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On entire solutions of certain Fermat type differential-difference equations

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Abstract. The article's main focus is on the precise forms of the transcendental entire solutions to a few specific differential-difference equations of the Fermat type that support the finite order constraint. Some earlier results provided by Liu et al, reference [12, 11] are improved throughout this concerning the existence of our results. The sharpness of some claims in this work is supported by several examples.

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1. Introduction, Definitions and Results

This article discusses the fundamental symbols and traditional conclusions of Nevanlinna's value distribution theory of meromorphic functions, and we assume that readers are already familiar with all of those concepts (see, for instance, [6, 9, 18, 19]) in the complex plane $\mathbb{C}$. We investigate meromorphic solutions of some functional equations that are analogous to Fermat varieties throughout this study. We start with a classical functional equation of the Fermat type

$$f(z)^n + g(z)^n = 1, \tag{1.1}$$

it is analogous to the Fermat diophantine equation $x^n + y^n = 1$ over a function field, where n is an integer. The Fermat type functional equation (1.1) was first studied by Gross [2], who established that the equation has no transcendental meromorphic solutions when $n \geq 4$. Next, Montel [13] shown that, when $n \geq 3$, the equation (1.1) has no transcendental entire solutions. Iyer [1] examined the Fermat type solutions for $n = 2$ in 1939 and came to the conclusion that the equation (1.1) has only the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is an entire function.

In 1970, Yang [16] also worked at the Fermat type functional equation with the following form:

$$a(z)f(z)^n + b(z)g(z)^m = 1, \tag{1.2}$$

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where a and b are small functions with regard to f. Recall that if $T(r, \alpha) = S(r, f)$, $S(r, f)$ represents any quantity satisfying $S(r, f) = o(T(r, f)), r \to \infty$ outside of a possible exceptional set of finite logarithmic measure, then α is a small function with regard to f. It is shown that there exist no non-constant entire solutions to the equation (1.2) for two positive integers m, n satisfying $\frac{1}{m} + \frac{1}{n} < 1$. Yang's result fails to exist any non-constant entire solutions f and g if the assumption $m > 2, n > 2$ is satisfied. However, we can find the expressions of entire solutions when $m = n = 2$ and f, g have any particular relation.

Many researchers attempt to characterize the solutions of the differential, difference, differential-difference, or partial differential equations of the Fermat type based on recent works of difference analogs of the Fermat type (See, [4, 5, 8, 10, 12, 14, 15] for example).

In 2007, Tang-Liao [14] investigated the entire solutions of the following differential equation

$$f(z)^2 + \{P(z)f^{(k)}(z)\}^2 = Q(z),$$

where P, Q are non-zero polynomials. In 2013, Liu-Yang [12] examined the existence of the type equation

$$f(z)^2 + \{P(z)(f(z+c) - f(z))\}^2 = Q(z), \tag{1.3}$$

where P, Q are non-zero polynomials and admit there is no finite order transcendental entire solution.

Recall that the order f is defined as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

In this article, our main intention is to consider a more specialized version of the non-linear difference equation (1.3). More specifically, we wish to investigate the result in detail as follows.

Theorem 1.1. *Take l be a positive integer and set*

$$L_f = \sum_{k=0}^{l} b_k f^{(k)},$$

where $b_0, b_1, ..., b_l (\neq 0)$ are constants. If $P(z), Q(z)$ are non-zero polynomials, $c \in \mathbb{C}\backslash\{0\}$ and if f is a transcendental entire solution of the following differential-difference equation

$$f(z)^2 + P(z)^2(L_f(z+c) - L_f(z))^2 = Q(z) \tag{1.4}$$

with finite order, then f is of the form

$$f(z) = \frac{Q_1(z)e^{az+b} + Q_2(z)e^{-az-b}}{2},$$

where $a(\neq 0), b$ are constants, Q_1 and Q_2 are non-zero polynomials with $Q_1Q_2 = Q$. Moreover, if $q(z) = \sum_{k=0}^{l} b_k z^k$, then one of the following results must happen:
(i) If $e^{ac} \neq 1$.
(a) $P(z)$ must be constant if either $q(a) \neq 0$ or $q(-a) \neq 0$, or if either Q_1 or Q_2 is a constant;
(b) $P(z)$ must be degree one if either $q(a) = 0, q'(a) \neq 0$ or $q(-a) = 0, q'(-a) \neq 0$ and if both Q_1, Q_2 are non-constant polynomials. Otherwise, $\deg P \geq 2$.
(ii) If $e^{ac} = 1$.
(c) $P(z)$ must be degree one if either $q(a) \neq 0$ or $q(-a) \neq 0$ and if both Q_1, Q_2 are non-constant polynomials;
(d) $P(z)$ must be degree two if either $q(a) = 0, q'(a) \neq 0$ or $q(-a) = 0, q'(-a) \neq 0$ and if both Q_1, Q_2 are non-constant polynomials. Otherwise, $\deg P \geq 3$.

Example 1.2. Take $a = i, b = 0, c = \frac{-\pi}{2}$. Then $f(z) = \frac{e^{iz}+e^{-iz}}{2}$ satisfies the equation

$$f(z)^2 + \left\{\frac{1}{2}(f(z+c) - f(z)) + \frac{1}{2}(f'(z+c) - f'(z))\right\}^2 = 1.$$

Example 1.3. Take $a = 1, b = 0, c = \pi i$. Then $f(z) = \frac{ze^{z}+ze^{-z}}{2}$ satisfies the equation

$$f(z)^2 - \frac{z^2}{16}\left\{(f''(z+c) - f''(z)) - (f^{(4)}(z+c) - f^{(4)}(z))\right\}^2 = z^2.$$

Example 1.4. Take $a = 2, b = 0, c = \pi i$. Then $f(z) = \frac{ze^{2z}+ze^{-2z}}{2}$ satisfies the equation

$$f(z)^2 - \frac{z^2}{100c^2}\left\{(f'(z+c) - f'(z)) + (f'''(z+c) - f'''(z))\right\}^2 = z^2.$$

Example 1.5. Take $a = 1, b = 0, c = 2\pi i$. Then $f(z) = \frac{z^2e^{z}+z^2e^{-z}}{2}$ satisfies the equation

$$f(z)^2 - \frac{z^4}{16c^2}\left\{(f''(z+c) - f''(z)) - (f^{(4)}(z+c) - f^{(4)}(z))\right\}^2 = z^4.$$

Example 1.6. Take $a = 1, b = 0, c = 2\pi i$. Then $f(z) = \frac{z^3e^{z}+z^3e^{-z}}{2}$ satisfies the equation

$$f(z)^2 - \frac{z^6}{576c^2}\left\{(f'(z+c) - f'(z)) - 2(f'''(z+c) - f'''(z)) + (f^{(5)}(z+c) - f^{(5)}(z))\right\}^2 = z^6.$$

In the following, we want to know what would happen if $P(z)$ is substituted with $\frac{1}{P(z)}$ in the expression (1.4). In this instance, the differential-difference equation has the form

$$f(z)^2 + \frac{1}{P(z)^2}(L_f(z+c) - L_f(z))^2 = Q(z), \tag{1.5}$$

where $P(z), Q(z)$ are non-zero polynomials, and the following result is obtained.

Theorem 1.7. *If the differential-difference equation in the form* (1.5) *admits a transcendental entire solution f with finite order, then*

$$f(z) = \frac{Q_1(z)e^{az+b} + Q_2(z)e^{-az-b}}{2},$$

where $a(\neq 0), b$ are constants, Q_1 and Q_2 are non-zero polynomials with $Q_1Q_2 = Q$ and $P(z)$ must be constant if either $q(a) \neq 0$ or $q(-a) \neq 0$, or if either Q_1 or Q_2 is a constant, where $q(z) = \sum_{k=0}^{l} b_k z^k$ and $e^{ac} \neq 1$.

