circumstances arise.

(I) Suppose $g \in \mathbb{C}$ and $e^g \equiv K \in \mathbb{C} \setminus \{0\}$. Let $a_1 \frac{\partial f(z)}{\partial z_1} = \frac{1}{\sqrt{2}}(U + V)$, $a_2 f(z) + a_3 f(z + c) + a_4 \frac{\partial^2 f(z)}{\partial z_1^2} = \frac{1}{\sqrt{2}}(U - V)$, where $U, V \in \mathscr{E}^{<\infty}(\mathbb{C}^n)$. Now (1.9) can be written as

$$(3.1) \qquad \begin{aligned} &(1 + \alpha)U^2 + (1 - \alpha)V^2 = K \\ &\Rightarrow \left(\sqrt{1 + \alpha}U + i\sqrt{1 - \alpha}V\right)\left(\sqrt{1 + \alpha}U - i\sqrt{1 - \alpha}V\right) = K. \end{aligned}$$

As $U, V \in \mathscr{E}^{<\infty}(\mathbb{C}^n)$, so $\sqrt{1 + \alpha}U \pm i\sqrt{1 - \alpha}V \in \mathscr{E}^{<\infty}(\mathbb{C}^n)$ and have no zeros in $\mathbb{C}^n$. So in view of the Lemma 2.4 from (3.1), we have

$$\sqrt{1 + \alpha}U + i\sqrt{1 - \alpha}V = K_1 e^{P(z)} \quad \text{and} \quad \sqrt{1 + \alpha}U - i\sqrt{1 - \alpha}V = K_2 e^{-P(z)},$$

where $K_1, K_2 \in \mathbb{C} \setminus \{0\}$ such that $K_1 K_2 = K$ and $P(z)$ is an entire function in $\mathbb{C}^n$. Thus, we have

$$(3.2) \qquad \sqrt{1 + \alpha}U = \frac{K_1 e^{P(z)} + K_2 e^{-P(z)}}{2} \quad \text{and} \quad \sqrt{1 - \alpha}V = \frac{K_1 e^{P(z)} - K_2 e^{-P(z)}}{2i}.$$

Since $U, V \in \mathscr{E}^{<\infty}(\mathbb{C}^n)$, so in view of Lemmas 2.5, 2.6 and 2.8, we get from (3.2) that $P(z)$ is a polynomial in $\mathbb{C}^n$. The following cases arise separately.

Case 1. If $P(z)$ is constant, then from (3.2), we have $\sqrt{1+\alpha}U$ and $\sqrt{1-\alpha}V$ are both constants, say ϕ_1 and ϕ_2 respectively, where $\phi_1, \phi_2 \in \mathbb{C}$ with $\phi_1^2+\phi_2^2 = K$. Thus

$$(3.3) \qquad a_1\frac{\partial f(z)}{\partial z_1} = \frac{1}{\sqrt{2}}\left(\frac{\phi_1}{\sqrt{1+\alpha}} + \frac{\phi_2}{\sqrt{1-\alpha}}\right) = \phi_3 \text{ and}$$

$$(3.4) \qquad a_2 f(z) + a_3 f(z+c) + a_4\frac{\partial^2 f}{\partial z_1^2} = \frac{1}{\sqrt{2}}\left(\frac{\phi_1}{\sqrt{1+\alpha}} - \frac{\phi_2}{\sqrt{1-\alpha}}\right) = \phi_4,$$

where $\phi_3, \phi_4 \in \mathbb{C}$ with $\phi_3^2 + 2\alpha\phi_3\phi_4 + \phi_4^2 = K$. From (3.3) and (3.4), we have $f(z+c) = -\frac{a_2}{a_3}f(z) + \frac{\phi_4}{a_3}$. Since $\frac{\partial f}{\partial z_1} = \frac{\phi_3}{a_1}$, we express $f(z)$ as

$$f(z) = \begin{cases} G_1(y_1) + \frac{\phi_4}{a_3\tau}\omega + k_0, & \text{if } a_2 + a_3 = 0, \\ e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)} G_2(y_1) + \frac{\phi_4}{a_2+a_3}, & \text{if } a_2 + a_3 \neq 0, \end{cases}$$

where $k_0 \in \mathbb{C}$, $\omega = \sum_{j=1}^n z_j$, $\tau = \sum_{j=1}^n c_j \neq 0$, $\tau_1 = \sum_{j=2}^n c_j \neq 0$, $G_1(y_1)$ (resp. $G_2(y_1)$) is a finite order transcendental entire (resp. a finite order entire) periodic function in $z_2, z_3, \cdots, z_n$ with period $s_1 \in \mathbb{C}^{n-1} \setminus \mathbb{O}$ and $\phi_3 = \begin{cases} \frac{a_1\phi_4}{a_3\tau}, & \text{if } a_2 + a_3 = 0, \\ 0, & \text{if } a_2 + a_3 \neq 0. \end{cases}$

Case 2. If $P(z)$ is a non-constant polynomial in $\mathbb{C}^n$, then from (3.2), we get

$$(3.5) \qquad \frac{\partial f(z)}{\partial z_1} = \frac{1}{\sqrt{2}a_1}\left(A_1K_1e^{P(z)} + A_2K_2e^{-P(z)}\right) \text{ and}$$

$$(3.6) \qquad a_2 f(z) + a_3 f(z+c) + a_4\frac{\partial^2 f}{\partial z_1^2} = \frac{1}{\sqrt{2}}\left(A_2K_1e^{P(z)} + A_1K_2e^{-P(z)}\right),$$

where A_1, A_2 are in (1.13). Differentiating partially with respect to z_1 on both sides of the first equation of (3.5), we get

$$(3.7) \qquad \frac{\partial^2 f(z)}{\partial z_1^2} = \frac{1}{\sqrt{2}a_1}\left(A_1K_1e^{p(z)} - A_2K_2e^{-P(z)}\right)\frac{\partial P(z)}{\partial z_1}.$$

From (3.6) and (3.7), we deduce that

$$(3.8) \qquad a_2 f(z) + a_3 f(z+c) = K_1\left(\frac{A_2}{\sqrt{2}} - \frac{a_4A_1}{\sqrt{2}a_1}\frac{\partial P(z)}{\partial z_1}\right)e^{P(z)} + K_2\left(\frac{A_1}{\sqrt{2}} + \frac{a_4A_2}{\sqrt{2}a_1}\frac{\partial P(z)}{\partial z_1}\right)e^{-P(z)}.$$

Differentiating partially (3.8) with respect to z_1, we get

$$(3.9) \qquad a_2\frac{\partial f(z)}{\partial z_1} + a_3\frac{\partial f(z+c)}{\partial z_1} = K_1e^{P(z)}\left(\frac{A_2}{\sqrt{2}}\frac{\partial P(z)}{\partial z_1} - \frac{a_4A_1}{\sqrt{2}a_1}\left(\frac{\partial P(z)}{\partial z_1}\right)^2 - \frac{a_4A_1}{\sqrt{2}a_1}\times\frac{\partial^2 P(z)}{\partial z_1^2}\right) + K_2e^{-P(z)}\left(\frac{a_4A_2}{\sqrt{2}a_1}\frac{\partial^2 P(z)}{\partial z_1^2} - \frac{A_1}{\sqrt{2}}\frac{\partial P(z)}{\partial z_1} - \frac{a_4A_2}{\sqrt{2}a_1}\left(\frac{\partial P(z)}{\partial z_1}\right)^2\right).$$

From (3.5) and (3.9), we get

$$(3.10) \qquad M(z)e^{P(z)-P(z+c)} + N(z)e^{-P(z)-P(z+c)} - \frac{A_2K_2}{A_1K_1}e^{-2P(z+c)} \equiv 1,$$

where

$$M(z)=\frac{a_1}{a_3}\left(\frac{A_2}{A_1}\frac{\partial P(z)}{\partial z_1}-\frac{a_4}{a_1}\left(\frac{\partial P(z)}{\partial z_1}\right)^2-\frac{a_4}{a_1}\frac{\partial^2 P(z)}{\partial z_1^2}-\frac{a_2}{a_1}\right)\quad\text{and}\tag{3.11}$$

$$N(z)=\frac{a_1K_2}{a_3K_1}\left(\frac{a_4A_2}{a_1A_1}\frac{\partial^2 P(z)}{\partial z_1^2}-\frac{\partial P(z)}{\partial z_1}-\frac{a_4A_2}{a_1A_1}\left(\frac{\partial P(z)}{\partial z_1}\right)^2-\frac{a_2A_2}{a_1A_1}\right).\tag{3.12}$$

From (3.10), it is clear that both $M(z)$ and $N(z)$ are not simultaneously identically zero, otherwise we arise at a contradiction. Let $M_1(z)\not\equiv 0$ and $N_1(z)\equiv 0$. Then from (3.10), we get

$$M_1(z)e^{P(z)-P(z+c)}-\frac{A_2K_2}{A_1K_1}e^{-2P(z+c)}\equiv 1$$

$$\Rightarrow\qquad M_1(z)e^{P(z)}-\frac{A_2K_2}{A_1K_1}e^{-P(z+c)}-e^{P(z+c)}\equiv 0.\tag{3.13}$$

From (3.13), it is easy to see that $P(z)-P(z+c)$ is a non-constant polynomial, otherwise $e^{-2P(z+c)}$ is identically a constant, which contradicts the fact that $P(z)$ is a non-constant polynomial. Also, $P(z)+P(z+c)$ is a non-constant polynomial. In view of Lemma 2.7 and (3.13), we get $M_1(z)\equiv 0$, which is a contradiction. Similarly, we get a contradiction whenever $M(z)\equiv 0$ and $N(z)\not\equiv 0$. Therefore, $M(z)\not\equiv 0$ and $N(z)\not\equiv 0$. Since $e^{-P(z)-P(z+c)}$ and $e^{-2P(z+c)}$ are not identically constants and it is easy to see that

$$\begin{aligned}&N\left(r,M(z)e^{P(z)-P(z+c)}\right)=N\left(r,N(z)e^{-P(z)-P(z+c)}\right)\\&=N\left(r,-\frac{A_2K_2}{A_1K_1}e^{-2P(z+c)}\right)=N\left(r,0;M(z)e^{P(z)-P(z+c)}\right)\\&=N\left(r,0;N(z)e^{-P(z)-P(z+c)}\right)=N\left(r,0;-\frac{A_2K_2}{A_1K_1}e^{-2P(z+c)}\right)=S\left(r,\xi(z)\right),\end{aligned}$$

where $\xi(z)$ is either $\frac{A_2K_2}{A_1K_1}e^{-2P(z+c)}$ or $N(z)e^{-P(z)-P(z+c)}$. In view of Lemma 2.3 and (3.10), we get

$$M(z)e^{P(z)-P(z+c)}\equiv 1\Rightarrow e^{P(z+c)-P(z)}\equiv M(z).\tag{3.14}$$

In view of (3.10) and (3.14), we obtain

$$N(z)e^{-P(z)-P(z+c)}\equiv\frac{A_2K_2}{A_1K_1}e^{-2P(z+c)}\Rightarrow e^{P(z)-P(z+c)}\equiv\frac{A_1K_1}{A_2K_2}N(z).\tag{3.15}$$

