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Golden slant lightlike submanifolds of golden semi-Riemannian manifolds

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Abstract. We introduce the study of golden slant lightlike submanifolds of golden semi-Riemannian manifolds. After defining a golden slant lightlike submanifold of a golden semi-Riemannian manifold, we present a non-trivial example of such lightlike submanifolds. Then we derive a necessary and sufficient condition for the induced connection on a golden slant lightlike submanifold of a golden semi-Riemannian manifold to be a metric connection. Further, we establish some characterization results for the integrability of distributions arising in golden slant lightlike submanifolds of golden semi-Riemannian manifolds. Finally, we discuss the concept of minimal golden slant lightlike submanifolds.

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1. Introduction

The theory of lightlike submanifolds is one of the most significant contributions in differential geometry. In case of lightlike submanifolds, the tangent bundle has non-empty intersection with the normal bundle. This primary difference makes the study of lightlike submanifolds different and more complicated from the case of Riemannian submanifolds. The basic theory of lightlike submanifolds was developed by Duggal-Bejancu [6] and has been further developed by Duggal-Sahin and many others (see [5], [7], [9], [17] etc.). In recent years, many significant applications of lightlike submanifolds have been found in the study of black holes, Killing horizons and radiation and electromagnetic fields etc. (c.f. [6], [10]).

On the other hand, the concept of an almost complex manifold (resp., an almost product manifold) with an almost complex structure $P^2 = -I$ (resp., an almost product structure $P^2 = I$) has been widely studied. Further, as a generalization of these structures, Yano [18] established the f-structure which is a tensor field of the type (1,1) satisfying the equation $f^3 + f = 0$. On a similar note, Crasmareanu and Hretcanu ([3], [4]) introduced a new kind of structure with the structural polynomial $f(x) = x^2 - x - Id = 0$ and studied its properties on a Riemannian manifold. This new structure $P^2 = P + I$ is known as a

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golden structure. In 2017, Poyraz and Yasar [14] introduced the golden structure in lightlike hypersurfaces of semi-Riemannian manifolds. Then, in 2019, Erdogan [8] and Poyraz and Yasar [15] developed the study of golden structure in lightlike submanifolds. Further, Poyraz [13] studied golden GCR-lightlike submanifolds of a golden semi-Riemannian manifold. After that, the concept of golden structure in pseudo slant lightlike submanifolds and semi-slant lightlike submanifolds was studied by Acet [1] and Kumar and Yadav [11]. Recently, Pruthi and Kumar studied golden generic lightlike submanifolds of a golden semi-Riemannian manifold in [16].
Therefore, in the present paper, we study golden slant lightlike submanifolds of golden semi-Riemannian manifolds. After giving the definition of a golden slant lightlike submanifolds of a golden semi-Riemannian manifold, we provide a non-trivial example of this class of lightlike submanifolds. Then, we derive a characterization result for the induced connection on a golden slant lightlike submanifold of a golden semi-Riemannian manifold to be a metric connection. Further, we find necessary and sufficient conditions for the integrability of distributions associated with golden slant lightlike submanifolds and also discuss the geometry of leaves of the distributions. Finally, we discuss the concept of minimal golden slant lightlike submanifolds of golden semi-Riemannian manifolds.

2. Preliminaries

2.1. Geometry of lightlike submanifolds

Assume a submanifold (K_n, g) of a semi-Riemannian manifold $(\bar{K}_{m+n}, \bar{g})$ such that $\bar{g}$ is a metric with index q satisfying $m, n \geq 1$ and $m+n-1 \geq q \geq 1$. If the metric $\bar{g}$ is degenerate on TK, then T_pK and $T_pK^\perp$, both are degenerate and there exists a radical subspace $Rad(T_pK)$ such that $Rad(T_pK) = T_pK \cap T_pK^\perp$. If $Rad(TK) : p \in K \to Rad(T_pK)$ is a smooth distribution on K with rank $r > 0$ and $1 \leq r \leq n$, then K is called an r-lightlike submanifold of $\bar{K}$ [6]. While the radical distribution $Rad(TK)$ of TK is defined as

$$Rad(TK) = \cup_{p\in K}\{\xi \in T_pK | g(u, \xi) = 0, \ \forall \ u \in T_pK, \xi \neq 0\}.$$