In 2015, Liu-Dong [11] examined the entire solution to the following Fermat type difference equation

$$D^2 f(z)^2 + (Af(z+c) + Bf(z))^2 = 1. \tag{1.6}$$

Liu-Dong then showed that if the equation (1.6) contains transcendental entire solutions of finite order, then $A^2 = B^2 + D^2$. They also examined the equation

$$f(z)^2 + (Af^{(m)}(z) + Bf^{(n)}(z))^2 = 1, \tag{1.7}$$

which admits transcendental entire solutions with finite order when $m+n$ is an even number and m, n are odd.

In that study, we are given transcendental entire solutions in a form of Fermat-type differential-difference equations with a more extended form of the problem (1.7) and proved the following conclusion.

Theorem 1.8. *Take m, n two positive integers. Let $a(z), b(z)$ be non-zero rational functions and also let α, c be non-zero complex numbers. If the following equation*

$$f(z)^2 + (a(z)f^{(m)}(z+c) + b(z)f^{(n)}(z))^2 = \alpha^2 \tag{1.8}$$

admits a transcendental entire solution with finite order, then $a(z), b(z)$ reduce to the constant a, b respectively, and $f = \alpha \sin(Az+B)$, where B is a constant and $e^{iAc} = \frac{i - b(iA)^n}{a(iA)^m}$. Also satisfy the following conditions:

(a) n is even, then $a^2A^{2m} = b^2A^{2n} + 1$,
(b) n is odd, then $a^2A^{2m} = b^2A^{2n} + 2i^{n+1}bA^n + 1$.

Remark 1.9. It is simple to see that Theorem 1.8 extends the conclusion in (1.6) for the situation $m = n = 0$ and $a(z) = A, b(z) = B$, since $A^2 = B^2 + 1$.

The following examples exhibit that each case in Theorem 1.8 actually occurs.

Example 1.10. Let m, n be even. Take $A = 1, B = 0$ and $c = -i \ln \frac{1-i}{\sqrt{2}}$. Then $f(z) = \alpha \sin z$ satisfies the equation

$$f(z)^2 + \left\{\sqrt{2}f''(z+c) - f''(z)\right\}^2 = \alpha^2.$$

Example 1.11. Let m, n be odd. Take $A = \sqrt{2}i, B = 2$ and $c = \frac{-\pi i}{\sqrt{2}}$. Then $f(z) = \alpha \sin(\sqrt{2}iz + 2)$ satisfies the equation

$$f(z)^2 + \left\{-\frac{i}{\sqrt{2}}f'(z+c) - \frac{i}{\sqrt{2}}f'''(z)\right\}^2 = \alpha^2.$$

Example 1.12. Let m be even and n be odd. Take $A = i, B = 1$ and $c = \frac{-\pi i}{2}$. Then $f(z) = \alpha \sin(iz + 1)$ satisfies the equation

$$f(z)^2 + \left\{2f''(z+c) + if'(z)\right\}^2 = \alpha^2.$$

Example 1.13. Let m be odd and n be even. Take $A = 1, B = 1$ and $c = \frac{\pi}{3}$. Then $f(z) = \alpha \sin(z + 1)$ satisfies the equation

$$f(z)^2 + \left\{2f'(z+c) - \sqrt{3}f''(z)\right\}^2 = \alpha^2.$$

Corollary 1.14. *Take m, n two positive integers. Let $a(z), b(z)$ be non-zero rational functions and also let α, c be non-zero complex numbers. The equation* (1.8) *has no transcendental meromorphic solution of f with finite order if $a(z), b(z)$ are non-constant rational functions.*

Corollary 1.15. *Take m, n two positive integers. Let $a(z), b(z)$ be non-zero rational functions and also let α, c be non-zero complex numbers. The equation* (1.8) *does not have a transcendental meromorphic solution of f with finite order if $a(z), b(z)$ are non-zero constants a, b and*

(i) $a^2 A^{2m} \neq b^2 A^{2n} + 1$,
(ii) $a^2 A^{2m} \neq b^2 A^{2n} + 2i^{n+1} b A^n + 1$.

Also, in 2015 Liu-Dong [11] further considered the entire solutions of the Fermat type difference equation

$$[a_1 f(z+c) + b_1 f(z)]^2 + [a_2 f(z+c) + b_2 f(z)]^2 = 1 \tag{1.9}$$

and produced the results shown below.

Theorem 1.16 ([11], Theorem 1.15)**.** *Let a_1, a_2, b_1, b_2 be non-zero complex numbers. If the equation* (1.9) *contains transcendental entire solutions of finite order, then $a_1^2 + a_2^2 = b_1^2 + b_2^2$ and*

$$f(z) = \frac{a_2 \cos(aiz + bi) + a_1 \sin(aiz + bi)}{a_2 b_1 - a_1 b_2},$$

where $a(\neq 0)$ and b are constants.

Remark 1.17. If any one of a_1, a_2, b_1, b_2 is zero, then we can draw the conclusion from the result in (1.6).

We generalize the results from Theorem (1.16) regarding the difference equation (1.9) and arrive at the following theorem.

Theorem 1.18. *Take complex numbers a_1, a_2, b_1, b_2, c with $c \neq 0$ and k being a positive integer. If*

$$[a_1 f^{(k)}(z) + b_1 f(z+c)]^2 + [a_2 f^{(k)}(z) + b_2 f(z+c)]^2 = 1 \tag{1.10}$$

has a transcendental entire solution f with finite order, then one of the subsequent situations happens:

(*i*) *If $a_1 = 0$, then $a^{2k} a_2^2 = (-1)^k (b_1^2 + b_2^2)$ and*

$$f(z) = \frac{e^{az+b-ac} + e^{-az-b+ac}}{2b_1}, e^{ac} = -\frac{a^k a_2}{ib_1 + b_2}.$$

(*ii*) *If $b_1 = 0$, then $b_2^2 = (-1)^k a^{2k} (a_1^2 + a_2^2)$ and*

$$f(z) = \frac{e^{az+b} + (-1)^k e^{-az-b}}{2a_1 a^k}, e^{ac} = -\frac{a^k (ia_1 + a_2)}{b_2}.$$

(*iii*) *If $a_2 = 0$, then $a^{2k} a_1^2 = (-1)^k (b_1^2 + b_2^2)$ and*

$$f(z) = \frac{e^{az+b-ac} + e^{-az-b+ac}}{2b_2}, e^{ac} = -\frac{a^k a_1}{b_1 + ib_2}.$$

(*iv*) *If $b_2 = 0$, then $b_1^2 = (-1)^k a^{2k} (a_1^2 + a_2^2)$ and*

$$f(z) = \frac{e^{az+b} + (-1)^k e^{-az-b}}{2a_2 a^k}, e^{ac} = -\frac{a^k (a_1 + ia_2)}{b_1}.$$

(*v*) *If none of a_1, a_2, b_1, b_2 is zero, then $a^{2k}(a_1^2 + a_2^2) = (-1)^k (b_1^2 + b_2^2)$ and*

$$f(z) = \frac{Ae^{az+b} + Be^{-az-b}}{2a^k}, e^{ac} = -\frac{a^k (ia_1 + a_2)}{(ib_1 + b_2)},$$

where $A = \frac{ib_1 + b_2}{a_1 b_2 - a_2 b_1}$, $B = (-1)^k \frac{-ib_1 + b_2}{a_1 b_2 - a_2 b_1}$, and b are constants, respectively.

It is simple to determine that Theorem 1.18 extends Theorem 1.16 for the case $k = 0$.