From (3.14) and (3.15), it is clear that $P(z+c)-P(z)$ is a constant, say k. Since each c_i is non-zero for $n\geq 3$, it is easy to see that $P(z)=\sum_{j=1}^n b_jz_j+g_1(t)+b_{n+1}$, where $g_1(t)$ is a polynomial in $t:=d_1z_1+d_2z_2+\ldots+d_nz_n$ such that $d_1c_1+d_2c_2+\ldots+d_nc_n=0$, $b_i,d_i,b_{n+1}\in\mathbb{C}$ $(1\leq i\leq n)$. From (3.14), we have

$$\frac{A_2}{A_1}(b_1+d_1g_1'(t))-\frac{a_4}{a_1}\left(b_1+d_1g_1'(t)\right)^2-\frac{a_4d_1^2}{a_1}g_1''(t)-\frac{a_2}{a_1}\equiv\frac{a_3}{a_1}e^{\sum_{j=1}^n b_jc_j}.$$

By comparing the degrees on both sides, we get that $\deg(g_1(t)) \leq 1$. For simplicity, we still denote $P(z) = \sum_{j=1}^{n} b_j z_j + b_{n+1}$, where $b_j \in \mathbb{C}$ $(1 \leq j \leq n+1)$ such that $\sum_{j=1}^{n} b_j c_j = k$. From (3.14) and (3.15), we deduce that

$$(3.16) \qquad \begin{cases} e^{\sum_{j=1}^{n} b_j c_j} = \frac{a_1}{a_3}\left(\frac{A_2 b_1}{A_1} - \frac{a_4 b_1^2}{a_1} - \frac{a_2}{a_1}\right) \text{ and} \\ e^{-\sum_{j=1}^{n} b_j c_j} = -\frac{a_1}{a_3}\left(\frac{A_1 b_1}{A_2} + \frac{a_4}{a_1} b_1^2 + \frac{a_2}{a_1}\right), \end{cases}$$

where $b_j \in \mathbb{C}$ $(1 \leq j \leq n+1)$. Now we will discuss the following two cases.

Sub-case 2.1. If $b_1 = 0$, then from (3.5) and (3.16), we get $a_2 = \pm a_3$ and

$$(3.17) \quad f(z) = \frac{1}{\sqrt{2}a_1}\left(A_1 K_1 e^{\sum_{j=2}^{n} b_j z_j + b_{n+1}} + A_2 K_2 e^{-\sum_{j=2}^{n} b_j z_j - b_{n+1}}\right) z_1 + \Phi(y_1),$$

where $\Phi(y_1)$ is a finite order entire function of $z_2, z_3, \cdots, z_n$. Using (3.16) and (3.17), we get from (3.6) that

$$a_2\Phi(y_1) + a_3\Phi(y_1 + s_1) \equiv \frac{(a_1 A_2 + a_2 c_1 A_1) K_1}{\sqrt{2}a_1} e^{\sum_{j=2}^{n} b_j z_j + b_{n+1}} + \frac{(a_1 A_1 + a_2 c_1 A_2) K_2}{\sqrt{2}a_1} e^{-\sum_{j=2}^{n} b_j z_j - b_{n+1}}.$$

We express $\Phi(y_1)$ as

$$\Phi(y_1) = e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\phi(y_1) - \frac{(a_1 A_2 + a_2 c_1 A_1) K_1}{\sqrt{2}a_1 a_2 \tau_1}\omega_1 e^{\sum_{j=2}^{n} b_j z_j + b_{n+1}} - \frac{(a_1 A_1 + a_2 c_1 A_2) K_2}{\sqrt{2}a_1 a_2 \tau_1}\omega_1 e^{-\sum_{j=2}^{n} b_j z_j - b_{n+1}},$$

where $\tau_1 = \sum_{j=2}^{n} c_j$ and $\phi(y_1)$ is a finite order entire periodic function with period s_1.

Sub-case 2.2. If $b_1 \neq 0$, then from (3.5), we get

$$(3.18) \quad f(z) = \frac{1}{\sqrt{2}a_1 b_1}\left(A_1 K_1 e^{\sum_{j=1}^{n} b_j z_j + b_{n+1}} - A_2 K_2 e^{-\sum_{j=1}^{n} b_j z_j - b_{n+1}}\right) + \Psi(y_1),$$

where $\Psi(y_1)$ is a finite order entire function of $z_2, z_3, \cdots, z_n$. Using (3.16) and (3.18), we get from (3.6) that $\Psi(y_1 + s_1) \equiv -\frac{a_2}{a_3}\Psi(y_1)$. If $a_2 + a_3 = 0$, then $\Psi(y_1)$ is a finite order entire periodic function with period s_1. If $a_2 + a_3 \neq 0$, then we express $\Psi(y_1)$ as $\Psi(y_1) \equiv e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\psi(y_1)$, where $\tau_1 = \sum_{j=2}^{n} c_j$, $\psi(y_1)$ is a finite order transcendental entire periodic function of $z_2, z_3, \cdots, z_n$ with period $s_1 \in \mathbb{C}^{n-1} \setminus \mathbb{O}$.

(II) Suppose $g(z)$ is a non-constant polynomial in $\mathbb{C}^n$. Let $a_1\frac{\partial f(z)}{\partial z_1} = \frac{1}{\sqrt{2}}(U + V)$, $a_2 f(z) + a_3 f(z+c) + a_4\frac{\partial^2 f(z)}{\partial z_1^2} = \frac{1}{\sqrt{2}}(U - V)$, where $U, V \in \mathscr{E}^{<\infty}(\mathbb{C}^n)$. Now (1.9) can be written as

$$(3.19) \qquad \left(\frac{\sqrt{1+\alpha}U}{e^{\frac{g(z)}{2}}} + i\frac{\sqrt{1-\alpha}V}{e^{\frac{g(z)}{2}}}\right)\left(\frac{\sqrt{1+\alpha}U}{e^{\frac{g(z)}{2}}} - i\frac{\sqrt{1-\alpha}V}{e^{\frac{g(z)}{2}}}\right) = 1.$$

As $U, V \in \mathscr{E}^{<\infty}(\mathbb{C}^n)$, so $\frac{\sqrt{1+\alpha}U}{e^{\frac{g(z)}{2}}} \pm i\frac{\sqrt{1-\alpha}V}{e^{\frac{g(z)}{2}}} \in \mathscr{E}_T^{<\infty}(\mathbb{C}^n)$ and have no zeros in $\mathbb{C}^n$. So, in view of Lemma 2.4 and (3.19), we get

$$\frac{\sqrt{1+\alpha}U}{e^{\frac{g(z)}{2}}} + i\frac{\sqrt{1-\alpha}V}{e^{\frac{g(z)}{2}}} = K_1 e^{P(z)} \text{ and } \frac{\sqrt{1+\alpha}U}{e^{\frac{g(z)}{2}}} - i\frac{\sqrt{1-\alpha}V}{e^{\frac{g(z)}{2}}} = K_2 e^{-P(z)},$$

where $K_1, K_2 \in \mathbb{C}\setminus\{0\}$ such that $K_1K_2 = 1$ and $P(z)$ is an entire function in $\mathbb{C}^n$. Thus, we have
(3.20)

$$\sqrt{1+\alpha}U = \frac{K_1e^{\gamma_1(z)} + K_2e^{\gamma_2(z)}}{2} \text{ and } \sqrt{1-\alpha}V = \frac{K_1e^{\gamma_1(z)} - K_2e^{\gamma_2(z)}}{2i},$$

where $\gamma_1(z) = P(z) + \frac{g(z)}{2}$ and $\gamma_2(z) = -P(z) + \frac{g(z)}{2}$. Since $U, V \in \mathscr{E}^{<\infty}(\mathbb{C}^n)$ and $g(z)$ is a non-constant polynomial in $\mathbb{C}^n$, so in view of Lemmas 2.5, 2.6 and 2.8, we get from (3.20) that $P(z)$ is a polynomial in $\mathbb{C}^n$. Thus from (3.20), we deduce that

$$(3.21) \qquad a_1\frac{\partial f(z)}{\partial z_1} = \frac{1}{\sqrt{2}}\left(\frac{K_1e^{\gamma_1(z)} + K_2e^{\gamma_2(z)}}{2\sqrt{1+\alpha}} + \frac{K_1e^{\gamma_1(z)} - K_2e^{\gamma_2(z)}}{2i\sqrt{1-\alpha}}\right)$$
$$= \frac{1}{\sqrt{2}}\left(A_1K_1e^{\gamma_1(z)} + A_2K_2e^{\gamma_2(z)}\right) \text{ and}$$

$$(3.22) \qquad a_2f(z) + a_3f(z+c) + a_4\frac{\partial^2 f}{\partial z_1^2} = \frac{1}{\sqrt{2}}\left(A_2K_1e^{\gamma_1(z)} + A_1K_2e^{\gamma_2(z)}\right),$$

where A_1, A_2 are in (1.13). Differentiating partially with respect to z_1 on both sides of the first equation of (3.21), we get

$$(3.23) \qquad \frac{\partial^2 f(z)}{\partial z_1^2} = \frac{A_1K_1e^{\gamma_1(z)}\frac{\partial \gamma_1(z)}{\partial z_1} + A_2K_2e^{\gamma_2(z)}\frac{\partial \gamma_2(z)}{\partial z_1}}{\sqrt{2}a_1}.$$

Using (3.23), we deduce from (3.22) that

$$(3.24) \qquad a_2f(z) + a_3f(z+c) = K_1e^{\gamma_1(z)}\left(\frac{A_2}{\sqrt{2}} - \frac{a_4A_1}{\sqrt{2}a_1}\frac{\partial\gamma_1(z)}{\partial z_1}\right)$$
$$+K_2e^{\gamma_2(z)}\left(\frac{A_1}{\sqrt{2}} - \frac{a_4A_2}{\sqrt{2}a_1}\frac{\partial\gamma_2(z)}{\partial z_1}\right).$$

Differentiating (3.24) partially with respect to z_1, we get

$$a_2\frac{\partial f(z)}{\partial z_1} + a_3\frac{\partial f(z+c)}{\partial z_1} = K_1e^{\gamma_1(z)}\left(\frac{A_2}{\sqrt{2}}\frac{\partial\gamma_1(z)}{\partial z_1} - \frac{a_4A_1}{\sqrt{2}a_1}\left(\frac{\partial\gamma_1(z)}{\partial z_1}\right)^2 - \frac{a_4A_1}{\sqrt{2}a_1}\times\right.$$
$$(3.25) \left.\frac{\partial^2\gamma_1(z)}{\partial z_1^2}\right) + K_2e^{\gamma_2(z)}\left(\frac{A_1}{\sqrt{2}}\frac{\partial\gamma_2(z)}{\partial z_1} - \frac{a_4A_2}{\sqrt{2}a_1}\left(\frac{\partial\gamma_2(z)}{\partial z_1}\right)^2 - \frac{a_4A_2}{\sqrt{2}a_1}\frac{\partial^2\gamma_2(z)}{\partial z_1^2}\right).$$