Further, let $S(TK)$ be the screen distribution in TK such that $TK = Rad(TK) \perp S(TK)$ and $S(TK^\perp)$ be the screen transversal vector bundle in $TK^\perp$ such that $TK^\perp = Rad(TK) \perp S(TK^\perp)$.
Moreover there exists a local null frame $\{N_i\}$ of null sections with values in the orthogonal complement of $S(TK^\perp)$ in $S(TK^\perp)^\perp$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \text{ for any } i, j \in \{1, 2, .., r\},$$

where $\{\xi_i\}$ is any local basis of $\Gamma(Rad(TK))$. It implies that $tr(TK)$ and $ltr(TK)$ respectively are the vector bundles in $T\bar{K}|_K$ and $S(TK^\perp)^\perp$ with the property

$$tr(TK) = ltr(TK) \perp S(TK^\perp) \tag{2.1}$$

and

$$T\bar{K}|_K = TK \oplus tr(TK) = S(TK) \perp (Rad(TK) \oplus ltr(TK)) \perp S(TK^\perp).$$

Let us denote by $\bar{\nabla}$ and ∇ the Levi-Civita connection on $\bar{K}$ and torsion-free linear connection on K, respectively. Then, the Gauss and Weingarten formulae are given as

$$\bar{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h^l(Y_1, Y_2) + h^s(Y_1, Y_2), \tag{2.2}$$

$$\bar{\nabla}_{Y_1} N = -A_N Y_1 + \nabla^l_{Y_1} N + D^s(Y_1, N), \tag{2.3}$$

$$\bar{\nabla}_{Y_1} W = -A_W Y_1 + D^l(Y_1, W) + \nabla^s_{Y_1} W, \tag{2.4}$$

where $Y_1, Y_2 \in \Gamma(TK), N \in \Gamma(ltr(TK))$ and $W \in \Gamma(S(TK^\perp))$. Further, employing Eqs. (2.2) and (2.4), we derive

$$g(A_W Y_1, Y_2) = \bar{g}(h^s(Y_1, Y_2), W) + \bar{g}(Y_2, D^l(Y_1, W)). \tag{2.5}$$

Let us denote the projection morphism of TK on $S(TK)$ by η. Then, we have

$$\nabla_{Y_1} \eta Y_2 = \nabla^*_{Y_1} \eta Y_2 + h^*(Y_1, \eta Y_2), \quad \nabla_{Y_1} \xi = -A^*_\xi Y_1 + \nabla^{*t}_{Y_1} \xi, \tag{2.6}$$

where $\{h^*(Y_1, \eta Y_2), \nabla^{*t}_{Y_1} \xi\} \in \Gamma(Rad(TK))$ and $\{\nabla^*_{Y_1} \eta Y_2, A^*_\xi Y_1\} \in \Gamma(S(TK))$. Further, employing Eqs. (2.3), (2.4) and (2.6), we get

$$\bar{g}(h^l(Y_1, \eta Y_2), \xi) = g(A^*_\xi Y_1, \eta Y_2). \tag{2.7}$$

As $\bar{\nabla}$ is a metric connection on $\bar{K}$, therefore for any $Y_1, Y_2, Y_3 \in \Gamma(TK)$, one has

$$(\nabla_{Y_1} g)(Y_2, Y_3) = \bar{g}(h^l(Y_1, Y_2), Y_3) + \bar{g}(h^l(Y_1, Y_3), Y_2), \tag{2.8}$$

which implies that ∇ is not always a metric connection on K.

2.2. Golden semi-Riemannian manifolds

A semi-Riemannian manifold $(\bar{K}, \bar{g})$ with a golden structure P on $\bar{K}$ is said to be a golden semi-Riemannian manifold [12] if

$$P^2 = P + I, \quad \bar{g}(PY_1, Y_2) = \bar{g}(Y_1, PY_2), \tag{2.9}$$

for $Y_1, Y_2 \in \Gamma(T\bar{K})$. If P is a golden structure, then it can be seen that

$$\bar{g}(PY_1, PY_2) = \bar{g}(PY_1, Y_2) + \bar{g}(Y_1, Y_2), \tag{2.10}$$

for $Y_1, Y_2 \in \Gamma(T\bar{K})$.

3. Golden slant lightlike submanifolds

Firstly, we state the following two essential lemmas given in [11] , which will be useful in forthcoming part of paper.

Lemma 3.1. *Consider an r-lightlike submanifold K of golden semi-Riemannian manifold $\bar{K}$ with index $2q$ and $PRad(TK)$ is a distribution on K along with $Rad(TK) \cap PRad(TK) = \{0\}$. Then, $Pltr(TK)$ is a subbundle of $S(TK)$ such that $PRad(TK) \cap Pltr(TK) = \{0\}$.*

Lemma 3.2. *For an r-lightlike submanifold K of golden semi-Riemannian manifold $\bar{K}$, along with the assumption of Lemma 3.1, (provided, $r = q$), any complementary distribution to $PRad(TK) \oplus Pltr(TK)$ in $S(TK)$ must be Riemannian.*

Definition 3.3. A q-lightlike submanifold K of a golden semi-Riemannian manifold $\bar{K}$ with index $2q$ is called a golden slant lightlike submanifold of $\bar{K}$ if

(A) $Rad(TK)$ is a distribution on K such that $PRad(TK) \cap Rad(TK) = \{0\}$.