Example 1.19. Take $a = i, b = 0, c = \frac{\pi}{4}$. Then $f(z) = \frac{e^{i(z-\frac{\pi}{4})}+e^{-i(z-\frac{\pi}{4})}}{2}$ satisfies the equation

$$f(z+c)^2 + (-\sqrt{2}f'(z) + f(z+c))^2 = 1.$$

Example 1.20. Take $a = \frac{-i}{\sqrt{2}}, b = 0, c = \frac{-\pi}{2\sqrt{2}}$. Then $f(z) = \frac{-i}{\sqrt{2}}(e^{\frac{-i}{\sqrt{2}}z} - e^{\frac{i}{\sqrt{2}}z})$ satisfies the equation

$$(-f'(z))^2 + (f'(z) + f(z+c))^2 = 1.$$

Example 1.21. Take $a = 2^{\frac{1}{4}}, b = 0, c = \frac{3\pi i}{2^{\frac{9}{4}}}$. Then $f(z) = \frac{e^{2^{\frac{1}{4}}(z-c)}+e^{-2^{\frac{1}{4}}(z-c)}}{2}$ satisfies the equation

$$f(z+c)^2 + (f''(z) + f(z+c))^2 = 1.$$

Example 1.22. Take $a = 2^{-\frac{1}{4}}, b = 0, c = \frac{3\pi i}{2^{\frac{7}{4}}}$. Then $f(z) = \frac{e^{2^{-\frac{1}{4}}z}+e^{-2^{-\frac{1}{4}}z}}{-\sqrt{2}}$ satisfies the equation

$$f''(z)^2 + (f''(z) + f(z+c))^2 = 1.$$

Example 1.23. Take $a = -i, b = 0, c = \pi$. Then $f(z) = \frac{(\frac{i+1}{2})e^{-iz}+(\frac{i-1}{2})e^{iz}}{-2i}$ satisfies the equation

$$(f'(z) + f(z+c))^2 + (-f'(z) + f(z+c))^2 = 1.$$

Example 1.24. Take $a = i, b = 0, c = \frac{-\pi}{2}$. Then $f(z) = \frac{(\frac{1-i}{2})e^{iz}+(\frac{1+i}{2})e^{-iz}}{-2}$ satisfies the equation

$$(f''(z) + f(z+c))^2 + (f''(z) - f(z+c))^2 = 1.$$

Corollary 1.25. *Take non-zero complex numbers a_1, a_2, b_1, b_2, c, and let k be a positive integer. Under the constraint $a^{2k}(a_1^2 + a_2^2) \neq (-1)^k(b_1^2 + b_2^2)$, the equation* (1.10) *has no transcendental meromorphic solution f with finite order.*

2. Some lemmas

Lemma 2.1 ([18], Lemma 5.1)**.** *If f is a non-constant periodic meromorphic function, then $\rho(f) \geq 1$.*

Lemma 2.2 ([17], Lemma 1)**.** *Let f_1 and f_2 be two meromorphic functions, and let a, b_1, b_2 be small functions of f_1 and f_2 satisfying $ab_1b_2 \not\equiv 0$ and $b_1f_1 + b_2f_2 = a$. Then*

$$T(r, f_1) \leq \overline{N}(r, f_1) + \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + S(r, f_1).$$

Lemma 2.3 ([3], Lemma 5.1). *Let $a_k(z)$ be entire functions of finite order ρ and let $g_k(z)$ be entire functions such that $g_l(z)-g_k(z)(l \neq k)$ are transcendental entire functions or polynomials of degree greater than ρ. Then*

$$\sum_{k=1}^{n} a_k(z)e^{g_k(z)} = a_0(z)$$

holds only when

$$a_0(z) = a_1(z) = ... = a_n(z) \equiv 0$$

Lemma 2.4 ([7], Lemma 2.5). *Let $b_k(z)$ be meromorphic functions of finite order ρ such that $b_k(z)$ has only finitely many poles for each k. Let $g_k(z)$ be entire functions such that $g_l(z) - g_k(z)(l \neq k)$ are transcendental entire functions or polynomials of degree greater than ρ. Then*

$$\sum_{k=1}^{n} b_k(z)e^{g_k(z)} = b_0(z)$$

holds only when

$$b_0(z) = b_1(z) = ... = b_n(z) \equiv 0$$

3. Proof of Theorem 1.1

Proof. Assume that f is a transcendental entire solution of finite order that satisfies the requirements of the system (1.4). The system (1.4) can therefore be rewritten as follows

$$\begin{aligned} &[f(z) + iP(z)(L_f(z+c) - L_f(z))] \\ &\times[f(z) - iP(z)(L_f(z+c) - L_f(z))] = Q(z). \end{aligned} \tag{3.1}$$

Therefore, both of the expressions $f(z)+iP(z)(L_f(z+c)-L_f(z))$ and $f(z)-iP(z)(L_f(z+c) - L_f(z))$ have finitely many zeros, then h is a non-constant polynomial function that has the property of

$$\begin{aligned} f(z) + iP(z)(L_f(z+c) - L_f(z)) &= Q_1(z)e^{h(z)}, \\ f(z) - iP(z)(L_f(z+c) - L_f(z)) &= Q_2(z)e^{-h(z)}, \end{aligned}$$

where Q_1, Q_2 are non-constant polynomials with $Q_1Q_2 = Q$. In view of above, we deduce that

$$\begin{aligned} f(z) &= \frac{Q_1(z)e^{h(z)} + Q_2(z)e^{-h(z)}}{2}, \\ L_f(z+c) - L_f(z) &= \frac{Q_1(z)e^{h(z)} - Q_2(z)e^{-h(z)}}{2iP(z)}. \end{aligned} \tag{3.2}$$

By differentiating the first equation in (3.2), it follows that

$$f^{(k)}(z) = \frac{U_k(z)e^{h(z)} + V_k(z)e^{-h(z)}}{2}, \tag{3.3}$$

where

$$U_k = Q_1^{(k)} + kQ_1^{(k-1)}h' + ... + kQ_1'[(h')^{k-1} + L_{k-1}(h', h'', ..., h^{(l)})] \\ + Q_1[(h')^k + L_k(h', h'', ..., h^{(l)})],$$

$$V_k = Q_2^{(k)} + kQ_2^{(k-1)}(-h)' + ... + kQ_2'[(-h')^{k-1} + H_{k-1}(-h', -h'', ..., -h^{(l)})] \\ + Q_2[(-h')^k + H_k(-h', -h'', ..., -h^{(l)})],$$

in which $L_k, L_{k-1}, H_k, H_{k-1}$ are differential polynomials of $h', h'', ..., h^{(l)}$ such that $\deg L_k \leq k$, $\deg L_{k-1} \leq k-1$, $\deg H_k \leq k$, $\deg H_{k-1} \leq k-1$. In view of the results (3.2) and (3.3), we obtain that

$$W_1(z)e^{h(z+c)} + W_2(z)e^{-h(z+c)} + W_3(z)e^{h(z)} = W_4(z)e^{-h(z)}, \tag{3.4}$$

where

$$W_1(z) = P(z)\sum_{k=0}^{l} b_k U_k(z+c), \qquad W_2(z) = P(z)\sum_{k=0}^{l} b_k V_k(z+c),$$

$$W_3(z) = iQ_1(z) - P(z)\sum_{k=0}^{l} b_k U_k(z), \quad W_4(z) = iQ_2(z) + P(z)\sum_{k=0}^{l} b_k V_k(z).$$

Our claim that $W_j(z) \not\equiv 0$ $(j = 1,2,3,4)$. It is obvious that no three in $W_j(z)$ $(j = 1,2,3,4)$ can be zero together. If one of $W_j(z)$ $(j = 1,2,3,4)$ is equal to zero, such as $W_4(z) \equiv 0$, the equation (3.4) yields

$$-\frac{W_3(z)}{W_2(z)}e^{h(z)+h(z+c)} - \frac{W_1(z)}{W_2(z)}e^{2h(z+c)} \equiv 1.$$

In view of Lemma 2.2, it follows that the function $H(z) := h(z+c)$ satisfies

$$\begin{aligned} T(r, e^{2H}) &\leq \overline{N}(r, e^{2H}) + \overline{N}\left(r, \frac{1}{e^{2H}}\right) + \overline{N}\left(r, \frac{1}{e^{H+h}}\right) + S(r, e^{2H}) \\ &= S(r, e^{2H}), \end{aligned}$$

which is impossible. If any two of $W_j(z)$ $(j = 1,2,3,4)$ is equal to zero, such as $W_1(z) \equiv 0$ and $W_3(z) \equiv 0$. Then (3.4) yields

$$e^{h(z)-H(z)} = \frac{W_4(z)}{W_2(z)}.$$

Then by the second main theorem, we have

$$\begin{aligned} T(r, e^{h-H}) &\leq \overline{N}\left(r, \frac{1}{e^{h-H}}\right) + \overline{N}(r, e^{h-H}) + \overline{N}\left(r, \frac{1}{e^{h-H}-1}\right) \\ &\quad + S(r, e^{h-H}) \\ &= S(r, e^{h-H}), \end{aligned}$$

which is impossible. Thus the claim is done.