From (3.21) and (3.25), we obtain

$$a_2\frac{A_1K_1e^{\gamma_1(z)}+A_2K_2e^{\gamma_2(z)}}{\sqrt{2}a_1}+a_3\frac{A_1K_1e^{\gamma_1(z+c)}+A_2K_2e^{\gamma_2(z+c)}}{\sqrt{2}a_1}$$
$$\equiv K_1e^{\gamma_1(z)}\left(\frac{A_2}{\sqrt{2}}\frac{\partial\gamma_1(z)}{\partial z_1}-\frac{a_4A_1}{\sqrt{2}a_1}\left(\frac{\partial\gamma_1(z)}{\partial z_1}\right)^2-\frac{a_4A_1}{\sqrt{2}a_1}\frac{\partial^2\gamma_1(z)}{\partial z_1^2}\right)$$
$$+K_2e^{\gamma_2(z)}\left(\frac{A_1}{\sqrt{2}}\frac{\partial\gamma_2(z)}{\partial z_1}-\frac{a_4A_2}{\sqrt{2}a_1}\left(\frac{\partial\gamma_2(z)}{\partial z_1}\right)^2-\frac{a_4A_2}{\sqrt{2}a_1}\frac{\partial^2\gamma_2(z)}{\partial z_1^2}\right),\text{ i.e.,}$$

$$M_1(z)e^{\gamma_1(z)-\gamma_1(z+c)}+N_1(z)e^{\gamma_2(z)-\gamma_1(z+c)}-\frac{A_2K_2}{A_1K_1}e^{\gamma_2(z+c)-\gamma_1(z+c)}\equiv 1, \tag{3.26}$$

where

$$M_1(z)=\frac{a_1}{a_3}\left(\frac{A_2}{A_1}\frac{\partial\gamma_1(z)}{\partial z_1}-\frac{a_4}{a_1}\left(\frac{\partial\gamma_1(z)}{\partial z_1}\right)^2-\frac{a_4}{a_1}\frac{\partial^2\gamma_1(z)}{\partial z_1^2}-\frac{a_2}{a_1}\right)\text{ and} \tag{3.27}$$

$$N_1(z)=\frac{a_1K_2}{a_3K_1}\left(\frac{\partial\gamma_2(z)}{\partial z_1}-\frac{a_4A_2}{a_1A_1}\left(\frac{\partial\gamma_2(z)}{\partial z_1}\right)^2-\frac{a_4A_2}{a_1A_1}\frac{\partial^2\gamma_2(z)}{\partial z_1^2}-\frac{a_2A_2}{a_1A_1}\right). \tag{3.28}$$

We will discuss the following cases.
Case 1. If $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a constant, i.e., $\gamma_2(z+c)-\gamma_1(z+c)$ is a constant, say $k\in\mathbb{C}$. Then $-P(z+c)+\frac{g(z+c)}{2}-P(z+c)-\frac{g(z+c)}{2}\equiv k\Rightarrow P(z+c)\equiv-\frac{k}{2}$, a constant. From (3.21) and (3.22), we have

$$a_1\frac{\partial f}{\partial z_1}=K_3e^{\frac{g(z)}{2}}\text{ and }a_2f(z)+a_3f(z+c)+a_4\frac{\partial^2 f}{\partial z_1^2}=K_4e^{\frac{g(z)}{2}}, \tag{3.29}$$

where $K_3=\frac{A_1K_1\xi+A_2K_2\xi^{-1}}{\sqrt{2}}$, $K_4=\frac{A_2K_1\xi+A_1K_2\xi^{-1}}{\sqrt{2}}$, $e^{-\frac{k}{2}}=\xi(\neq 0)$. If $K_3=0$, then by Case 1 of (I), $f(z)$ is easily obtainable. Therefore we consider $K_3\neq 0$. From (3.29), we deduce that

$$\frac{a_1}{a_3K_3}\left(\frac{K_4}{2}\frac{\partial g(z)}{\partial z_1}-\frac{a_4K_3}{4a_1}\left(\frac{\partial g(z)}{\partial z_1}\right)^2-\frac{a_4K_3}{2a_1}\frac{\partial^2 g(z)}{\partial z_1^2}-\frac{a_2K_3}{a_1}\right)\equiv e^{\frac{g(z+c)-g(z)}{2}}. \tag{3.30}$$

Since $g(z)$ is a non-constant polynomial, from (3.30) it is clear that $e^{\frac{g(z+c)-g(z)}{2}}$ must be a constant, i.e., $g(z+c)-g(z)$ is a constant, say $K_5\in\mathbb{C}$. By using similar arguments as Case 2 of (I), we have $g(z)=\sum_{j=1}^n b_jz_j+b_{n+1}$, where $b_j\in\mathbb{C}$ for $1\leq j\leq n+1$ with $\sum_{j=1}^n b_jc_j=K_5$. From (3.29) and (3.30), we deduce that

$$\frac{\partial f}{\partial z_1}=\frac{K_3}{a_1}e^{\frac{1}{2}\sum_{j=1}^n b_jz_j+\frac{1}{2}b_{n+1}} \tag{3.31}$$

$$\text{and}\quad\frac{a_1}{a_3K_3}\left(\frac{K_4b_1}{2}-\frac{a_4K_3}{4a_1}b_1^2-\frac{a_2K_3}{a_1}\right)\equiv e^{\frac{1}{2}\sum_{j=1}^n b_jc_j}. \tag{3.32}$$

The following cases arise separately.
Sub-case 1.1. If $b_1=0$, then from (3.31) and (3.32), we get

$$f(z)=\frac{K_3}{a_1}z_1e^{\frac{1}{2}\sum_{j=2}^n b_jz_j+\frac{1}{2}b_{n+1}}+\Psi_1(y_1)\text{ and }e^{\frac{1}{2}\sum_{j=2}^n b_jc_j}=-a_2/a_3,$$

where $\Psi_1(y_1)$ is a finite order entire function of $z_2, z_3 \ldots, z_n$. From (3.29), we deduce that $a_2\Psi_1(y_1) + a_3\Psi_1(y_1 + s_1) \equiv \frac{(a_1K_4+a_2c_1K_3)}{a_1} e^{\frac{1}{2}\sum_{j=2}^{n} b_j z_j + \frac{1}{2}b_{n+1}}$. We express $\Psi_1(y_1)$ as

$$\Psi_1(y_1) = e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\Psi_2(y_1) - \left(\frac{K_4}{a_2\tau_1} + \frac{K_3c_1}{a_1\tau_1}\right)\omega_1 e^{\frac{1}{2}\sum_{j=2}^{n} b_j z_j + \frac{1}{2}b_{n+1}},$$

where $\tau_1 = \sum_{j=2}^{n} c_j$, $\Psi_2(y_1)$ is a finite order entire periodic function of $z_2, z_3, \ldots, z_n$ with period $s_1 \in \mathbb{C}^{n-1} \setminus \mathbb{O}$.

Sub-case 1.2. If $b_1 \neq 0$, then from (3.31), we deduce that

$$f(z) = \frac{2K_3}{a_1b_1} e^{\frac{1}{2}\sum_{j=1}^{n} b_j z_j + \frac{1}{2}b_{n+1}} + g_2(y_1), \tag{3.33}$$

where $g_2(y_1)$ is a finite order entire function of $z_2, z_3, \ldots, z_n$. Using (3.32), (3.33) and from the second equation of (3.29), we deduce that $g_2(y_1 + s_1) \equiv -\frac{a_2}{a_3}g_2(y_1)$. We express $g_2(y_1)$ as $g_2(y_1) \equiv e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\Psi_3(y_1)$, where $\tau_1 = \sum_{j=2}^{n} c_j$, $\Psi_3(y_1)$ is a finite order entire periodic function with period s_1.

Case 2. Suppose $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is non-constant. From (3.26), it is clear that both $M_1(z)$ and $N_1(z)$ are not simultaneously identically zero, otherwise we arrive at a contradiction. Let $M_1(z) \not\equiv 0$ and $N_1(z) \equiv 0$. Then from (3.26), we have

$$M_1(z)e^{\gamma_1(z)-\gamma_1(z+c)} - \frac{A_2K_2}{A_1K_1}e^{\gamma_2(z+c)-\gamma_1(z+c)} \equiv 1,$$

$$\text{i.e., } M_1(z)e^{\gamma_1(z)} - \frac{A_2K_2}{A_1K_1}e^{\gamma_2(z+c)} - e^{\gamma_1(z+c)} \equiv 0. \tag{3.34}$$

From (3.34), it is clear that $\gamma_1(z) - \gamma_1(z+c)$ is not a constant. We also claim that $\gamma_2(z+c) - \gamma_1(z)$ is not a constant. If not, let $\gamma_2(z+c) - \gamma_1(z) \equiv k \Rightarrow \gamma_2(z+c) \equiv \gamma_1(z) + k$. Then from (3.34), we have

$$\left(M_1(z) - \frac{A_2K_2}{A_1K_1}e^{k}\right)e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1,$$

which is a contradiction, since $\gamma_1(z) - \gamma_1(z+c)$ is not a constant. By using Lemma 2.7 and (3.34), we get $M_1(z) \equiv 0$, which is a contradiction. Similarly, we get a contradiction when $M_1(z) \equiv 0$, $N_1(z) \not\equiv 0$. Hence $M_1(z) \not\equiv 0$ and $N_1(z) \not\equiv 0$. Since $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is non-constant and it is easy to see that

$$\begin{aligned}
&N(r, M_1(z)e^{\gamma_1(z)-\gamma_1(z+c)}) = N(r, N_1(z)e^{\gamma_2(z)-\gamma_1(z+c)})\\
&= N\left(r, \frac{A_2K_2}{A_1K_1}e^{\gamma_2(z+c)-\gamma_1(z+c)}\right) = N(r, 0; M_1(z)e^{\gamma_1(z)-\gamma_1(z+c)})\\
&= N(r, 0; N_1(z)e^{\gamma_2(z)-\gamma_1(z+c)}) = N\left(r, 0; \frac{A_2K_2}{A_1K_1}e^{\gamma_2(z+c)-\gamma_1(z+c)}\right)\\
&= S\left(r, \frac{A_2K_2}{A_1K_1}e^{\gamma_2(z+c)-\gamma_1(z+c)}\right).
\end{aligned}$$

In view of Lemma 2.3 and (3.26), we get that either $M_1(z)e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1$ or $N_1(z)e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1$.

Sub-case 2.1. Let $M_1(z)e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1$. From (3.26), we have

$$\frac{A_1K_1}{A_2K_2}N_1(z)e^{\gamma_2(z)-\gamma_2(z+c)} \equiv 1,$$

where $M_1(z)$, $N_1(z)$ are given in (3.27) and (3.28) respectively. From above two relations, we have both $\gamma_1(z)-\gamma_1(z+c)$ and $\gamma_2(z)-\gamma_2(z+c)$ are constants, say ξ_1 and ξ_2 respectively, where $\xi_1, \xi_2 \in \mathbb{C}$. By using similar arguments as Case 2 of (I), we have $\gamma_1(z) = \sum_{j=1}^n b_jz_j + b_{n+1}$ and $\gamma_2(z) = \sum_{j=1}^n d_jz_j + d_{n+1}$ with $\sum_{j=1}^n b_jc_j = -\xi_1$, $\sum_{j=1}^n d_jc_j = -\xi_2$. Thus,

$$\begin{cases} \frac{a_1}{a_3}\left(\frac{A_2}{A_1}b_1 - \frac{a_4}{a_1}b_1^2 - \frac{a_2}{a_1}\right) \equiv e^{\sum_{j=1}^n b_jc_j}, \\ \frac{a_1}{a_3}\left(\frac{A_1}{A_2}d_1 - \frac{a_4}{a_1}d_1^2 - \frac{a_2}{a_1}\right) \equiv e^{\sum_{j=1}^n d_jc_j}. \end{cases} \tag{3.35}$$

From (3.21), we have

$$\frac{\partial f}{\partial z_1} = \left(A_1K_1e^{\sum_{j=1}^n b_jz_j+b_{n+1}} + A_2K_2e^{\sum_{j=1}^n d_jz_j+d_{n+1}}\right)/(\sqrt{2}a_1). \tag{3.36}$$

The following cases arise separately.