(B) For each non-zero vector field Z tangent to D at $z \in U \subset K$, the angle $\theta(Z)$ between PZ and the vector space D_z is constant (known as slant angle), that is, it is independent of the choice of $z \in U \subset K$ and $Z \in D_z$, where D is the complementary distribution to $PRad(TK) \oplus Pltr(TK)$ in $S(TK)$.

In view of above definition, the tangent bundle is decomposed as

$$TK = Rad(TK) \perp (PRad(TK) \oplus Pltr(TK)) \perp D.$$

For $Y \in \Gamma(TK)$, we consider

$$PY = fY + \omega Y, \tag{3.1}$$

where $fY \in \Gamma(TK)$ and $\omega Y \in \Gamma(tr(TK))$. Similarly for $V \in \Gamma(tr(TK))$, we assume

$$PV = tV + nV, \tag{3.2}$$

where $tV \in \Gamma(TK)$ and $nV \in \Gamma(tr(TK))$. Then we have

$$f^2 = f + I - t\omega, \quad \omega = \omega f + n\omega, \tag{3.3}$$

$$n^2 = n + I - \omega t, \quad t = ft + tn.$$

Let ϕ_1, ϕ_2, ϕ_3 and ϕ_4 be the projections of TK on $Rad(TK)$, $P(Rad(TK))$, $P(ltr(TK))$ and D respectively. Then for $Y \in \Gamma(TK)$, we have

$$Y = \phi_1 Y + \phi_2 Y + \phi_3 Y + \phi_4 Y, \tag{3.4}$$

applying P to Eq. (3.4), we obtain

$$PY = P\phi_1 Y + P\phi_2 Y + P\phi_3 Y + P\phi_4 Y,$$

which gives

$$PY = f\phi_1 Y + f\phi_2 Y + \omega\phi_3 Y + f\phi_4 Y + \omega\phi_4 Y. \tag{3.5}$$

Moreover Eq. (3.5) can be written as

$$PY = fY + \omega\phi_3 Y + \omega\phi_4 Y,$$

where $fY = f\phi_1 Y + f\phi_2 Y + f\phi_4 Y$.
On differentiating Eq. (3.5) and then considering the components of $Rad(TK)$, $PRad(TK)$, $Pltr(TK)$, $ltr(TK)$, D and $S(TK^{\perp})$, we obtain

$$\begin{aligned}\phi_1(\nabla_{Y_1} P\phi_1 Y_2) + \phi_1(\nabla_{Y_1} P\phi_2 Y_2) + \phi_1(\nabla_{Y_1} f\phi_4 Y_2) &= \phi_1(A_{\omega\phi_3 Y_2} Y_1)\\ &+ \phi_1(A_{\omega\phi_4 Y_2} Y_1) + P\phi_2 \nabla_{Y_1} Y_2.\end{aligned} \tag{3.6}$$

$$\begin{aligned}\phi_2(\nabla_{Y_1} P\phi_1 Y_2) + \phi_2(\nabla_{Y_1} P\phi_2 Y_2) + \phi_2(\nabla_{Y_1} f\phi_4 Y_2) &= \phi_2(A_{\omega\phi_3 Y_2} Y_1)\\ &+ \phi_2(A_{\omega\phi_4 Y_2} Y_1) + P\phi_1 \nabla_{Y_1} Y_2.\end{aligned} \tag{3.7}$$

$$\begin{aligned}\phi_3(\nabla_{Y_1} P\phi_1 Y_2) + \phi_3(\nabla_{Y_1} P\phi_2 Y_2) + \phi_3(\nabla_{Y_1} f\phi_4 Y_2) &= \phi_3(A_{\omega\phi_3 Y_2} Y_1)\\ &+ \phi_3(A_{\omega\phi_4 Y_2} Y_1) + Ph^l(Y_1, Y_2).\end{aligned} \tag{3.8}$$

$$\begin{aligned}h^l(Y_1, P\phi_1 Y_2) + h^l(Y_1, P\phi_2 Y_2) + h^l(Y_1, f\phi_4 Y_2) =& \omega\phi_3 \nabla_{Y_1} Y_2 - \nabla^l_{Y_1} \omega\phi_3 Y_2\\ &- D^l(Y_1, \omega\phi_4 Y_2).\end{aligned} \tag{3.9}$$

$$\begin{aligned}\phi_4(\nabla_{Y_1} P\phi_1 Y_2) + \phi_4(\nabla_{Y_1} P\phi_2 Y_2) + \phi_4(\nabla_{Y_1} f\phi_4 Y_2) &= \phi_4(A_{\omega\phi_3 Y_2} Y_1)\\ &+ \phi_4(A_{\omega\phi_4 Y_2} Y_1) + f\phi_4 \nabla_{Y_1} Y_2 + th^s(Y_1, Y_2).\end{aligned} \tag{3.10}$$

$$\begin{aligned}h^s(Y_1, P\phi_1 Y_2) + h^s(Y_1, P\phi_2 Y_2) + h^s(Y_1, f\phi_4 Y_2) &= \omega\phi_4 \nabla_{Y_1} Y_2 - \nabla^s_{Y_1} \omega\phi_4 Y_2\\ &- D^s(Y_1, \omega\phi_3 Y_2) + nh^s(Y_1, Y_2).\end{aligned} \tag{3.11}$$