Then from (3.4), one can obtain

$$W_1(z)e^{f_1(z)} + W_2(z)e^{f_2(z)} + W_3(z)e^{f_3(z)} = W_4(z), \tag{3.5}$$

in which

$$f_1(z) = h(z) + h(z+c),\ f_2(z) = h(z) - h(z+c),\ f_3(z) = 2h(z).$$

Then applying Lemma 2.3 to the system (3.5), either $f_2(z) = f_3(z) - f_1(z) = h(z) - h(z+c)$ or $f_1(z) = f_3(z) - f_2(z) = h(z) + h(z+c)$ is a constant.

If $h(z) + h(z+c)$ is a constant, then $h(z)$ must be a constant. Otherwise, $0 = \deg[h(z) + h(z+c)] = \deg h(z) \geq 1$, which is not possible. Because h is a non-constant polynomial, there is a contradiction in that.

Therefore, $h(z) - h(z+c)$ must be a constant. Following that, we obtain $h'(z) \equiv h'(z+c)$. This demonstrates that the function $h'(z)$ is periodic and has period c. Since $\rho(h') = \rho(h) < 1$, it deduce from Lemma 2.1 that $h' = a$, where $a(\neq 0)$ is a constant, and hence $h(z) = az + b$, where b is a constant.

Then, by the equation (3.2), we can deduce the result

$$f(z) = \frac{Q_1(z)e^{az+b} + Q_2(z)e^{-az-b}}{2}.$$

Next, we want to figure out the polynomial P. After that, put $h = az + b$ into (3.5). Then, one can be rewritten as follows

$$W_1(z)e^{2az+2b+ac} + W_2(z)e^{-ac} + W_3(z)e^{2az+2b} = W_4(z).$$

This also gives

$$\begin{cases} W_1(z)e^{ac} & = -W_3(z), \\ W_2(z)e^{-ac} & = W_4(z). \end{cases}$$

Now using the expressions of W_j $(j = 1,2,3,4)$, the aforementioned equalities become

(3.6)

$$\begin{aligned} P(z)\sum_{k=0}^{l} b_k \Big\{ & a^k(Q_1(z+c)e^{ac} - Q_1(z)) + ka^{k-1}(Q_1'(z+c)e^{ac} - Q_1'(z)) + \ldots \\ & \ldots + (Q_1^{(k)}(z+c)e^{ac} - Q_1^{(k)}(z))\Big\} = -iQ_1(z) \end{aligned}$$

and

$$\begin{aligned} P(z)\sum_{k=0}^{l} b_k \Big\{ & (-a)^k(Q_2(z+c)e^{-ac} - Q_2(z)) + k(-a)^{k-1}(Q_2'(z+c)e^{-ac} \\ & - Q_2'(z)) + \ldots + (Q_2^{(k)}(z+c)e^{-ac} - Q_2^{(k)}(z))\Big\} = iQ_2(z). \end{aligned} \tag{3.7}$$

The following two cases are then examined:

Case 1. When $e^{ac} \neq 1$. There are two subcases happened:

Subcase 1.1. First we suppose that either $q(a) \neq 0$ or $q(-a) \neq 0$, where $q(z) = \sum_{k=0}^{l} b_k z^k$; if $q(a) \neq 0$, comparing the highest degree coefficient in the system (3.6), we determine that

$$P(z)q(a)(e^{ac} - 1) = -i.$$

Also, if $q(-a) \neq 0$, comparing the highest degree coefficient in the system (3.7), we find that

$$P(z)q(-a)(e^{-ac} - 1) = i.$$

This shows P is a constant. These results also happened if either Q_1 or Q_2 is a constant.

Subcase 1.2. If either $q(a) = 0,\ q'(a) \neq 0$ or $q(-a) = 0, q'(-a) \neq 0$ and both Q_1, Q_2 are non-constant polynomials. First we say, $q(a) = 0, q'(a) \neq 0$. Then by (3.6), we have

$$P(z)\sum_{k=0}^{l} b_k \Big\{ k a^{k-1}(Q_1'(z+c)e^{ac} - Q_1'(z)) + ... \\ ... + (Q_1^{(k)}(z+c)e^{ac} - Q_1^{(k)}(z)) \Big\} = -iQ_1(z).$$

By comparing the coefficients of the above system we have P is linear. We must also have $q(-a) = 0, q'(-a) \neq 0$. If $q(-a) = 0, q'(-a) \neq 0$, then by comparing the coefficients of equation (3.7) we can get the similar conclusion.

Otherwise, either $q(a) = 0, q'(a) = 0$ or $q(-a) = 0, q'(-a) = 0$ and both Q_1, Q_2 are non-constant polynomials. First we say, $q(a) = 0, q'(a) = 0$. Then the equation (3.6) becomes

$$P(z)\sum_{k=0}^{l} b_k \Big\{ k(k-1)a^{k-2}(Q_1''(z+c)e^{ac} - Q_1''(z)) + ... \\ ... + (Q_1^{(k)}(z+c)e^{ac} - Q_1^{(k)}(z)) \Big\} = -iQ_1(z).$$

From this by comparing the coefficients, we must have $\deg P \geq 2$. This case must have $\deg Q_1 \geq 2$. In the next when $q(-a) = 0, q'(-a) = 0$, we can obtain $\deg P \geq 2$, by comparing the coefficients in the equation (3.7) and also have $\deg Q_2 \geq 2$.

Case 2. When $e^{ac} = 1$. Afterward, rewrite equations (3.6) and (3.7) as follows

$$P(z)\sum_{k=0}^{l} b_k \Big\{ a^k(Q_1(z+c) - Q_1(z)) + k a^{k-1}(Q_1'(z+c) - Q_1'(z)) + ... \\ ... + (Q_1^{(k)}(z+c) - Q_1^{(k)}(z)) \Big\} = -iQ_1(z) \tag{3.8}$$

and

(3.9)
$$P(z)\sum_{k=0}^{l} b_k\Big\{(-a)^k(Q_2(z+c)-Q_2(z))+k(-a)^{k-1}(Q_2'(z+c)-Q_2'(z))+...$$
$$...+(Q_2^{(k)}(z+c)-Q_2^{(k)}(z))\Big\}=iQ_2(z).$$

The following two subcases are discussed.

Subcase 2.1. If either $q(a)\neq 0$ or $q(-a)\neq 0$, where $q(z)=\sum_{k=0}^{l} b_k z^k$ and both Q_1, Q_2 are non-constant polynomials. First, if $q(a)\neq 0$, then comparing the coefficients in the equation of (3.8), we find that P is linear. Next for the case $q(-a)\neq 0$, we can get a similar conclusion by comparing the coefficients in the equation (3.9).

Subcase 2.2. If either $q(a)=0, q'(a)\neq 0$ or $q(-a)=0, q'(-a)\neq 0$ and both Q_1, Q_2 are non-constant polynomials. First we say, $q(a)=0, q'(a)\neq 0$, Then by (3.8), we have

$$P(z)\sum_{k=0}^{l} b_k\Big\{ka^{k-1}(Q_1'(z+c)-Q_1'(z))+...$$
$$...+(Q_1^{(k)}(z+c)-Q_1^{(k)}(z))\Big\}=-iQ_1(z).$$

Comparing the coefficients of the above equation shows that $\deg P=2$. For this case, we must have $\deg Q_1\geq 2$. In the next case we have $q(-a)=0, q'(-a)\neq 0$. If $q(-a)=0, q'(-a)\neq 0$, then by comparing the coefficients in the equation (3.9) we can obtain $\deg P=2$ and also have $\deg Q_2\geq 2$.