Sub-case 2.1.1. If $b_1 = d_1 = 0$, then from (3.35) and (3.36), we deduce that

$$\begin{aligned} f(z) = {} & \frac{A_1K_1e^{\sum_{j=2}^n b_jz_j+b_{n+1}}}{\sqrt{2}a_1}z_1 + \frac{A_2K_2e^{\sum_{j=2}^n d_jz_j+d_{n+1}}}{\sqrt{2}a_1}z_1 + \Psi_2(y_1) \text{ and} \\ & e^{\sum_{j=2}^n b_jc_j} = -a_2/a_3 = e^{\sum_{j=2}^n d_jc_j}, \end{aligned} \tag{3.37}$$

where $\Psi_2(y_1)$ is a finite order entire function of $z_2, z_3, \ldots, z_n$. Using (3.37) in (3.22), we get

$$\begin{aligned} a_2\Psi_2(y_1) + a_3\Psi_2(y_1+s_1) \equiv {} & \frac{(a_1A_2 + a_2c_1A_1)\,K_1e^{\sum_{j=2}^n b_jz_j+b_{n+1}}}{\sqrt{2}a_1} \\ & + \frac{(a_1A_1 + a_2c_1A_2)\,K_2e^{\sum_{j=2}^n d_jz_j+d_{n+1}}}{\sqrt{2}a_1}. \end{aligned}$$

We express $\Psi_2(y_1)$ as

$$\begin{aligned} \Psi_2(y_1) \equiv {} & e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\Psi_3(y_1) - \frac{(a_1A_2 + a_2c_1A_1)\,K_1e^{\sum_{j=2}^n b_jz_j+b_{n+1}}\omega_1}{\sqrt{2}a_1a_2\tau_1} \\ & + \frac{(a_1A_1 + a_2c_1A_2)\,K_2e^{\sum_{j=2}^n d_jz_j+d_{n+1}}\omega_1}{\sqrt{2}a_1a_2\tau_1}, \end{aligned}$$

where $\tau_1 = \sum_{j=2}^n c_j$ and $\Psi_3(y_1)$ is a finite order entire periodic function with period s_1.

Sub-case 2.1.2. If $b_1 = 0, d_1 \neq 0$, then from (3.36), we deduce that

$$f(z) = \frac{A_1K_1e^{\sum_{j=2}^n b_jz_j+b_{n+1}}}{\sqrt{2}a_1}z_1 + \frac{A_2K_2e^{\sum_{j=1}^n d_jz_j+d_{n+1}}}{\sqrt{2}a_1d_1} + \Psi_4(y_1), \tag{3.38}$$

where $\Psi_4(y_1)$ is a finite order entire function of $z_2, z_3, \ldots, z_n$. Similarly, from (3.22) and (3.38), we deduce that

$$\Psi_4(y_1) \equiv e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\Psi_5(y_1) - \frac{(a_1A_2 + a_2c_1A_1)\,K_1e^{\sum_{j=2}^n b_jz_j + b_{n+1}}}{\sqrt{2}a_1a_2\tau_1}\omega_1,$$

where $\tau_1 = \sum_{j=2}^n c_j$ and $\Psi_5(y_1)$ is a finite order entire periodic function with period s_1.

Sub-case 2.1.3. If $b_1 \neq 0$ and $d_1 = 0$, then by using similar arguments as Sub-case 2.1.2 of (II), we deduce from (3.22) and (3.36) that

$$\begin{aligned} f(z) &= \frac{A_1K_1e^{\sum_{j=1}^n b_jz_j + b_{n+1}}}{\sqrt{2}a_1b_1} + \frac{A_2K_2e^{\sum_{j=2}^n d_jz_j + d_{n+1}}}{\sqrt{2}a_1}z_1 \\ &+ e^{\frac{\omega_1}{\tau_1}\log\left(-\frac{a_2}{a_3}\right)}\Psi_6(y_1) - \frac{(a_1A_1 + a_2c_1A_2)\,K_1e^{\sum_{j=2}^n d_jz_j + d_{n+1}}}{\sqrt{2}a_1a_2\tau_1}\omega_1, \end{aligned}$$

$\Psi_6(y_1)$ is a finite order entire periodic function with period s_1.

Sub-case 2.1.4. If $b_1 \neq 0$ and $d_1 \neq 0$, then from (3.36), we deduce that

$$f(z) = \frac{A_1K_1e^{\sum_{j=1}^n b_jz_j + b_{n+1}}}{\sqrt{2}a_1b_1} + \frac{A_2K_2e^{\sum_{j=1}^n d_jz_j + d_{n+1}}}{\sqrt{2}a_1d_1} + \Psi_7(y_1), \tag{3.39}$$

where $\Psi_7(y_1)$ is a finite order entire function of $z_2, z_3, \ldots, z_n$. Using (3.35) and (3.36), we get from (3.24) that $g_6(y_1 + s_1) = -\frac{a_2}{a_3}g_6(y_1)$. We express $\Psi_7(y_1)$ as

$$\Psi_7(y_1) = e^{\frac{\sum_{j=2}^n z_j}{\sum_{j=2}^n c_j}\log\left(-\frac{a_2}{a_3}\right)}\Psi_8(y_1),$$

where $\Psi_8(y_1)$ is a finite order entire periodic function with period $s_1 \in \mathbb{C}^{n-1}\setminus\mathbb{O}$. Also $g(z) = \sum_{j=1}^n (b_j + d_j)z_j + b_{n+1} + d_{n+1}$.

Sub-case 2.2. Let $N_1(z)e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1$. From (3.26), we have

$$\frac{A_1K_1}{A_2K_2}M_1(z)e^{\gamma_1(z)-\gamma_2(z+c)} \equiv 1, \tag{3.40}$$

where $M_1(z)$, $N_1(z)$ are given in (3.27) and (3.28) respectively. Therefore from above two relations, we have both $\gamma_2(z) - \gamma_1(z+c)$ and $\gamma_1(z) - \gamma_2(z+c)$ are constants, say ξ_1 and ξ_2 respectively, where $\xi_1, \xi_2 \in \mathbb{C}$. Now $\gamma_1(z) - \gamma_1(z + 2c) = (\gamma_1(z) - \gamma_2(z+c)) + (\gamma_2(z+c) - \gamma_1(z+2c)) \equiv \xi_1 + \xi_2$ and $\gamma_2(z) - \gamma_2(z+2c) = (\gamma_2(z) - \gamma_1(z+c)) + (\gamma_1(z+c) - \gamma_2(z+2c)) \equiv \xi_1 + \xi_2$. By using similar arguments as Case 2 of (I), we have $\gamma_1(z) = \sum_{j=1}^n b_jz_j + b_{n+1}$ and $\gamma_2(z) = \sum_{j=1}^n d_jz_j + d_{n+1}$ with $\sum_{j=1}^n 2b_jc_j = -\xi_1 - \xi_2$, $\sum_{j=1}^n 2d_jc_j = -\xi_1 - \xi_2$. From (3.40), we have

$$\left(\frac{A_2}{A_1}b_1 - \frac{a_4}{a_1}b_1^2 - \frac{a_2}{a_1}\right)e^{\sum_{j=1}^n (b_j - d_j)z_j + b_{n+1} - d_{n+1}} \equiv \frac{a_3A_2K_2}{a_1A_1K_1}e^{\sum_{j=1}^n d_jc_j}. \tag{3.41}$$

From (3.41), it is clear that $b_j = d_j$ for $1 \leq j \leq n$. Therefore $\gamma_1(z+c) - \gamma_2(z+c) = \sum_{j=1}^n b_j(z_j + c_j) + b_{n+1} - \sum_{j=1}^n b_j(z_j + c_j) - d_{n+1} = b_{n+1} - d_{n+1}$, which is a constant that contradicts the fact that $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is non-constant. This completes the proof. $\square$

It is worth mentioning that, if $\mathcal{P}(z)$ is a polynomial in $\mathbb{C}^2$ such that $\mathcal{P}(z+c)-\mathcal{P}(z)=\xi$, where $\xi\in\mathbb{C}$ and $c\in\mathbb{C}^2\setminus\{(0,0)\}$, then $\mathcal{P}(z)$ must be of the form $\mathcal{P}(z)=a_1z_1+a_2z_2+\varphi(t)+A$, where $\varphi(t)$ is a polynomial in $t:=d_1z_1+d_2z_2$ such that $d_1c_1+d_2c_2=0$, $a_i,d_i,A\in\mathbb{C}$ $(1\le i\le 2)$. Thus $\mathcal{P}(z)$ is not necessarily linear. But in $\mathbb{C}^n(n\ge 3)$ the above conclusion may not hold. For example, let $\mathcal{P}(z)=a_1z_1z_3+a_2z_2z_3$ and $c=(c_1,c_2,c_3)\in\mathbb{C}^3$ such that $c_1\ne 0$, $c_2\ne 0$, $c_3=0$ and $a_1c_1+a_2c_2=0$. Then $\mathcal{P}(z+c)-\mathcal{P}(z)$ is a constant, but we can't express $\mathcal{P}(z)$ as $a_1z_1+a_2z_2+\varphi(t)+A$. Thus in $\mathbb{C}^n$, if $\mathcal{P}(z+c)-\mathcal{P}(z)$ is constant, then we express $\mathcal{P}(z)$ as $\mathcal{P}(z)=\sum_{j=1}^n a_jz_j+\varphi(t)+A$ only when each c_i is non-zero, where $\varphi(t)$ is a polynomial in $t:=\sum_{j=1}^n d_jz_j$ such that $\sum_{j=1}^n d_jc_j=0$, $a_i,d_i,A\in\mathbb{C}$ $(1\le i\le n)$.