Lemma 3.4. *For a golden slant lightlike submanifold K of a golden semi-Riemannian manifold $\bar{K}$, one has $\omega\phi_4 Y \in \Gamma(S(TK^{\perp}))$ for $Y \in \Gamma(TK)$.*

Proof. For $Y \in \Gamma(TK)$, $\omega\phi_4 Y \in \Gamma(S(TK^{\perp}))$, if and only if, $\bar{g}(\omega\phi_4 Y, \xi) = 0$ for $\xi \in \Gamma(Rad(TK))$. Therefore, $\bar{g}(\omega\phi_4 Y, \xi) = \bar{g}(P\phi_4 Y - f\phi_4 Y, \xi) = \bar{g}(P\phi_4 Y, \xi) = g(\phi_4 Y, P\xi) = 0$ implies $\omega\phi_4 Y$ has no components in $ltr(TK)$. Hence, the result follows. □

From above lemma, it is clear that

$$T\bar{K}|_K = S(TK) \perp \{Rad(TK) \oplus ltr((TK)\} \perp \{\omega(D) \perp \mu\}.$$

Theorem 3.5. *(Existence Theorem) Consider K, a q-lightlike submanifold of a golden semi-Riemannian manifold $\bar{K}$. Then K is a golden slant llightlike submanifold, if and only if,* *(i) $Pltr(TK)$ is a distribution on K,*

(ii) $f^2Y = \lambda(PY + Y)$,

(iii) $t\omega Y = (1-\lambda)(fY + Y) - \lambda(\omega Y)$,

for $Y \in \Gamma(D)$, where $\lambda = \cos^2\theta$ and $\lambda \in [0, 1)$.

Proof. Assume that K is a golden slant lightlike submanifold of $\bar{K}$. Then from Lemma 3.1, we have $Pltr(TK)$ is also a distribution on K such that $Pltr(TK) \subset S(TK)$, which proves (i). In addition, for $Y \in \Gamma(D)$, the angle between PY and D is constant. Thus, we have

$$\cos\theta = \frac{|fY|}{|PY|}, \tag{3.12}$$

which gives

$$\cos^2\theta = \frac{|fY|^2}{|PY|^2} = \frac{g(fY, fY)}{\bar{g}(PY, PY)} = \frac{g(Y, f^2Y)}{\bar{g}(Y, P^2Y)}. \tag{3.13}$$

Therefore, we have

$$g(Y, f^2Y) = \cos^2\theta(Y)\bar{g}(Y, P^2Y). \tag{3.14}$$

As we know that θ is constant on slant distribution D, thus here we conclude that $g(Y, f^2Y) = \lambda\bar{g}(Y, P^2Y) = \bar{g}(Y, \lambda P^2Y)$, which further gives

$$\bar{g}(Y, (f^2 - \lambda P^2)Y) = 0. \tag{3.15}$$

Since $(f^2 - \lambda P^2)Y \in \Gamma(D)$ and non-degeneracy of D implies $(f^2 - \lambda P^2)Y = 0$, then we get

$$f^2Y = \lambda P^2Y = \lambda(PY + Y), \tag{3.16}$$

for $Y \in \Gamma(D)$. Hence the assertion (ii) proved.
Next, for $Y \in \Gamma(D)$, employing Eq. (3.3), we have

$$t\omega Y = -f^2Y + fY + Y. \tag{3.17}$$

Since K is a golden slant lightlike submanifold, therefore using Eq. (3.16), we obtain

$$\begin{aligned} t\omega Y &= -\lambda(PY + Y) + fY + Y \\ &= -\lambda(fY + Y) - \lambda(\omega Y) + fY + Y \\ &= (1-\lambda)(fY + Y) - \lambda(\omega Y), \end{aligned} \tag{3.18}$$

which proves the assertion (iii).
Conversely, let K be a q-lightlike submanifold such that conditions (i), (ii) and (iii) are satisfied. Therefore condition (i) implies that $PRad(TK)$ is a distribution on K. Further, from Lemma 3.2, it is clear that the complementary