Otherwise, if either $q(a)=0, q'(a)=0$ or $q(-a)=0, q'(-a)=0$ and both Q_1, Q_2 are non-constant polynomials. First we say, $q(a)=0, q'(a)=0$, then by (3.8) we have

$$P(z)\sum_{k=0}^{l} b_k\Big\{k(k-1)a^{k-2}(Q_1''(z+c)-Q_1''(z))+...$$
$$...+(Q_1^{(k)}(z+c)-Q_1^{(k)}(z))\Big\}=-iQ_1(z),$$

this shows that $\deg P\geq 3$, by comparing the coefficients. This must happen when $\deg Q_1\geq 3$. In the next when $q(-a)=0, q'(-a)=0$, we can obtain $\deg P\geq 3$, by comparing the coefficients in the equation (3.9) and also have $\deg Q_2\geq 3$. Thus, the Theorem 1.1 is completely finished. $\square$

4. Proof of Theorem 1.7

If $P(z)$ is replaced by $\frac{1}{P(z)}$, as in Theorem 1.1, one can obtain from (3.6) and (3.7) as

$$\sum_{k=0}^{l} b_k\Big\{a^k(Q_1(z+c)e^{ac}-Q_1(z))+ka^{k-1}(Q_1'(z+c)e^{ac}-Q_1'(z))+\ldots \tag{4.1}$$
$$\ldots+(Q_1^{(k)}(z+c)e^{ac}-Q_1^{(k)}(z))\Big\}=-iP(z)Q_1(z)$$

and

$$\sum_{k=0}^{l} b_k\Big\{(-a)^k(Q_2(z+c)e^{-ac}-Q_2(z))+k(-a)^{k-1}(Q_2'(z+c)e^{-ac} \tag{4.2}$$
$$-Q_2'(z))+\ldots+(Q_2^{(k)}(z+c)e^{-ac}-Q_2^{(k)}(z))\Big\}=iP(z)Q_2(z).$$

Similar to the **Case 1** in Theorem 1.1, we can get P is a constant if either $q(a)\neq 0$ or $q(-a)\neq 0$, where $q(z)=\sum_{k=0}^{l} b_k z^k$ and similar conclusion holds if either Q_1 or Q_2 is a constant. Furthermore, similar to the **Case 2** in Theorem 1.1, we can get a contradiction from (4.1) and (4.2). This concludes the proof of Theorem 1.7.

5. Proof of Theorem 1.8

Proof. Assuming f is a transcendental entire solution of (1.8) with finite order, we obtain

$$\left[\frac{f(z)}{\alpha}\right]^2+\left[\frac{a(z)f^{(m)}(z+c)+b(z)f^{(n)}(z)}{\alpha}\right]^2=1.$$

Based on this relationship, by Iyer's result [1] we write

$$f(z)=\alpha\sinh(z),\ a(z)f^{(m)}(z+c)+b(z)f^{(n)}(z)=\alpha\cosh(z), \tag{5.1}$$

in which h is a non-constant polynomial.

By differentiating the first equation in (5.1), we have

$$f^{(k)}(z)=\frac{\alpha(M_k(z)e^{ih(z)}-N_k(z)e^{-ih(z)})}{2i}, \tag{5.2}$$

where

$$\begin{aligned} M_k &= (ih')^k+F_{k-1}(h',h'',\ldots,h^{(k)})\\ N_k &= (-ih')^k+G_{k-1}(-h',-h'',\ldots,-h^{(k)}), \end{aligned}$$

in which F_{k-1},G_{k-1} are differential polynomials of $h',h'',\ldots,h^{(k)}$ such that $\deg F_{k-1}\leq k-1,\ \deg G_{k-1}\leq k-1$.

In view of (5.2) and the second equation in (5.1), we may derive

$$H_1 e^{ih(z+c)} + H_2 e^{-ih(z+c)} + H_3 e^{ih(z)} = H_4 e^{-ih(z)}, \tag{5.3}$$

where

$$H_1 = a(z)M_m(z+c), \quad H_2 = -a(z)N_m(z+c)$$
$$H_3 = b(z)M_n(z) - i, \quad H_4 = b(z)N_n(z) + i.$$

We may deduce that $H_j \not\equiv 0$ $(j = 1, 2, 3, 4)$ in the same way as in Theorem 1.1.

From (5.3), we now have

$$H_1 e^{f_1(z)} + H_2 e^{f_2(z)} + H_3 e^{f_3(z)} = H_4, \tag{5.4}$$

where

$$f_1(z) = ih(z) + ih(z+c), \; f_2(z) = ih(z) - ih(z+c), \; f_3(z) = 2ih(z).$$

Now applying Lemma 2.4, we find from (5.4) that $f_2(z) = f_3(z) - f_1(z) = ih(z) - ih(z+c)$ or $f_1(z) = f_3(z) - f_2(z) = ih(z) + ih(z+c)$ is a constant.

If $ih(z) + ih(z+c)$ is a constant, then $h(z)$ must be a constant. Otherwise, $0 = \deg[h(z) + h(z+c)] = \deg h(z) \geq 1$, which is not possible. Because h is a non-constant polynomial, there is a contradiction in that.

Therefore, $ih(z) - ih(z+c)$ must be a constant. Following that, we obtain $h'(z) \equiv h'(z+c)$. This demonstrates that the function $h'(z)$ is periodic and has period c. Since $\rho(h') = \rho(h) < 1$, it deduces from Lemma 2.1 that $h' = A$, where $A(\neq 0)$ is a constant, and hence $h(z) = Az + B$, where B is a constant.

Then, using the equation (5.1), it produces the desired result

$$f(z) = \alpha \sin(Az + B).$$

Now put $h = Az + B$ into (5.4), the following result can be found

$$\begin{cases} H_1 e^{iAc} & = -H_3, \\ H_2 e^{-iAc} & = H_4. \end{cases} \tag{5.5}$$

These two equalities result in

$$a(z)^2 A^{2m} = b(z)^2 A^{2n} + b(z)(i^{n+1} + (-i)^{n+1})A^n + 1. \tag{5.6}$$

We may now write the following form using the equation (1.8),

$$f(z)^2 + a(z)^2 (f^{(m)}(z+c))^2 + b(z)^2 (f^{(n)}(z))^2 + 2a(z)b(z)f^{(m)}(z+c)f^{(n)}(z) = \alpha^2.$$

By using the result in (5.6) to the equation above, we may obtain

$$I_1(z) e^{2iAz} + I_2(z) e^{-2iAz} = I_3(z), \tag{5.7}$$

where

$$\begin{aligned}
I_1(z) &= \alpha^2 A^{2m} e^{2iB} + i^{2m}\alpha^2\{A^{2m+2n}b(z)^2 + A^{2m+n}(i^{n+1}+(-i)^{n+1})b(z)\\
&\quad +A^{2m}\}e^{2iAc+2iB} + i^{2n}\alpha^2 A^{2m+2n}e^{2iB}b(z)^2\\
&\quad +2i^{m+n}A^{3m+n}\alpha^2 e^{2iB+iAc}a(z)b(z),\\
I_2(z) &= \alpha^2 A^{2m} e^{-2iB} + i^{2m}\alpha^2\{A^{2m+2n}b(z)^2 + A^{2m+n}(i^{n+1}+(-i)^{n+1})b(z)\\
&\quad +A^{2m}\}e^{-2iAc-2iB} + i^{2n}\alpha^2 A^{2m+2n}e^{-2iB}b(z)^2\\
&\quad +2(-i)^{m+n}A^{3m+n}\alpha^2 e^{-2iB-iAc}a(z)b(z),\\
I_3(z) &= 2\alpha^2\{2A^{2m+2n}b(z)^2 + A^{2m+n}(i^{n+1}+(-i)^{n+1})b(z)\}\\
&\quad +2i^{m+n}\alpha^2 A^{3m+n}((-1)^n e^{iAc} + (-1)^m e^{-iAc})a(z)b(z).
\end{aligned}$$