Proof of Theorem 1.17. Let $f\in\mathscr{E}_T^{<\infty}(\mathbb{C}^n)$ satisfy (1.12), where $g(z)$ is a polynomial in $\mathbb{C}^n$. Now the following circumstances arise.
(I) Suppose $g(z)$ is a non-constant polynomial. Let $a_1\Delta f(z)+a_2\frac{\partial f(z)}{\partial z_1}=\frac{1}{\sqrt{2}}(U+V)$, $a_3\Delta f(z)+a_4\frac{\partial f(z)}{\partial z_2}=\frac{1}{\sqrt{2}}(U-V)$, where $U,V\in\mathscr{E}^{<\infty}(\mathbb{C}^n)$. By using similar arguments as in (II) of Theorem 1.14., we get

$$a_1\Delta f(z)+a_2\frac{\partial f(z)}{\partial z_1}=\frac{1}{\sqrt{2}}\left(A_1K_1e^{\gamma_1(z)}+A_2K_2e^{\gamma_2(z)}\right)\quad\text{and}\tag{3.42}$$

$$a_3\Delta f(z)+a_4\frac{\partial f(z)}{\partial z_2}=\frac{1}{\sqrt{2}}\left(A_2K_1e^{\gamma_1(z)}+A_1K_2e^{\gamma_2(z)}\right),\tag{3.43}$$

where A_1,A_2 are in (1.13), $K_1,K_2\in\mathbb{C}\setminus\{0\}$ with $K_1K_2=1$, $\gamma_1(z)=P(z)+\frac{g(z)}{2}$, $\gamma_2(z)=-P(z)+\frac{g(z)}{2}$ and $P(z)$ is a polynomial in $\mathbb{C}^n$. From (3.42) and (3.43), we deduce that

$$a_2a_3\frac{\partial f(z)}{\partial z_1}-a_1a_4\frac{\partial f(z)}{\partial z_2}=\frac{a_3A_1-a_1A_2}{\sqrt{2}}K_1e^{\gamma_1(z)}+\frac{a_3A_2-a_1A_1}{\sqrt{2}}K_2e^{\gamma_2(z)}.\tag{3.44}$$

Note that $\frac{\partial^2 f(z)}{\partial z_1\partial z_2}=\frac{\partial^2 f(z)}{\partial z_2\partial z_1}$. Differentiating partially (3.42) and (3.43) with respect to z_2 and z_1 respectively, we get

$$a_1\frac{\partial\Delta f(z)}{\partial z_2}+a_2\frac{\partial^2 f(z)}{\partial z_2\partial z_1}=\frac{A_1K_1e^{\gamma_1(z)}\frac{\partial\gamma_1(z)}{\partial z_2}+A_2K_2e^{\gamma_2(z)}\frac{\partial\gamma_2(z)}{\partial z_2}}{\sqrt{2}}\quad\text{and}\tag{3.45}$$

$$a_3\frac{\partial\Delta f(z)}{\partial z_1}+a_4\frac{\partial^2 f(z)}{\partial z_1\partial z_2}=\frac{A_2K_1e^{\gamma_1(z)}\frac{\partial\gamma_1(z)}{\partial z_1}+A_1K_2e^{\gamma_2(z)}\frac{\partial\gamma_2(z)}{\partial z_1}}{\sqrt{2}}.\tag{3.46}$$

From (3.45) and (3.46), we deduce that

$$\begin{aligned}a_1a_4\frac{\partial\Delta f(z)}{\partial z_2}-a_2a_3\frac{\partial\Delta f(z)}{\partial z_1}&=\left(\frac{a_4A_1}{\sqrt{2}}\frac{\partial\gamma_1(z)}{\partial z_2}-\frac{a_2A_2}{\sqrt{2}}\frac{\partial\gamma_1(z)}{\partial z_1}\right)K_1e^{\gamma_1(z)}\\&+\left(\frac{a_4A_2}{\sqrt{2}}\frac{\partial\gamma_2(z)}{\partial z_2}-\frac{a_2A_1}{\sqrt{2}}\frac{\partial\gamma_2(z)}{\partial z_1}\right)K_2e^{\gamma_2(z)}.\end{aligned}\tag{3.47}$$

Using (3.44), we get from (3.47) that

$$\frac{a_1A_2-a_3A_1}{\sqrt{2}}K_1e^{\gamma_1(z+c)}+\frac{a_1A_1-a_3A_2}{\sqrt{2}}K_2e^{\gamma_2(z+c)}+\frac{a_3A_1-a_1A_2}{\sqrt{2}}K_1e^{\gamma_1(z)}$$
$$+\frac{a_3A_2-a_1A_1}{\sqrt{2}}K_2e^{\gamma_2(z)}\equiv\left(\frac{a_4A_1}{\sqrt{2}}\frac{\partial\gamma_1(z)}{\partial z_2}-\frac{a_2A_2}{\sqrt{2}}\frac{\partial\gamma_1(z)}{\partial z_1}\right)K_1e^{\gamma_1(z)}$$
$$+\left(\frac{a_4A_2}{\sqrt{2}}\frac{\partial\gamma_2(z)}{\partial z_2}-\frac{a_2A_1}{\sqrt{2}}\frac{\partial\gamma_2(z)}{\partial z_1}\right)K_2e^{\gamma_2(z)}\Rightarrow$$

$$\Gamma_1(z)\frac{e^{\gamma_1(z)}}{e^{\gamma_1(z+c)}}-\frac{(a_1A_1-a_3A_2)K_2}{(a_1A_2-a_3A_1)K_1}\frac{e^{\gamma_2(z+c)}}{e^{\gamma_1(z+c)}}+\Gamma_2(z)\frac{e^{\gamma_2(z)}}{e^{\gamma_1(z+c)}}\equiv 1, \tag{3.48}$$

where

$$\Gamma_1(z)=\frac{a_4A_1\frac{\partial\gamma_1(z)}{\partial z_2}-a_2A_2\frac{\partial\gamma_1(z)}{\partial z_1}-(a_3A_1-a_1A_2)}{a_1A_2-a_3A_1}$$

and

$$\Gamma_2(z)=\frac{K_2}{K_1}\frac{a_4A_2\frac{\partial\gamma_2(z)}{\partial z_2}-a_2A_1\frac{\partial\gamma_2(z)}{\partial z_1}-(a_3A_2-a_1A_1)}{a_1A_2-a_3A_1}.$$

We will discuss the following cases.

Case 1. Suppose $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a constant. By using similar arguments as in Case 1 of (II) of Theorem 1.14, we obtain $P(z+c)\equiv-\frac{k}{2}$, where $k\in\mathbb{C}$. From (3.42) and (3.43), we have

$$a_1\Delta f(z)+a_2\frac{\partial f(z)}{\partial z_1}=K_3e^{\frac{g(z)}{2}} \text{ and } a_3\Delta f(z)+a_4\frac{\partial f(z)}{\partial z_2}=K_4e^{\frac{g(z)}{2}}, \tag{3.49}$$

where $K_3=\frac{A_1K_1\xi+A_2K_2\xi^{-1}}{\sqrt{2}}$, $K_4=\frac{A_2K_1\xi+A_1K_2\xi^{-1}}{\sqrt{2}}$, $e^{-\frac{k}{2}}=\xi(\neq 0)$. From (3.49), we have

$$a_2a_3\frac{\partial f(z)}{\partial z_1}-a_1a_4\frac{\partial f(z)}{\partial z_2}=(a_3K_3-a_1K_4)\,e^{\frac{g(z)}{2}}. \tag{3.50}$$

From (3.49) and (3.50), we deduce that

$$e^{\frac{g(z+c)-g(z)}{2}}\equiv\frac{a_4K_3\frac{\partial g(z)}{\partial z_2}-a_2K_4\frac{\partial g(z)}{\partial z_1}+(a_1K_4-a_3K_3)}{a_1K_4-a_3K_3}, \tag{3.51}$$

where $a_1K_4-a_3K_3\neq 0$. Since $g(z)$ is a non-constant polynomial, from (3.51) it is clear that $e^{\frac{g(z+c)-g(z)}{2}}$ must be a constant, i.e., $g(z+c)-g(z)$ is a constant. By using similar arguments as in Case 2 of (I) of Theorem 1.14, we have $g(z)=\sum_{j=1}^n b_jz_j+b_{n+1}$, where $b_j\in\mathbb{C}$ for $1\leq j\leq n+1$. From (3.51), we deduce that

$$e^{\frac{1}{2}\sum_{j=1}^n b_jc_j}-1\equiv\frac{a_4K_3b_2-a_2K_4b_1}{a_1K_4-a_3K_3}. \tag{3.52}$$

The Lagrange's auxiliary equations [35, Chapter 2] of (3.50) are

$$\frac{dz_1}{a_2a_3}=\frac{dz_2}{-a_1a_4}=\frac{dz_3}{0}=\cdots=\frac{dz_n}{0}=\frac{df}{(a_3K_3-a_1K_4)\,e^{\frac{1}{2}\sum_{j=1}^n b_jz_j+\frac{1}{2}b_{n+1}}}.$$

Note that $e_2 = a_1a_4z_1 + a_2a_3z_2$, $e_i = z_i$ $(3 \leq i \leq n)$ and

$$\begin{aligned} df &= \frac{(a_3K_3 - a_1K_4)\, e^{\frac{1}{2}\sum_{j=1}^{n} b_jz_j + \frac{1}{2}b_{n+1}}}{a_2a_3} dz_1 \\ &= \frac{(a_3K_3 - a_1K_4)\, e^{\frac{1}{2}\left(b_1z_1 + b_2\left(\frac{e_2 - a_1a_4z_1}{a_2a_3}\right) + b_3e_3 + \cdots + b_ne_n + b_{n+1}\right)}}{a_2a_3} dz_1 \\ \Rightarrow f(z) &= \frac{2\,(a_3K_3 - a_1K_4)\, e^{\frac{1}{2}\sum_{j=1}^{n} b_jz_j + \frac{1}{2}b_{n+1}}}{b_1a_2a_3 - b_2a_1a_4} + e_1. \end{aligned}$$

Note that after integration with respect to z_1, replacing e_2 by $a_1a_4z_1 + a_2a_3z_2$, e_3 by $z_3, \ldots,$ e_n by z_n, where $e_j \in \mathbb{C}$ $(1 \leq j \leq n)$. Hence the solution is $\Psi(e_1, e_2, \ldots, e_n) = 0$. For simplicity, we suppose

$$f(z) = \frac{2\,(a_3K_3 - a_1K_4)\, e^{\frac{1}{2}\sum_{j=1}^{n} b_jz_j + \frac{1}{2}b_{n+1}}}{(a_2a_3b_1 - a_1a_4b_2)} + h_2(y), \tag{3.53}$$

where $a_2a_3b_1 - a_1a_4b_2 \neq 0$, $a_3K_3 - a_1K_4 \neq 0$, $h_2(y)$ is a finite order entire function in $a_1a_4z_1 + a_2a_3z_2, z_3, \ldots, z_n$ with $a_2a_3\frac{\partial h_2(y)}{\partial z_1} = a_1a_4\frac{\partial h_2(y)}{\partial z_2}$. Using (3.52) and (3.53), we get from (3.49) that

$$h_2(y+s) - h_2(y) = \frac{a_4b_2K_3 - a_2b_1K_4}{a_2a_3b_1 - a_1a_4b_2} e^{\frac{1}{2}\sum_{j=1}^{n} b_jz_j + \frac{1}{2}b_{n+1}} - \frac{a_2}{a_1}\frac{\partial h_2(y)}{\partial z_1}.$$

Case 2. Suppose $e^{\gamma_2(z+c) - \gamma_1(z+c)}$ is non-constant. From (3.48), it is clear that both $\Gamma_1(z)$ and $\Gamma_2(z)$ are not simultaneously identically zero, otherwise we get a contradiction. Using the same arguments as in Case 2 of (II) of Theorem 1.14, we deduce from (3.48) that $\Gamma_1(z) \not\equiv 0$ and $\Gamma_2(z) \not\equiv 0$. Since $e^{\gamma_2(z+c) - \gamma_1(z+c)}$ is non-constant, it is easy to see that

$$\begin{aligned} & N\left(r, \Gamma_1(z)e^{\gamma_1(z) - \gamma_1(z+c)}\right) = N\left(r, 0; \Gamma_1(z)e^{\gamma_1(z) - \gamma_1(z+c)}\right) \\ &= N\left(r, \Gamma_2(z)e^{\gamma_2(z) - \gamma_1(z+c)}\right) = N\left(r, 0, \Gamma_2(z)e^{\gamma_2(z) - \gamma_1(z+c)}\right) \\ &= N\left(r, \frac{(a_1A_1 - a_3A_2)K_2}{(a_1A_2 - a_3A_1)K_1} e^{\gamma_2(z+c) - \gamma_1(z+c)}\right) \\ &= N\left(r, 0; \frac{(a_1A_1 - a_3A_2)K_2}{(a_1A_2 - a_3A_1)K_1} e^{\gamma_2(z+c) - \gamma_1(z+c)}\right) \\ &= S\left(r, \frac{(a_1A_1 - a_3A_2)K_2}{(a_1A_2 - a_3A_1)K_1} e^{\gamma_2(z+c) - \gamma_1(z+c)}\right). \end{aligned}$$

In view of Lemma 2.3 and (3.48), we get that either $\Gamma_1(z)e^{\gamma_1(z) - \gamma_1(z+c)} \equiv 1$ or $\Gamma_2(z)e^{\gamma_2(z) - \gamma_1(z+c)} \equiv 1$. Now the following two cases arise.