distribution of $PRad(TK) \oplus Pltr((TK)$ in $S(TK)$, is Riemannian, therefore for $Y \in \Gamma(D)$, we have

$$\cos\theta = \frac{\bar{g}(PY, fY)}{|PY||fY|} = \frac{g(Y, f^2Y)}{|PY||fY|} = \lambda\frac{\bar{g}(Y, P^2Y)}{|PY||fY|} = \lambda\frac{\bar{g}(PY, PY)}{|PY||fY|},$$

which further gives

$$cos\theta = \lambda\frac{|PY|}{|fY|} \tag{3.19}$$

and also

$$cos\theta = \frac{|fY|}{|PY|}. \tag{3.20}$$

Therefore, from Eqs. (3.19) and (3.20), we conclude that $\cos^2\theta$ is constant and K is golden slant lightlike submanifold. Hence, the proof is completed. □

Example 3.6. Let $(\bar{K}, \bar{g}) = (\mathbb{R}^{14}_2, \bar{g})$ be a semi-Euclidean space of signature $(+, +, +, -, +, -, +, +, +, +, +, +, +, +)$ w.r.t. the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10}, \partial x_{11}, \partial x_{12}, \partial x_{13}, \partial x_{14}\}$. Consider a golden structure P defined by

$$\begin{aligned}P(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}) = (&x_1 + x_2, x_1, x_3 + x_4,\\ &x_3, x_5 + x_6, x_5, (x_7 - x_8)\sin\alpha - (x_{11} + x_{12})\cos\alpha, -x_8\sin\alpha\\ &- x_{11}\cos\alpha, (x_9 - x_{10})\sin\alpha - (x_{13} + x_{14})\cos\alpha, -x_{10}\sin\alpha\\ &- x_{13}\cos\alpha, (x_7 + x_8)\cos\alpha + (x_{12} - x_{11})\sin\alpha, x_7\cos\alpha + x_{12}\sin\alpha,\\ &(x_9 + x_{10})\cos\alpha + (x_{14} - x_{13})\sin\alpha, x_9\cos\alpha + x_{14}\sin\alpha),\end{aligned}$$

where $\alpha \in \left(0, \frac{\pi}{2}\right)$. Consider K, a 5-dimensional submanifold of $(\mathbb{R}^{14}_2, \bar{g})$ given by

$$x^1 = u^2, \quad x^2 = u^1, \quad x^3 = u^5\cos\theta, \quad x^4 = u^1\cos\theta, \quad x^5 = u^5\sin\theta,$$

$$x^6 = u^1\sin\theta, \quad x^7 = u^4\cos\theta, \quad x^8 = u^3\cos\theta, \quad x^9 = u^4\sin\theta,$$

$$x^{10} = u^3\sin\theta, \quad x^{11} = u^4, \quad x^{12} = u^3, \quad x^{13} = u^4, \quad x^{14} = 0,$$

where $\theta \in \mathbb{R} - \left\{\frac{n\pi}{2},\ n \in Z\right\}$. Then TK is spanned by Z_1, Z_2, Z_3, Z_4, Z_5, where

$$Z_1 = \partial x_2 + \cos\theta\partial x_4 + \sin\theta\partial x_6, \quad Z_2 = \partial x_1, \quad Z_3 = \cos\theta\partial x_8 + \sin\theta\partial x_{10} + \partial x_{12},$$

$$Z_4 = \cos\theta\partial x_7 + \sin\theta\partial x_9 + \partial x_{11} + \partial x_{13}, \quad Z_5 = \cos\theta\partial x_3 + \sin\theta\partial x_5.$$

Thus, K is a 1-lightlike submanifold with $Rad(TK) = Span\{Z_1\}$ and $ltr(TK)$ is spanned by

$$N_1 = \frac{1}{2}\{\partial x_2 - \cos\theta\partial x_4 - \sin\theta\partial x_6\}.$$

It follows that $PZ_1 = Z_2 + Z_5$ and $PN_1 = \frac{1}{2}\{Z_2 - Z_5\}$, which implies that $PRad(TK)$ and $Pltr(TK)$ are distributions on N. Hence $D = Span\{Z_3, Z_4\}$ is a slant distribution w.r.t. P with slant angle α. Further by direct calculations, $S(TK^{\perp})$ is spanned by

$$W = \cos\theta\partial x_7 + \sin\theta\partial x_9 - \partial x_{13} + \partial x_{14}.$$

Therefore, K is a proper golden slant lightlike submanifold of $\mathbb{R}^{14}_2$.

From Eq. (2.8), one may note that the induced connection on a lightlike submanifold is not necessarily a metric connection, in general. In this context, we give the following result enabling the induced connection ∇ on a golden slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{K}$ to be a metric connection.