Applying Lemma 2.4 in the equation (5.7), we have

$$I_1(z)\equiv 0,\ I_2(z)\equiv 0,\ I_3(z)\equiv 0. \tag{5.8}$$

After simplicity in the third equation of (5.8), we have

$$2A^n b(z) + (i^{n+1}+(-i)^{n+1}) = -i^{m+n}A^m((-1)^n e^{iAc} + (-1)^m e^{-iAc})a(z). \tag{5.9}$$

The first two equations of (5.8) yield

$$\begin{aligned}
&2i^n A^{m+n}((-1)^{m+n}e^{-iAc} - e^{iAc})a(z)b(z)\\
&= i^m\{A^{2n}b(z)^2 + A^n(i^{n+1}+(-i)^{n+1})b(z)+1\}(e^{2iAc}-e^{-2iAc}).
\end{aligned}$$

Using this equation along with (5.9), we obtain

$$Ca(z)^2 + Da(z) = E, \tag{5.10}$$

where

$$\begin{aligned}
C &= 4i^{m+2n}A^{2m}((-1)^n e^{iAc} + (-1)^m e^{-iAc})((-1)^{m+n}e^{-iAc} - e^{iAc})\\
&\quad + i^{3m+2n}A^{2m}((-1)^n e^{iAc} + (-1)^m e^{-iAc})^2(e^{2iAc}-e^{-2iAc}),\\
D &= 4i^n A^m(i^{n+1}+(-i)^{n+1})((-1)^{m+n}e^{-iAc} - e^{iAc})\\
E &= i^m\{(i^{n+1}+(-i)^{n+1})^2 - 4\}(e^{2iAc}-e^{-2iAc}).
\end{aligned}$$

This shows from the equation (5.10) that $a(z)$ is a constant, say a. Furthermore, $b(z)$ reduce to a constant b. Afterward, (5.6) becomes

$$a^2A^{2m} = b^2A^{2n} + b(i^{n+1}+(-i)^{n+1})A^n + 1. \tag{5.11}$$

We talk about two cases:

Case (1): When n be even. Following that, by (5.11), we have

$$a^2A^{2m} = b^2A^{2n} + 1.$$

Case (2): When n be odd. Following that, by (5.11), we have

$$a^2A^{2m} = b^2A^{2n} + 2i^{n+1}bA^n + 1.$$

Furthermore, we may deduce $e^{iAc} = \frac{i-b(iA)^n}{a(iA)^m}$ from the first equation in (5.5).

This concludes the complete proof of the Theorem 1.8. □

6. Proof of Theorem 1.17

If any one of a_1, a_2, b_1, b_2 is zero, we established the following four cases.

Case 1: When $a_1 = 0$. Following the same factorization process as in Theorem 1.1, we obtain

$$\begin{aligned} f(z+c) &= \frac{e^{h(z)} + e^{-h(z)}}{2b_1}, \\ a_2 f^{(k)}(z) + b_2 f(z+c) &= \frac{e^{h(z)} - e^{-h(z)}}{2i}, \end{aligned} \tag{6.1}$$

in which h is a non-constant polynomial.

Now eliminating f from (6.1), we get

$$S(z)e^{h(z-c)} + T(z)e^{-h(z-c)} + (ib_1 + b_2)e^{h(z)} = (ib_1 - b_2)e^{-h(z)}, \tag{6.2}$$

where

$$S(z) = a_2 h_1(z-c), \ T(z) = a_2 h_2(z-c)$$

and

$$h_1(z-c) = (h'(z-c))^k + S_{1,k-1}(h'(z-c), h''(z-c), ..., h^{(k)}(z-c)),$$

$$h_2(z-c) = (-h'(z-c))^k + T_{1,k-1}(-h'(z-c), -h''(z-c), ..., -h^{(k)}(z-c)),$$

where $S_{1,k-1}, T_{1,k-1}$ are differential polynomials of $h', h'', ..., h^{(k)}$ with degree less than equal to $k-1$. Next, we distinguish four subcases.

Subcase 1.1. When $S(z) \equiv 0$ and $T(z) \equiv 0$.

Then equation (6.2) yields

$$(ib_1 + b_2)e^{2h(z)} = ib_1 - b_2,$$

which shows that h is a constant, this is a contradiction, so **Subcase 1.1** is excluded.

Subcase 1.2. When $S(z) \equiv 0$ and $T(z) \not\equiv 0$.

Hence $b_1 \neq -ib_2$, otherwise we have from (6.2) that

$$T(z)e^{h(z)-h(z-c)} = -2b_2,$$

which gets a contradiction using the second main theorem. Equation (6.2) then implies that

$$\frac{T(z)}{ib_1 - b_2} e^{h(z)-h(z-c)} + \frac{ib_1 + b_2}{ib_1 - b_2} e^{2h(z)} = 1.$$

Then, according to Lemma 2.2, we have

$$\begin{aligned} T(r, e^{2h}) &\leq \overline{N}\left(r, \frac{1}{e^{2h}}\right) + \overline{N}(r, e^{2h}) + \overline{N}\left(r, \frac{1}{e^{h(z)-h(z+c)}}\right) \\ &\quad + S(r, e^{2h}) \\ &= S(r, e^{2h}), \end{aligned}$$

which is a contradiction, so **Subcase 1.2** is excluded.

Subcase 1.3. When $S(z) \not\equiv 0$ and $T(z) \equiv 0$.

Hence $b_1 \neq -ib_2$, as in **Subcase 1.2**. Then (6.2) reduces to

$$\frac{S(z)}{ib_1 - b_2} e^{h(z)+h(z-c)} + \frac{ib_1 + b_2}{ib_1 - b_2} e^{2h(z)} = 1,$$

and using Lemma 2.2, we have

$$\begin{aligned} T(r, e^{2h}) &\leq \overline{N}\left(r, \frac{1}{e^{2h}}\right) + \overline{N}(r, e^{2h}) + \overline{N}\left(r, \frac{1}{e^{h(z)+h(z-c)}}\right) \\ &\quad + S(r, e^{2h}) \\ &= S(r, e^{2h}), \end{aligned}$$

which is impossible, so **Subcase 1.3** is excluded.

Subcase 1.4. When $S(z) \not\equiv 0$ and $T(z) \not\equiv 0$.

Now (6.2) becomes

$$S(z)e^{f_1(z)} + T(z)e^{f_2(z)} + (ib_1 + b_2)e^{f_3(z)} = ib_1 - b_2, \tag{6.3}$$

where

$$f_1(z) = h(z) + h(z-c), f_2(z) = h(z) - h(z-c), f_3(z) = 2h(z).$$

Now applying Lemma 2.3, we find from (6.3) that $f_2(z) = f_3(z) - f_1(z) = h(z) - h(z-c)$ or $f_1(z) = f_3(z) - f_2(z) = h(z) + h(z-c)$ is a constant.

If $h(z) + h(z-c)$ is a constant, then $h(z)$ must be a constant. Otherwise, $0 = \deg[h(z) + h(z-c)] = \deg h(z) \geq 1$, which is not possible. Because h is a non-constant polynomial, there is a contradiction in that.

Therefore, $h(z) - h(z-c)$ must be a constant. Following that, we obtain $h'(z) \equiv h'(z-c)$. This demonstrates that the function $h'(z)$ is periodic and has period c. Since $\rho(h') = \rho(h) < 1$, it deduces from Lemma 2.1 that $h' = a$, where $a(\neq 0)$ is a constant, and hence $h(z) = az + b$, where b is a constant.