Sub-case 2.1. Suppose $\Gamma_1(z)e^{\gamma_1(z) - \gamma_1(z+c)} \equiv 1$. From (3.48), we have

$$\frac{a_4A_2\frac{\partial \gamma_2(z)}{\partial z_2} - a_2A_1\frac{\partial \gamma_2(z)}{\partial z_1} - (a_3A_2 - a_1A_1)}{a_1A_1 - a_3A_2} \equiv e^{\gamma_2(z+c) - \gamma_2(z)}.$$

By using similar arguments as in Sub-case 2.1 of (II) of Theorem 1.14, we obtain $\gamma_1(z) = \sum_{j=1}^{n} b_jz_j + b_{n+1}$, $\gamma_2(z) = \sum_{j=1}^{n} d_jz_j + d_{n+1}$ and $g(z) = \sum_{j=1}^{n}(b_j +$

$d_j)z_j + b_{n+1} + d_{n+1}$, where $b_j, d_j \in \mathbb{C}$ for $1 \leq j \leq n+1$ such that

$$\begin{cases} e^{\sum_{j=1}^{n} b_j c_j} - 1 \equiv \frac{a_4 b_2 A_1 - a_2 b_1 A_2}{a_1 A_2 - a_3 A_1} \\ e^{\sum_{j=1}^{n} d_j c_j} - 1 \equiv \frac{a_4 d_2 A_2 - a_2 d_1 A_1}{a_1 A_1 - a_3 A_2}. \end{cases} \tag{3.54}$$

From (3.44), we have

$$a_2 a_3 \frac{\partial f}{\partial z_1} - a_1 a_4 \frac{\partial f}{\partial z_2} = \frac{a_3 A_1 - a_1 A_2}{\sqrt{2}} K_1 e^{\sum_{j=1}^{n} b_j z_j + b_{n+1}} + \frac{a_3 A_2 - a_1 A_1}{\sqrt{2}} K_2 e^{\sum_{j=1}^{n} d_j z_j + d_{n+1}}. \tag{3.55}$$

The Lagrange's auxiliary equations [35, Chapter 2] of (3.55) are

$$\frac{dz_1}{a_2 a_3} = \frac{dz_2}{-a_1 a_4} = \frac{dz_3}{0} = \cdots = \frac{dz_n}{0} = \frac{df}{\frac{a_3 A_1 - a_1 A_2}{\sqrt{2}} K_1 e^{\sum_{j=1}^{n} b_j z_j + b_{n+1}} + \frac{a_3 A_2 - a_1 A_1}{\sqrt{2}} K_2 e^{\sum_{j=1}^{n} d_j z_j + d_{n+1}}}.$$

Note that $e_2 = a_1 a_4 z_1 + a_2 a_3 z_2$, $e_i = z_i$ $(3 \leq i \leq n)$ and

$$\begin{aligned} df &= \frac{(a_3 A_1 - a_1 A_2) K_1 e^{\sum_{j=1}^{n} b_j z_j + b_{n+1}} + (a_3 A_2 - a_1 A_1) K_2 e^{\sum_{j=1}^{n} d_j z_j + d_{n+1}}}{\sqrt{2} a_2 a_3} dz_1 \\ &= \frac{(a_3 A_1 - a_1 A_2) K_1 e^{b_1 z_1 + b_2 \left(\frac{e_2 - a_1 a_4 z_1}{a_2 a_3}\right) + b_3 e_3 + \cdots + b_n e_n + b_{n+1}}}{\sqrt{2} a_2 a_3} dz_1 \\ &+ \frac{(a_3 A_2 - a_1 A_1) K_2 e^{d_1 z_1 + d_2 \left(\frac{e_2 - a_1 a_4 z_1}{a_2 a_3}\right) + d_3 e_3 + \cdots + d_n e_n + d_{n+1}}}{\sqrt{2} a_2 a_3} dz_1. \end{aligned}$$

Therefore

$$f(z) = \frac{(a_3 A_1 - a_1 A_2) K_1 e^{\sum_{j=1}^{n} b_j z_j + b_{n+1}}}{\sqrt{2}(a_2 a_3 b_1 - a_1 a_4 b_2)} + \frac{(a_3 A_2 - a_1 A_1) K_2 e^{\sum_{j=1}^{n} d_j z_j + d_{n+1}}}{\sqrt{2}(a_2 a_3 d_1 - a_1 a_4 d_2)} + e_1,$$

where $a_2 a_3 b_1 - a_1 a_4 b_2 \neq 0$ and $a_2 a_3 d_1 - a_1 a_4 d_2 \neq 0$. Note that after integration with respect to z_1, replacing e_2 by $a_1 a_4 z_1 + a_2 a_3 z_2$, e_3 by $z_3, \ldots, e_n$ by z_n, where $e_j \in \mathbb{C}$ $(1 \leq j \leq n)$. Hence the solution is $\Psi_1(e_1, e_2, \ldots, e_n) = 0$. For simplicity, we suppose

$$f(z) = \frac{(a_3 A_1 - a_1 A_2) K_1 e^{\sum_{j=1}^{n} b_j z_j + b_{n+1}}}{\sqrt{2}(a_2 a_3 b_1 - a_1 a_4 b_2)} + \frac{(a_3 A_2 - a_1 A_1) K_2 e^{\sum_{j=1}^{n} d_j z_j + d_{n+1}}}{\sqrt{2}(a_2 a_3 d_1 - a_1 a_4 d_2)} + \chi(y), \tag{3.56}$$

where $a_2 a_3 b_1 - a_1 a_4 b_2 \neq 0$, $a_2 a_3 d_1 - a_1 a_4 d_2 \neq 0$, $\chi(y)$ is a finite order entire function in $a_1 a_4 z_1 + a_2 a_3 z_2, z_3, \ldots, z_n$ with $a_2 a_3 \frac{\partial \chi(y)}{\partial z_1} = a_1 a_4 \frac{\partial \chi(y)}{\partial z_2}$. Using (3.54) and (3.56), we get from (3.42) and (3.43) that

$$\chi(y+s) - \chi(y) = -\frac{a_2}{a_1} \frac{\partial \chi(y)}{\partial z_1} = -\frac{a_4}{a_3} \frac{\partial \chi(y)}{\partial z_2}.$$

Sub-case 2.2. Suppose $\Gamma_2(z)e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1$. From (3.48), we have

$$\frac{a_4A_1\frac{\partial\gamma_1(z)}{\partial z_2} - a_2A_2\frac{\partial\gamma_1(z)}{\partial z_1} - (a_3A_1 - a_1A_2)}{a_1A_1 - a_3A_2}\frac{K_1}{K_2}e^{\gamma_1(z)-\gamma_2(z+c)} \equiv 1.$$

Using the same arguments as in Sub-case 2.2 of (II) of Theorem 1.14, we arrive at a contradiction.

(II) Suppose $g \in \mathbb{C}$ and $e^g \equiv K \in \mathbb{C}\setminus\{0\}$. By similar arguments as in (I) of Theorem 1.14, we get

$$a_1\Delta f(z) + a_2\frac{\partial f(z)}{\partial z_1} = \frac{1}{\sqrt{2}}\left(A_1K_1e^{P(z)} + A_2K_2e^{-P(z)}\right) \text{ and} \tag{3.57}$$

$$a_3\Delta f(z) + a_4\frac{\partial f(z)}{\partial z_2} = \frac{1}{\sqrt{2}}\left(A_2K_1e^{P(z)} + A_1K_2e^{-P(z)}\right), \tag{3.58}$$

where A_1, A_2 are in (1.13), $K_1, K_2 \in \mathbb{C}\setminus\{0\}$ with $K_1K_2 = K$ and $P(z)$ is a polynomial in $\mathbb{C}^n$. Now the following cases arise separately.

Case 1. Suppose $P(z)$ is a constant. From (3.57) and (3.58), we have

$$a_1\Delta f(z) + a_2\frac{\partial f(z)}{\partial z_1} = \phi_1 \text{ and } a_3\Delta f(z) + a_4\frac{\partial f(z)}{\partial z_2} = \phi_2, \tag{3.59}$$

where $\phi_1, \phi_2 \in \mathbb{C}$ with $\phi_1^2 + 2\alpha\phi_1\phi_2 + \phi_2^2 = K$. From (3.59), we deduce that

$$a_2a_3\frac{\partial f(z)}{\partial z_1} - a_1a_4\frac{\partial f(z)}{\partial z_2} = a_3\phi_1 - a_1\phi_2. \tag{3.60}$$

The Lagrange's auxiliary equations [35, Chapter 2] of (3.60) are

$$\frac{dz_1}{a_2a_3} = \frac{dz_2}{-a_1a_4} = \frac{dz_3}{0} = \cdots = \frac{dz_n}{0} = \frac{df}{a_3\phi_1 - a_1\phi_2}.$$

Note that $a_1a_4z_1 + a_2a_3z_2 = d_2$, $d_j = z_j$ for $3 \le j \le n$ and $df = \frac{a_3\phi_1 - a_1\phi_2}{a_2a_3}dz_1 \Rightarrow f(z) = \frac{a_3\phi_1 - a_1\phi_2}{a_2a_3}z_1 + d_1$, where $d_j \in \mathbb{C}$ for $1 \le j \le n$. Hence the solution is $\Psi(d_1, d_2, \cdots, d_n) = 0$. For simplicity, we suppose that

$$f(z) = \frac{a_3\phi_1 - a_1\phi_2}{a_2a_3}z_1 + G(y), \tag{3.61}$$

where $G(y)$ is a finite order transcendental entire function in $a_1a_4z_1 + a_2a_3z_2, z_3, \ldots, z_n$ with $a_2a_3\frac{\partial G(y)}{\partial z_1} = a_1a_4\frac{\partial G(y)}{\partial z_2}$. From (3.59) and (3.61), we deduce that

$$\begin{aligned} G(y+s) - G(y) &= \frac{\phi_1}{a_1} - \frac{a_3\phi_1 - a_1\phi_2}{a_2a_3}\left(\frac{a_2}{a_1} + c_1\right) - \frac{a_2}{a_1}\frac{\partial G(y)}{\partial z_1} \\ &= \frac{\phi_2}{a_3} - \frac{a_3\phi_1 - a_1\phi_2}{a_2a_3}c_1 - \frac{a_4}{a_3}\frac{\partial G(y)}{\partial z_2}. \end{aligned}$$