Theorem 3.7. *For a golden slant lightlike submanifold K of a golden semi-Riemannian manifold $\bar{K}$, the induced connection ∇ is a metric connection, if and only if,*

$$P(\nabla^*_Y P\xi + h^*(Y, P\xi)) - \nabla^*_Y P\xi - h^*(Y, P\xi) \in \Gamma(Rad(TK))$$

and

$$th(Y, P\xi) = 0,$$

for $Y \in \Gamma(TK)$ and $\xi \in \Gamma(Rad(TK))$.

Proof. For $Y \in \Gamma(TK)$ and $\xi \in \Gamma(Rad(TK))$, employing Eq. (2.9), we have

$$\bar{\nabla}_Y \xi = P\bar{\nabla}_Y P\xi - \bar{\nabla}_Y P\xi.$$

Further, using Eqs. (2.2) and (3.2), we derive

$$\begin{aligned}\nabla_Y \xi + h(Y, \xi) &= P(\nabla_Y P\xi + h(Y, P\xi)) - (\nabla_Y P\xi + h(Y, P\xi)) \\ &= P\nabla_Y P\xi + th(Y, P\xi) + nh(Y, P\xi) - (\nabla_Y P\xi + h(Y, P\xi)).\end{aligned}$$

Then, equating the tangential components of above equation, we obtain

$$\nabla_Y \xi = P\nabla_Y P\xi + th(Y, P\xi) - \nabla_Y P\xi. \tag{3.21}$$

Now, using Eq. (2.6) in Eq. (3.21), we get

$$\nabla_Y \xi = P(\nabla^*_Y P\xi + h^*(Y, P\xi)) - \nabla^*_Y P\xi - h^*(Y, P\xi) + th(Y, P\xi). \tag{3.22}$$

Hence from Eq. (3.22), $\nabla_Y \xi \in \Gamma(Rad(TK))$, if and only if, $P(\nabla^*_Y P\xi + h^*(Y, P\xi)) - \nabla^*_Y P\xi - h^*(Y, P\xi) \in \Gamma(Rad(TK))$ and $th(Y, P\xi) = 0$. This completes the proof. □

Theorem 3.8. *Assume that K is a golden slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{K}$. Then the Nijenhius torsion tensor field of golden structure P vanishes.*

Proof. For $Y_1, Y_2 \in \Gamma(TK)$, employing Eq. (2.9), the Nijenhius tensor field becomes

$$\begin{aligned}N(Y_1,Y_2) =&[PY_1,PY_2]+P^2[Y_1,Y_2]-P[PY_1,Y_2]-P[Y_1,PY_2]\\ =&[PY_1,PY_2]+P[Y_1,Y_2]+[Y_1,Y_2]-P[PY_1,Y_2]-P[Y_1,PY_2]\\ =&\bar{\nabla}_{PY_1}PY_2-\bar{\nabla}_{PY_2}PY_1+P\{\bar{\nabla}_{Y_1}Y_2-\bar{\nabla}_{Y_2}Y_1\}+\bar{\nabla}_{Y_1}Y_2-\bar{\nabla}_{Y_2}Y_1\\ &-P\{\bar{\nabla}_{PY_1}Y_2-\bar{\nabla}_{Y_2}PY_1\}-P\{\bar{\nabla}_{Y_1}PY_2-\bar{\nabla}_{PY_2}Y_1\}.\\ =&P\bar{\nabla}_{Y_1}Y_2-P\bar{\nabla}_{Y_2}Y_1+\bar{\nabla}_{Y_1}Y_2-\bar{\nabla}_{Y_2}Y_1+P\bar{\nabla}_{Y_2}PY_1-P\bar{\nabla}_{Y_1}PY_2\\ =&P\bar{\nabla}_{Y_1}Y_2-P\bar{\nabla}_{Y_2}Y_1+\bar{\nabla}_{Y_1}Y_2-\bar{\nabla}_{Y_2}Y_1+P^2\bar{\nabla}_{Y_2}Y_1-P^2\bar{\nabla}_{Y_1}Y_2\\ =&P\bar{\nabla}_{Y_1}Y_2-P\bar{\nabla}_{Y_2}Y_1+\bar{\nabla}_{Y_1}Y_2-\bar{\nabla}_{Y_2}Y_1+P\bar{\nabla}_{Y_2}Y_1+\bar{\nabla}_{Y_2}Y_1\\ &-P\bar{\nabla}_{Y_1}Y_2-\bar{\nabla}_{Y_1}Y_2\\ =&0.\end{aligned}$$