Therefore, it follows from (6.1) that

$$f(z) = \frac{e^{az+b-ac} + e^{-az-b+ac}}{2b_1}.$$

Now put $h(z) = az + b$ into (6.3), we obtain

$$S(z)e^{2az+2b-ac} + T(z)e^{ac} + (ib_1 + b_2)e^{2az+2b} = ib_1 - b_2,$$

this yields

$$\begin{cases} S(z)e^{-ac} &= -ib_1 - b_2, \\ T(z)e^{ac} &= ib_1 - b_2. \end{cases}$$

Using the expressions of $S(z)$ and $T(z)$, the above two equations yield

$$a^{2k}a_2^2 = (-1)^k(b_1^2 + b_2^2).$$

Furthermore, we have $e^{ac} = -\frac{a^k a_2}{ib_1+b_2}$. Thus, the proof of **Case 1** is done.

Case 2: When $b_1 = 0$. Proceeding in the same as in the above factorization, we obtain

$$\begin{aligned} f^{(k)}(z) &= \frac{e^{h(z)} + e^{-h(z)}}{2a_1}, \\ a_2 f^{(k)}(z) + b_2 f(z+c) &= \frac{e^{h(z)} - e^{-h(z)}}{2i}, \end{aligned} \tag{6.4}$$

in which h is a non-constant polynomial.

Afterward, by differentiating (6.4), we get

$$\begin{aligned} f^{(2k)}(z) &= \frac{S_1(z)e^{h(z)} + S_2(z)e^{-h(z)}}{2a_1}, \\ a_2 f^{(2k)}(z) + b_2 f^{(k)}(z+c) &= \frac{S_1(z)e^{h(z)} - S_2(z)e^{-h(z)}}{2i}, \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} S_1(z) &= (h'(z))^k + S_{1,k-1}(h', h'', ..., h^{(k)}), \\ S_2(z) &= (-h'(z))^k + S_{2,k-1}(-h', -h'', ..., -h^{(k)}), \end{aligned}$$

in which $S_{1,k-1}, S_{2,k-1}$ are differential polynomials of $h', h'', ..., h^{(k)}$ with degree less than equal to $k-1$.

Now eliminating f from (6.4) and (6.5), we obtain

$$b_2 e^{h(z+c)} + b_2 e^{-h(z+c)} + T_1(z)e^{h(z)} = T_2(z)e^{-h(z)}, \tag{6.6}$$

where

$$T_1(z) = (ia_1 + a_2)S_1(z),\ T_2(z) = (ia_1 - a_2)S_2(z).$$

Next, we distinguish four subcases.

Subcase 2.1. When $T_1(z) \equiv 0$ and $T_2(z) \equiv 0$.

Then equation (6.6) produces

$$e^{2h(z+c)} = -1,$$

which shows that h is a constant, which is a contradiction, so **Subcase 2.1** is precluded.

Subcase 2.2. When $T_1(z) \equiv 0$ and $T_2(z) \not\equiv 0$.

Then, by (6.6), we conclude that

$$\frac{b_2}{T_2(z)} e^{h(z)+h(z+c)} + \frac{b_2}{T_2(z)} e^{h(z)-h(z+c)} = 1,$$

and by Lemma 2.2 the function $H_1(z) := h(z) - h(z+c)$ satisfies

$$\begin{aligned} T(r, e^{H_1}) &\le \overline{N}\left(r, \frac{1}{e^{H_1}}\right) + \overline{N}(r, e^{H_1}) + \overline{N}\left(r, \frac{1}{e^{2h-H_1}}\right) \\ &\quad + S(r, e^{H_1}) \\ &= S(r, e^{H_1}), \end{aligned}$$

which is a contradiction, so **Subcase 2.2** is precluded.

Subcase 2.3. When $T_1(z) \not\equiv 0$ and $T_2(z) \equiv 0$.

Thereafter, we reduce that by (6.6)

$$\frac{-b_2}{T_1(z)} e^{h(z+c)-h(z)} + \frac{-b_2}{T_1(z)} e^{-h(z)-h(z+c)} = 1,$$

and using Lemma 2.2 we have

$$\begin{aligned} T(r, e^{H_1-2h}) &\leq \overline{N}\left(r, \frac{1}{e^{H_1-2h}}\right) + \overline{N}(r, e^{H_1-2h}) + \overline{N}\left(r, \frac{1}{e^{-H_1}}\right) \\ &\quad + S(r, e^{H_1-2h}) \\ &= S(r, e^{H_1-2h}), \end{aligned}$$

which is impossible, so **Subcase 2.3** is precluded.

Subcase 2.4. When $T_1(z) \not\equiv 0$ and $T_2(z) \not\equiv 0$.

Now (6.6) changes into

$$b_2 e^{f_1(z)} + b_2 e^{f_2(z)} + T_1(z) e^{f_3(z)} = T_2(z), \tag{6.7}$$

where

$$f_1(z) = h(z) + h(z+c), f_2(z) = h(z) - h(z+c), f_3(z) = 2h(z).$$

Now applying Lemma 2.3, we find from (6.7) that $f_2(z) = f_3(z) - f_1(z) = h(z) - h(z+c)$ or $f_1(z) = f_3(z) - f_2(z) = h(z) + h(z+c)$ is a constant.

If $h(z) + h(z+c)$ is a constant, then $h(z)$ must be a constant. Otherwise, $0 = \deg[h(z) + h(z+c)] = \deg h(z) \geq 1$, which is not possible. Because h is a non-constant polynomial, there is a contradiction in that.

Therefore, $h(z) - h(z+c)$ must be a constant. Following that, we obtain $h'(z) \equiv h'(z+c)$. This demonstrates that the function $h'(z)$ is periodic and has period c. Since $\rho(h') = \rho(h) < 1$, it deduces from Lemma 2.1 that $h' = a$, where $a (\neq 0)$ is a constant, and hence $h(z) = az + b$, where b is a constant.

Therefore, it follows from (6.4) that

$$f(z) = \frac{e^{az+b} + (-1)^k e^{-az-b}}{2a_1 a^k} + A_{k-1}(z),$$

where $A_{k-1}(z)$ is a polynomial of z of degree less than equal to $k-1$. Furthermore, we conclude from the second equation in (6.4) that

$$f(z+c) = \left(-\frac{a_2}{2a_1 b_2} + \frac{1}{2ib_2}\right) e^{az+b} + \left(-\frac{a_2}{2a_1 b_2} - \frac{1}{2ib_2}\right) e^{-az-b}.$$

It deduces from the expressions of the above of $f(z+c) = f(z+c)$ that

$$\begin{cases} \frac{e^{ac}}{2a_1 a^k} &= -\frac{a_2}{2a_1 b_2} + \frac{1}{2ib_2}, \\ \frac{(-1)^k e^{-ac}}{2a_1 a^k} &= -\frac{a_2}{2a_1 b_2} - \frac{1}{2ib_2}, \\ A_{k-1}(z) &\equiv 0. \end{cases} \tag{6.8}$$

Thus

$$f(z)=\frac{e^{az+b}+(-1)^k e^{-az-b}}{2a_1a^k}.$$

The results of the first two equations in (6.8) show that

$$b_2^2=(-1)^k a^{2k}(a_1^2+a_2^2).$$

Furthermore, we have $e^{ac}=-\frac{a^k(ia_1+a_2)}{b_2}$. Thus, the proof of **Case 2** is done.