Case 2. Suppose $P(z)$ is a non-constant polynomial in $\mathbb{C}^n$. From (3.57) and (3.58), by using same procedure as in (I), we deduce that

$$a_2a_3\frac{\partial f(z)}{\partial z_1} - a_1a_4\frac{\partial f(z)}{\partial z_2} = \frac{a_3A_1 - a_1A_2}{\sqrt{2}}K_1e^{P(z)} + \frac{a_3A_2 - a_1A_1}{\sqrt{2}}K_2e^{-P(z)} \tag{3.62}$$

and

$$\frac{a_4A_1\frac{\partial P(z)}{\partial z_2}-a_2A_2\frac{\partial P(z)}{\partial z_1}-(a_3A_1-a_1A_2)}{a_1A_2-a_3A_1}e^{P(z)-P(z+c)}$$
$$+\frac{a_2A_1\frac{\partial P(z)}{\partial z_1}-a_4A_2\frac{\partial P(z)}{\partial z_2}-(a_3A_2-a_1A_1)}{a_1A_2-a_3A_1}\frac{K_2}{K_1}e^{-P(z)-P(z+c)}$$
$$-\frac{(a_1A_1-a_3A_2)K_2}{(a_1A_2-a_3A_1)K_1}e^{-2P(z+c)}\equiv 1. \tag{3.63}$$

In view of Lemmas 2.1 and 2.7, and by using similar arguments as in Case 2 of (I) of Theorem 1.14 in (3.63), we deduce that $P(z)=\sum_{j=1}^{n}b_jz_j+b_{n+1}$, where $b_j\in\mathbb{C}$ for $1\leq j\leq n+1$ such that

$$e^{\sum_{j=1}^{n}b_jc_j}-1\equiv\frac{a_4b_2A_1-a_2b_1A_2}{a_1A_2-a_3A_1}\quad\text{and}\quad e^{-\sum_{j=1}^{n}b_jc_j}-1\equiv\frac{a_2b_1A_1-a_4b_2A_2}{a_1A_1-a_3A_2}.$$

From (3.57), (3.58) and (3.62), by using similar procedure as in Sub-case 2.1 of (I), we get

$$f(z)=\frac{(a_3A_1-a_1A_2)K_1e^{\sum_{j=1}^{n}b_jz_j+b_{n+1}}-(a_3A_2-a_1A_1)K_2e^{-\sum_{j=1}^{n}b_jz_j-b_{n+1}}}{\sqrt{2}(a_2a_3b_1-a_1a_4b_2)}+\Psi(y),$$

where $b_j(1\leq j\leq n+1), K_1, K_2\in\mathbb{C}$ with $K_1K_2=K$, $a_2a_3b_1-a_1a_4b_2\neq 0$, $\Psi(y)$ is a finite order entire function in $a_1a_4z_1+a_2a_3z_2, z_3,\ldots,z_n$ with $a_2a_3\frac{\partial\Psi(y)}{\partial z_1}=a_1a_4\frac{\partial\Psi(y)}{\partial z_2}$ and $\Psi(y+s)-\Psi(y)=-\frac{a_2}{a_1}\frac{\partial\Psi(z)}{\partial z_1}=-\frac{a_4}{a_3}\frac{\partial\Psi(z)}{\partial z_2}$. This completes the proof. □

Proof of Theorem 1.19. Let $f\in\mathscr{E}_T^{<\infty}(\mathbb{C}^n)$ satisfy (1.10), where $g(z)$ is a polynomial in $\mathbb{C}^n$. Now the following circumstances arise.
(I) Suppose $g\in\mathbb{C}$ and $e^g\equiv K\in\mathbb{C}\setminus\{0\}$. By using similar arguments of (I) of Theorem 1.14, we get

$$\begin{cases} a_1f(z+c)=\frac{1}{\sqrt{2}}\left(A_1K_1e^{P(z)}+A_2K_2e^{-P(z)}\right) \text{ and}\\ a_2\frac{\partial f(z)}{\partial z_1}+a_3\frac{\partial^2 f(z)}{\partial z_1^2}=\frac{1}{\sqrt{2}}\left(A_2K_1e^{P(z)}+A_1K_2e^{-P(z)}\right),\end{cases}\tag{3.64}$$

where A_1, A_2 are in (1.13), $K_1, K_2\in\mathbb{C}\setminus\{0\}$ such that $K_1K_2=K$, and $P(z)$ is a polynomial in $\mathbb{C}^n$. It is clear that $P(z)$ must be a non-constant polynomial, otherwise it contradicts that $f(z)$ is transcendental. From first and second equations of (3.64), we deduce that

$$\frac{A_1}{A_2}\left(\frac{a_3}{a_1}\left\{\frac{\partial^2P(z)}{\partial z_1^2}+\left(\frac{\partial P(z)}{\partial z_1}\right)^2\right\}+\frac{a_2}{a_1}\frac{\partial P(z)}{\partial z_1}\right)\frac{e^{P(z)}}{e^{P(z+c)}}-\frac{A_1K_2}{A_2K_1}e^{-2P(z+c)}$$
$$+\frac{K_2}{K_1}\left(\frac{a_3}{a_1}\left\{\left(\frac{\partial P(z)}{\partial z_1}\right)^2-\frac{\partial^2P(z)}{\partial z_1^2}\right\}-\frac{a_2}{a_1}\frac{\partial P(z)}{\partial z_1}\right)e^{-P(z)-P(z+c)}\equiv 1. \tag{3.65}$$

In view of Lemmas 2.1 and 2.7, and by using similar arguments as in Case 2 of (I) of Theorem 1.14 in (3.65), we deduce that $P(z)=\sum_{j=1}^{n}b_jz_j+b_{n+1}$, where $b_j\in\mathbb{C}$ for $1\leq j\leq n+1$ such that

$$e^{\sum_{j=1}^{n}b_jc_j}=\frac{(a_3b_1^2+a_2b_1)A_1}{a_1A_2}\text{ and }e^{-\sum_{j=1}^{n}b_jc_j}=\frac{(a_3b_1^2-a_2b_1)A_2}{a_1A_1}. \tag{3.66}$$

From (3.66), it is clear that $b_1 \neq 0$ and from (3.64), we have

$$f(z) = \frac{1}{\sqrt{2}a_1}\left(A_1K_1e^{\sum_{j=1}^n b_j(z_j-c_j)+b_{n+1}} + A_2K_2e^{-\sum_{j=1}^n b_j(z_j-c_j)-b_{n+1}}\right).$$

(II) Suppose $g(z)$ is a non-constant polynomial in $\mathbb{C}^n$. By similar arguments as in (II) of Theorem 1.14, we get

$$\text{(3.67)} \qquad \begin{cases} a_1 f(z+c) = \frac{1}{\sqrt{2}}\left(A_1K_1e^{\gamma_1(z)} + A_2K_2e^{\gamma_2(z)}\right) \text{ and} \\ a_2\frac{\partial f(z)}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1^2} = \frac{1}{\sqrt{2}}\left(A_2K_1e^{\gamma_1(z)} + A_1K_2e^{\gamma_2(z)}\right), \end{cases}$$

where A_1, A_2 are in (1.13), $K_1, K_2 \in \mathbb{C}\setminus\{0\}$ such that $K_1K_2 = 1$, $\gamma_1(z) = P(z)+\frac{g(z)}{2}$, $\gamma_2(z) = -P(z)+\frac{g(z)}{2}$ and $P(z)$ is a polynomial in $\mathbb{C}^n$. From (3.67), we deduce that

$$\text{(3.68)} \quad \Gamma_3(z)e^{\gamma_1(z)-\gamma_1(z+c)} - \frac{A_1K_2}{A_2K_1}e^{\gamma_2(z+c)-\gamma_1(z+c)} + \Gamma_4(z)e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1,$$

where

$$\begin{cases} \Gamma_3(z) = \frac{A_1}{A_2}\left(\frac{a_3}{a_1}\left\{\frac{\partial^2\gamma_1}{\partial z_1^2} + \left(\frac{\partial\gamma_1}{\partial z_1}\right)^2\right\} + \frac{a_2}{a_1}\frac{\partial\gamma_1}{\partial z_1}\right) \text{ and} \\ \Gamma_4(z) = \frac{K_2}{K_1}\left(\frac{a_3}{a_1}\left\{\frac{\partial^2\gamma_2}{\partial z_1^2} + \left(\frac{\partial\gamma_2}{\partial z_1}\right)^2\right\} + \frac{a_2}{a_1}\frac{\partial\gamma_2}{\partial z_1}\right). \end{cases}$$

Now we discuss following cases.

Case 1. Suppose $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is a constant. By using similar arguments as in Case 1 of (II) of Theorem 1.14, we obtain $P(z+c) \equiv -\frac{k}{2}$, where $k \in \mathbb{C}$. From (3.67), we have

$$\text{(3.69)} \quad a_1 f(z+c) = K_3e^{\frac{g(z)}{2}} \text{ and } a_2\frac{\partial f(z)}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1^2} = K_4e^{\frac{g(z)}{2}},$$

where $K_3 = \frac{A_1K_1\xi + A_2K_2\xi^{-1}}{\sqrt{2}}$, $K_4 = \frac{A_2K_1\xi + A_1K_2\xi^{-1}}{\sqrt{2}}$, $e^{-\frac{k}{2}} = \xi \in \mathbb{C}\setminus\{0\}$. Clearly $K_3 \neq 0$. Now the following cases arise separately.

Sub-case 1.1. If $K_4 = 0$, then from (3.69), we deduce that $f(z) = \frac{K_3}{a_1}e^{\frac{g(z-c)}{2}}$, where $g(z)$ is a non-constant polynomial in $\mathbb{C}^n$ with $2a_2\frac{\partial g(z)}{\partial z_1} + a_3\left(\frac{\partial g(z)}{\partial z_1}\right)^2 + 2a_3\frac{\partial^2 g(z)}{\partial z_1^2} \equiv 0$.

Sub-case 1.2. If $K_4 \neq 0$, then from (3.69), we deduce that

$$\text{(3.70)} \qquad \frac{K_3}{K_4}\left(\frac{a_2}{2a_1}\frac{\partial g(z)}{\partial z_1} + \frac{a_3}{4a_1}\left\{\left(\frac{\partial g(z)}{\partial z_1}\right)^2 + 2\frac{\partial^2 g(z)}{\partial z_1^2}\right\}\right) \equiv e^{\frac{g(z+c)-g(z)}{2}}.$$

Since $g(z)$ is a non-constant polynomial, from (3.70) it is clear that $e^{\frac{g(z+c)-g(z)}{2}}$ must be a constant, i.e., $g(z+c) - g(z)$ is a constant, say $K_5 \in \mathbb{C}$. By using similar arguments as Case 2 of (I) of Theorem 1.14, we have $g(z) = \sum_{j=1}^n b_jz_j + b_{n+1}$, where $b_j \in \mathbb{C}$ for $1 \leq j \leq n+1$. From (3.70), we have

$$e^{\frac{1}{2}\sum_{j=1}^n b_jc_j} \equiv \frac{K_3}{K_4}\left(\frac{a_2}{2a_1}b_1 + \frac{a_3}{4a_1}b_1^2\right),$$

and hence we have $b_1 \neq 0$. From the first equation of (3.69), we have

$$(3.71)\quad f(z+c) = \frac{K_3}{a_1} e^{\frac{1}{2}\sum_{j=1}^{n} b_j z_j + \frac{1}{2} b_{n+1}} \Rightarrow f(z) = \frac{K_3}{a_1} e^{\frac{1}{2}\sum_{j=1}^{n} b_j (z_j - c_j) + \frac{1}{2} b_{n+1}},$$

where $b_j, K_3(\neq 0), K_4(\neq 0) \in \mathbb{C}$ for $1 \leq j \leq n+1$ with $b_1 \neq 0$ and $e^{\frac{1}{2}\sum_{j=1}^{n} b_j c_j} \equiv \frac{K_3}{K_4}\left(\frac{a_2}{2a_1} b_1 + \frac{a_3}{4a_1} b_1^2\right)$.