Hence, the result holds. ☐

Lemma 3.9. *Consider a golden slant lightlike submanifold K of a golden semi-Riemannian manifold $\bar{K}$. Then*

$$(\nabla_{Y_1}f)Y_2 = A_{\omega\phi_1 Y_2}Y_1 + A_{\omega\phi_4 Y_2}Y_1 + th(Y_1,Y_2) \tag{3.23}$$

and

$$\begin{aligned}\omega\nabla_{Y_1}Y_2 - h(Y_1,fY_2) + Ch(Y_1,Y_2) =&D^s(Y_1,\omega\phi_3Y_2) + D^l(Y_1,\omega\phi_4Y_2)\\ &+\nabla^s_{Y_1}\omega\phi_4Y_2 + \nabla^l_{Y_1}\omega\phi_3Y_2,\end{aligned}$$

where

$$(\nabla_{Y_1}f)Y_2 = \nabla_{Y_1}fY_2 - f\nabla_{Y_1}Y_2, \quad for \quad Y_1,Y_2 \in \Gamma(TK). \tag{3.24}$$

Proof. Considering Eqs. (2.2)-(2.4) and (3.1)-(3.2) and then equating the tangential and transversal components, the result follows. ☐

Theorem 3.10. *Assume that K is a golden slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{K}$. Then the slant distribution D is integrable, if and only if,*

$$\nabla_{Z_1}fZ_2 - A_{\omega\phi_4Z_2}Z_1 - th(Z_1,Z_2) - f\nabla_{Z_2}Z_1 \in \Gamma(D),$$

for $Z_1, Z_2 \in \Gamma(D)$.

Proof. For $Z_1, Z_2 \in \Gamma(D)$, employing Eqs. (3.23) and (3.24), we get

$$f[Z_1,Z_2] = \nabla_{Z_1}fZ_2 - A_{\omega\phi_4Z_2}Z_1 - th(Z_1,Z_2) - f\nabla_{Z_2}Z_1,$$

which proves the result. ☐

Theorem 3.11. *Let K be a golden slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{K}$. Then the slant distribution D is integrable, if and only if,* *(i)* $\phi_1(\nabla_{Z_1} fZ_2) - \phi_1(\nabla_{Z_2} fZ_1) = \phi_1(A_{\omega Z_2} Z_1) - \phi_1(A_{\omega Z_1} Z_2)$,

(ii) $\phi_2(\nabla_{Z_1} fZ_2) - \phi_2(\nabla_{Z_2} fZ_1) = \phi_2(A_{\omega Z_2} Z_1) - \phi_2(A_{\omega Z_1} Z_2)$,

(iii) $h^l(Z_1, fZ_2) - h^l(Z_2, fZ_1) = D^l(Z_2, \omega Z_1) - D^l(Z_1, \omega Z_2)$,

for $Z_1, Z_2 \in \Gamma(D)$.

Proof. For $Z_1, Z_2 \in \Gamma(D)$, employing Eq. (3.6), we have

$$\phi_1(\nabla_{Z_1} fZ_2) = \phi_1(A_{\omega Z_2} Z_1) + P\phi_2 \nabla_{Z_1} Z_2. \tag{3.25}$$

Then, interchanging the role of Z_1 and Z_2 in Eq. (3.25), we get

$$\phi_1(\nabla_{Z_2} fZ_1) = \phi_1(A_{\omega Z_1} Z_2) + P\phi_2 \nabla_{Z_2} Z_1. \tag{3.26}$$

Further, from Eqs. (3.25) and (3.26), we derive

$$\phi_1(\nabla_{Z_1} fZ_2) - \phi_1(\nabla_{Z_2} fZ_1) = \phi_1(A_{\omega Z_2} Z_1) - \phi_1(A_{\omega Z_1} Z_2) + P\phi_2[Z_1, Z_2]. \tag{3.27}$$

Next, from Eq. (3.7), we acquire

$$\phi_2(\nabla_{Z_1} fZ_2) = \phi_2(A_{\omega Z_2} Z_1) + P\phi_1 \nabla_{Z_1} Z_2. \tag{3.28}$$

On interchanging the role of Z_1 and Z_2 in Eq. (3.28), we obtain

$$\phi_2(\nabla_{Z_2} fZ_1) = \phi_2(A_{\omega Z_1} Z_2) + P\phi_1 \nabla_{Z_2} Z_1. \tag{3.29}$$

From Eqs. (3.28) and (3.29), we derive

$$\phi_2(\nabla_{Z_1} fZ_2) - \phi_2(\nabla_{Z_2} fZ_1) = \phi_2(A_{\omega Z_2} Z_1) - \phi_2(A_{\omega Z_1} Z_2) + P\phi_1[Z_1, Z_2]. \tag{3.30}$$

Then considering Eq. (3.9), we have

$$h^l(Z_1, fZ_2) = \omega\phi_3 \nabla_{Z_1} Z_2 - D^l(Z_1, \omega Z_2) \tag{3.31}$$

and interchanging the role of Z_1 and Z_2, Eq. (3.31) yields

$$h^l(Z_2, fZ_1) = \omega\phi_3 \nabla_{Z_2} Z_1 - D^l(Z_2, \omega Z_1). \tag{3.32}$$