Case 3. When $a_2=0$. In a manner similar to that of **Case 1**, we obtain

$$a^{2k}a_1^2=(-1)^k(b_1^2+b_2^2),\ f(z)=\frac{e^{az+b-ac}+e^{-az-b+ac}}{2b_2}$$

and $$e^{ac}=-\frac{a^k a_1}{b_1+ib_2}.$$

Case 4. When $b_2=0$. In a manner similar to that of **Case 2**, we obtain

$$b_1^2=(-1)^k a^{2k}(a_1^2+a_2^2),\ f(z)=\frac{e^{az+b}+(-1)^k e^{-az-b}}{2a_2a^k}$$

and $$e^{ac}=-\frac{a^k(a_1+ia_2)}{b_1}.$$

Case 5. When a_1,a_2,b_1,b_2 being the non-zero constants. We write into the following form

$$\begin{aligned} a_1f^{(k)}(z)+b_1f(z+c)&=\frac{e^{h(z)}+e^{-h(z)}}{2},\\ a_2f^{(k)}(z)+b_2f(z+c)&=\frac{e^{h(z)}-e^{-h(z)}}{2i}, \end{aligned} \tag{6.9}$$

in which h is a non-constant polynomial. Then, using the equations above, we obtain at

$$\begin{aligned} (a_1b_2-a_2b_1)f^{(k)}(z)&=\frac{ib_1+b_2}{2}e^{h(z)}-\frac{ib_1-b_2}{2}e^{-h(z)},\\ (a_2b_1-a_1b_2)f(z+c)&=\frac{ia_1+a_2}{2}e^{h(z)}-\frac{ia_1-a_2}{2}e^{-h(z)}. \end{aligned} \tag{6.10}$$

If $a_1b_2-a_2b_1=0$, then (6.10) yields $e^{2h(z)}=-1$, which shows that h is a constant, which is a contradiction.

If $a_1b_2-a_2b_1\neq 0$, then (6.10) conclude that

$$(ib_1+b_2)e^{h(z+c)}+(-ib_1+b_2)e^{-h(z+c)}+S_3(z)e^{h(z)}=T_3(z)e^{-h(z)}, \tag{6.11}$$

where

$$S_3(z)=(ia_1+a_2)h_1(z),\ T_3(z)=(ia_1-a_2)h_2(z)$$

and

$$\begin{aligned} h_1(z)&=(h'(z))^k+S_{3,k-1}(h',h'',...,h^{(k)}),\\ h_2(z)&=(-h'(z))^k+T_{3,k-1}(-h',-h'',...,-h^{(k)}), \end{aligned}$$

in which $S_{3,k-1}, T_{3,k-1}$ are differential polynomials of $h', h'', ..., h^{(k)}$ with degree less than equal to $k-1$. .

We can deduce that $S_3(z) \not\equiv 0, T_3(z) \not\equiv 0$ in the same way as in the **Case 2** above.

Now (6.11) becomes

$$(ib_1 + b_2)e^{f_1(z)} + (-ib_1 + b_2)e^{f_2(z)} + S_3(z)e^{f_3(z)} = T_3(z), \tag{6.12}$$

where

$$f_1(z) = h(z) + h(z+c), f_2(z) = h(z) - h(z+c), f_3(z) = 2h(z).$$

Now applying Lemma 2.3, we find from (6.12) that $f_2(z) = f_3(z) - f_1(z) = h(z) - h(z+c)$ or $f_1(z) = f_3(z) - f_2(z) = h(z) + h(z+c)$ is a constant.

If $h(z) + h(z+c)$ is a constant, then $h(z)$ must be a constant. Otherwise, $0 = \deg[h(z) + h(z+c)] = \deg h(z) \geq 1$, which is not possible. Because h is a non-constant polynomial, there is a contradiction in that.

Therefore, $h(z) - h(z+c)$ must be a constant. Following that, we obtain $h'(z) \equiv h'(z+c)$. This demonstrates that the function $h'(z)$ is periodic and has period c. Since $\rho(h') = \rho(h) < 1$, it deduces from Lemma 2.1 that $h' = a$, where $a(\neq 0)$ is a constant, and hence $h(z) = az + b$, where b is a constant.

From (6.10), it follows that

$$f(z) = \frac{(ib_1 + b_2)e^{az+b} + (-1)^k(-ib_1 + b_2)e^{-az-b}}{2(a_1b_2 - a_2b_1)a^k} + B_{k-1}(z),$$

where $B_{k-1}(z)$ is a polynomial of z of degree less than equal to $k-1$. The expression $f(z+c)$ in the equation (6.10) and $f(z)$ lead to the conclusion that

$$\begin{cases} \frac{ib_1+b_2}{2(a_1b_2-a_2b_1)} \frac{e^{ac}}{a^k} & = \frac{ia_1+a_2}{2(a_2b_1-a_1b_2)}, \\ \frac{-ib_1+b_2}{2(a_1b_2-a_2b_1)} \frac{(-1)^k e^{-ac}}{a^k} & = \frac{-ia_1+a_2}{2(a_2b_1-a_1b_2)}, \\ B_{k-1}(z) & \equiv 0. \end{cases} \tag{6.13}$$

Thus

$$f(z) = \frac{(ib_1 + b_2)e^{az+b} + (-1)^k(-ib_1 + b_2)e^{-az-b}}{2(a_1b_2 - a_2b_1)a^k}.$$

The results of the first two equations in (6.13) indicate that

$$a^{2k}(a_1^2 + a_2^2) = (-1)^k(b_1^2 + b_2^2).$$

Furthermore, we have $e^{ac} = -\frac{a^k(ia_1+a_2)}{(ib_1+b_2)}$. Theorem 1.18 has thus been fully proved.

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References

[1] Ganapathy Iyer, G. On certain functional equations. *J. Indian Math. Soc. 3* (1939), 312–315.

[2] Gross, F. On the equation $f^n + g^n = 1$. *Bull. Amer. Math. Soc. 72* (1966), 86–88.

[3] Gross, F. *Factorization of meromorphic functions*. Naval Research Laboratory, Mathematics Research Center, Washington, DC, 1972.

[4] Han, Q. On complex analytic solutions of the partial differential equation $(u_{z_1})^m + (u_{z_2})^m = u^m$. *Houston J. Math. 35*, 1 (2009), 277–289.

[5] Han, Q., and Lü, F. On the equation $f^n(z) + g^n(z) = e^{\alpha z+\beta}$. *Izv. Nats. Akad. Nauk Armenii Mat. 54*, 2 (2019), 90–96.

[6] Hayman, W. K. *Meromorphic functions*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1964.

[7] Hu, P., Wang, W., and Wu, L. Entire solutions of differential-difference equations of Fermat type. *Bull. Korean Math. Soc. 59*, 1 (2022), 83–99.

[8] Hu, P.-C., Li, P., and Yang, C.-C. *Unicity of meromorphic mappings*, vol. 1 of *Advances in Complex Analysis and its Applications*. Kluwer Academic Publishers, Dordrecht, 2003.

[9] Laine, I. *Nevanlinna theory and complex differential equations*, vol. 15 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1993.

[10] Li, B. Q. Entire solutions of $(u_{z_1})^m + (u_{z_2})^n = e^g$. *Nagoya Math. J. 178* (2005), 151–162.

[11] Liu, K., and Dong, X. Fermat type differential and difference equations. *Electron. J. Differential Equations* (2015), No. 159, 10.

[12] Liu, K., and Yang, L. On entire solutions of some differential-difference equations. *Comput. Methods Funct. Theory 13*, 3 (2013), 433–447.

[13] Montel, P. *Leçons sur les récurrences et leurs applications*. Gauthier-Villars, Paris, 1957. Recueillies et rédigées par Jacques Dufresnoy et Éloi Lefebvre.

[14] Tang, J.-F., and Liao, L.-W. The transcendental meromorphic solutions of a certain type of nonlinear differential equations. *J. Math. Anal. Appl. 334*, 1 (2007), 517–527.

[15] Wang, Q., Chen, W., and Hu, P. On entire solutions of two certain Fermat-type differential-difference equations. *Bull. Malays. Math. Sci. Soc. 43*, 4 (2020), 2951–2965.

[16] Yang, C.-c. A generalization of a theorem of P. Montel on entire functions. *Proc. Amer. Math. Soc. 26* (1970), 332–334.

[17] Yang, C.-C., and Li, P. On the transcendental solutions of a certain type of nonlinear differential equations. *Arch. Math. (Basel) 82*, 5 (2004), 442–448.

[18] Yang, C.-C., and Yi, H.-X. *Uniqueness theory of meromorphic functions*, vol. 557 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 2003.

[19] Yang, L. *Value distribution theory*. Springer-Verlag, Berlin; Science Press Beijing, Beijing, 1993. Translated and revised from the 1982 Chinese original.

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