Case 2. Suppose $e^{\gamma_2(z+c)-\gamma_1(z+c)}$ is non-constant. From (3.68), it is clear that both $\Gamma_3(z)$ and $\Gamma_4(z)$ are not simultaneously identically zero, otherwise we get a contradiction. By using the similar arguments as in Case 2 of (II) of Theorem 1.14 and in view of Lemma 2.3, we get from (3.68) that $\Gamma_3(z) \not\equiv 0$, $\Gamma_4(z) \not\equiv 0$. Furthermore,

$$\begin{aligned}
& N\left(r, \Gamma_3(z) e^{\gamma_1(z)-\gamma_1(z+c)}\right) = N\left(r, \Gamma_4(z) e^{\gamma_2(z)-\gamma_1(z+c)}\right) \\
&= N\left(r, \frac{A_1 K_2}{A_2 K_1} e^{\gamma_2(z+c)-\gamma_1(z+c)}\right) = N\left(r, 0; \Gamma_3(z) e^{\gamma_1(z)-\gamma_1(z+c)}\right) \\
&= N\left(r, 0; \Gamma_4(z) e^{\gamma_2(z)-\gamma_1(z+c)}\right) = N\left(r, 0; \frac{A_1 K_2}{A_2 K_1} e^{\gamma_2(z+c)-\gamma_1(z+c)}\right) \\
&= S\left(r, \frac{A_1 K_2}{A_2 K_1} e^{\gamma_2(z+c)-\gamma_1(z+c)}\right).
\end{aligned}$$

So, in view of Lemma 2.3 and (3.68), we get either $M_1(z) e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1$ or $N_1(z) e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1$. Now the following cases arise.

Sub-case 2.1. When $\Gamma_3(z) e^{\gamma_1(z)-\gamma_1(z+c)} \equiv 1$. From (3.68), we have

$$\frac{A_2}{A_1}\left(\frac{a_3}{a_1}\left\{\frac{\partial^2 \gamma_2}{\partial z_1^2} + \left(\frac{\partial \gamma_2}{\partial z_1}\right)^2\right\} + \frac{a_2}{a_1}\frac{\partial \gamma_2}{\partial z_1}\right) e^{\gamma_2(z)-\gamma_2(z+c)} \equiv 1.$$

By using similar arguments as in Case 2.1 of (I) of Theorem 1.14, we obtain $\gamma_1(z) = \sum_{j=1}^{n} b_j z_j + b_{n+1}$, $\gamma_2(z) = \sum_{j=1}^{n} d_j z_j + d_{n+1}$ and $g(z) = \sum_{j=1}^{n}(b_j + d_j) z_j + b_{n+1} + d_{n+1}$, where $b_j, d_j \in \mathbb{C}$ for $1 \leq j \leq n+1$ such that

$$(3.72)\quad e^{\sum_{j=1}^{n} b_j c_j} \equiv \frac{(a_3 b_1^2 + a_2 b_1) A_1}{a_1 A_2} \quad \text{and} \quad e^{\sum_{j=1}^{n} d_j c_j} \equiv \frac{(a_3 d_1^2 + a_2 d_1) A_2}{a_1 A_1}.$$

From (3.72), it is clear that b_1 and d_1 are both non-zero. From (3.67), we get

$$f(z) = \frac{A_1 K_1 e^{\sum_{j=1}^{n} b_j (z_j - c_j) + b_{n+1}} + A_2 K_2 e^{\sum_{j=1}^{n} d_j (z_j - c_j) + d_{n+1}}}{\sqrt{2} a_1},$$

where $b_j, d_j, K_1, K_2 \in \mathbb{C}$ for $1 \leq j \leq n+1$ with $b_1 \neq 0$, $d_1 \neq 0$, $K_1 K_2 = 1$, $e^{\sum_{j=1}^{n} b_j c_j} \equiv \frac{(a_3 b_1^2 + a_2 b_1) A_1}{a_1 A_2}$ and $e^{\sum_{j=1}^{n} d_j c_j} \equiv \frac{(a_3 d_1^2 + a_2 d_1) A_2}{a_1 A_1}$.

Sub-case 2.2. When $\Gamma_4(z) e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1$. From (3.68), we get

$$\frac{K_1}{K_2}\left(\frac{a_3}{a_1}\left(\frac{\partial^2 \gamma_1}{\partial z_1^2} + \left(\frac{\partial \gamma_1}{\partial z_1}\right)^2\right) + \frac{a_2}{a_1}\frac{\partial \gamma_1}{\partial z_1}\right) e^{\gamma_1(z)-\gamma_2(z+c)} \equiv 1.$$

Using the same arguments as in Sub-case 2.2 of (II) of Theorem 1.14, we arrive at a contradiction. This completes the proof. □

Proof of Theorem 1.22. Let $f \in \mathscr{E}_T^{<\infty}(\mathbb{C}^n)$ satisfy (1.11), where $g(z)$ is a polynomial in $\mathbb{C}^n$. Now the following circumstances arise.
(I) Suppose $g \in \mathbb{C}$ and $e^g \equiv K \in \mathbb{C} \setminus \{0\}$. By using similar arguments of (I) of Theorem 1.14, we get

$$\left\{\begin{array}{l} a_1 f(z+c) = \frac{1}{\sqrt{2}}\left(A_1K_1e^{P(z)} + A_2K_2e^{-P(z)}\right) \text{ and} \\ a_2\frac{\partial f(z)}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1 \partial z_2} = \frac{1}{\sqrt{2}}\left(A_2K_1e^{P(z)} + A_1K_2e^{-P(z)}\right), \end{array}\right. \tag{3.73}$$

where $K_1, K_2 \in \mathbb{C} \setminus \{0\}$ such that $K_1K_2 = K$, $A_1 = \frac{1}{2\sqrt{1+\alpha}} + \frac{1}{2i\sqrt{1-\alpha}}$, $A_2 = \frac{1}{2\sqrt{1+\alpha}} - \frac{1}{2i\sqrt{1-\alpha}}$ and $P(z)$ is a polynomial in $\mathbb{C}^n$. It is clear that $P(z)$ must be a non-constant polynomial, otherwise it contradicts that $f(z)$ is transcendental. From first and second equations of (3.73), we deduce that

$$\begin{aligned} &\frac{A_1}{A_2}\left[\frac{a_3}{a_1}\left(\frac{\partial^2 P(z)}{\partial z_1 \partial z_2} + \frac{\partial P(z)}{\partial z_1}\frac{\partial P(z)}{\partial z_2}\right) + \frac{a_2}{a_1}\frac{\partial P(z)}{\partial z_1}\right]\frac{e^{P(z)}}{e^{P(z+c)}} - \frac{A_1K_2}{A_2K_1}e^{-2P(z+c)} \\ &+\frac{K_2}{K_1}\left[\frac{a_3}{a_1}\left(\frac{\partial P(z)}{\partial z_1}\frac{\partial P(z)}{\partial z_2} - \frac{\partial^2 P(z)}{\partial z_1 \partial z_2}\right) - \frac{a_2}{a_1}\frac{\partial P(z)}{\partial z_1}\right]e^{-P(z)-P(z+c)} \equiv 1. \end{aligned} \tag{3.74}$$

In view of Lemmas 2.1 and 2.7, and by using similar arguments as in Case 2 of (I) of Theorem 1.14 in (3.74), we deduce that $P(z) = \sum_{j=1}^n b_j z_j + b_{n+1}$, where $b_j \in \mathbb{C}$ for $1 \le j \le n+1$ such that

$$e^{\sum_{j=1}^n b_j c_j} = \frac{(a_3b_1b_2 + a_2b_1)A_1}{a_1A_2} \text{ and } e^{-\sum_{j=1}^n b_j c_j} = \frac{(a_3b_1b_2 - a_2b_1)A_2}{a_1A_1}. \tag{3.75}$$

From (3.75), it is clear that $b_1 \neq 0$. Thus, from (3.64), we deduce that

$$f(z) = \frac{1}{\sqrt{2}a_1}\left(A_1K_1e^{\sum_{j=1}^n b_j(z_j - c_j) + b_{n+1}} + A_2K_2e^{-\sum_{j=1}^n b_j(z_j - c_j) - b_{n+1}}\right).$$

(II) Suppose $g(z)$ is a non-constant polynomial in $\mathbb{C}^n$. By similar arguments as in (II) of Theorem 1.14, we get

$$\left\{\begin{array}{l} a_1 f(z+c) = \frac{1}{\sqrt{2}}\left(A_1K_1e^{\gamma_1(z)} + A_2K_2e^{\gamma_2(z)}\right) \text{ and} \\ a_2\frac{\partial f(z)}{\partial z_1} + a_3\frac{\partial^2 f(z)}{\partial z_1 z_2} = \frac{1}{\sqrt{2}}\left(A_2K_1e^{\gamma_1(z)} + A_1K_2e^{\gamma_2(z)}\right), \end{array}\right. \tag{3.76}$$

where A_1, A_2 are in (1.13), $K_1, K_2 \in \mathbb{C} \setminus \{0\}$ such that $K_1K_2 = 1$, $\gamma_1(z) = P(z) + \frac{g(z)}{2}$, $\gamma_2(z) = -P(z) + \frac{g(z)}{2}$ and $P(z)$ is a polynomial in $\mathbb{C}^n$. From (3.76), we deduce that

$$\begin{aligned} &\frac{A_1}{A_2}\left[\frac{a_3}{a_1}\left(\frac{\partial^2 \gamma_1}{\partial z_1 \partial z_2} + \frac{\partial \gamma_1}{\partial z_1}\frac{\partial \gamma_1}{\partial z_2}\right) + \frac{a_2}{a_1}\frac{\partial \gamma_1}{\partial z_1}\right]\frac{e^{\gamma_1(z)}}{e^{\gamma_1(z+c)}} - \frac{A_1K_2}{A_2K_1}e^{\gamma_2(z+c)-\gamma_1(z+c)} \\ &+\frac{K_2}{K_1}\left[\frac{a_3}{a_1}\left(\frac{\partial^2 \gamma_2}{\partial z_1 \partial z_2} + \frac{\partial \gamma_2}{\partial z_1}\frac{\partial \gamma_2}{\partial z_2}\right) + \frac{a_2}{a_1}\frac{\partial \gamma_2}{\partial z_1}\right]e^{\gamma_2(z)-\gamma_1(z+c)} \equiv 1. \end{aligned} \tag{3.77}$$

The remaining part of the proof follows by using the same arguments as in proof of Theorem 1.19 and thus the conclusions of this Theorem are obtained. This completes the proof. □

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