From Eqs. (3.31) and (3.32), we obtain

$$h^l(Z_1, fZ_2) - h^l(Z_2, fZ_1) = \omega\phi_3[Z_1, Z_2] + D^l(Z_2, \omega Z_1) - D^l(Z_1, \omega Z_2). \tag{3.33}$$

Hence, the result follows from Eqs. (3.27), (3.30) and (3.33). □

Theorem 3.12. *For a golden slant lightlike submanifold K of a golden semi-Riemannian manifold $\bar{K}$, the anti-invariant distribution $Pltr(TK)$ is integrable, if and only if,*

$$A_{\omega Y_2} Y_1 + th(Y_1, Y_2) + f\nabla_{Y_2} Y_1 \in \Gamma(Pltr(TK)),$$

for $Y_1, Y_2 \in \Gamma(Pltr(TK))$.

Proof. For $Y_1, Y_2 \in \Gamma(Pltr(TK))$, employing Eqs. (3.23) and (3.24), we derive

$$f[Y_1, Y_2] = -A_{\omega Y_2} Y_1 - th(Y_1, Y_2) - f\nabla_{Y_2} Y_1,$$

which gives the result. □

Theorem 3.13. *Consider a golden slant lightlike submanifold K of a golden semi-Riemannian manifold $\bar{K}$. Then the radical distribution $Rad(TK)$ is integrable, if and only if,*

(i) $\phi_1(\nabla_{\xi_1} P\xi_2) = \phi_1(\nabla_{\xi_2} P\xi_1)$ *and* $\phi_4(\nabla_{\xi_1} f\phi_4\xi_2) = \phi_4(\nabla_{\xi_2} f\phi_4\xi_1)$,

(ii) $h^l(\xi_1, P\xi_2) = h^l(\xi_2, P\xi_1)$ *and* $h^s(\xi_1, P\xi_2) = h^s(\xi_2, P\xi_1)$,

for $\xi_1, \xi_2 \in \Gamma(Rad(TK))$.

Proof. For $\xi_1, \xi_2 \in \Gamma(Rad(TK))$, using Eq. (3.6), we derive

$$\phi_1(\nabla_{\xi_1} P\xi_2) = P\phi_2\nabla_{\xi_1}\xi_2. \tag{3.34}$$

On interchanging ξ_1 and ξ_2, Eq. (3.34) yields

$$\phi_1(\nabla_{\xi_2} P\xi_1) = P\phi_2\nabla_{\xi_2}\xi_1. \tag{3.35}$$

From Eqs. (3.34) and (3.35), we obtain

$$\phi_1(\nabla_{\xi_1} P\xi_2) - \phi_1(\nabla_{\xi_2} P\xi_1) = P\phi_2[\xi_1, \xi_2]. \tag{3.36}$$

Further, employing Eq. (3.10), we have

$$\phi_4(\nabla_{\xi_1} f\phi_4\xi_2) = f\phi_4\nabla_{\xi_1}\xi_2 + th^s(\xi_1, \xi_2). \tag{3.37}$$

By interchanging the role of ξ_1 and ξ_2 in Eq. (3.37), we get

$$\phi_4(\nabla_{\xi_2} f\phi_4\xi_1) = f\phi_4\nabla_{\xi_2}\xi_1 + th^s(\xi_2, \xi_1). \tag{3.38}$$

Then, from Eqs. (3.37) and (3.38), we obtain

$$\phi_4(\nabla_{\xi_1} f\phi_4\xi_2) - \phi_4(\nabla_{\xi_2} f\phi_4\xi_1) = f\phi_4[\xi_1, \xi_2]. \tag{3.39}$$

Now, from Eq. (3.9), we acquire

$$h^l(\xi_1, P\xi_2) = \omega\phi_3\nabla_{\xi_1}\xi_2. \tag{3.40}$$

By interchanging the role of ξ_1 and ξ_2 in Eq. (3.40), we get

$$h^l(\xi_2, P\xi_1) = \omega\phi_3\nabla_{\xi_2}\xi_1. \tag{3.41}$$

Then, using Eqs. (3.40) and (3.41), we derive

$$h^l(\xi_1, P\xi_2) - h^l(\xi_2, P\xi_1) = \omega\phi_3[\xi_1, \xi_2]. \tag{3.42}$$

Next, using Eq. (3.11), we have

$$h^s(\xi_1, P\xi_2) = \omega\phi_4\nabla_{\xi_1}\xi_2 + nh^s(\xi_1, \xi_2). \tag{3.43}$$

By interchanging the role of ξ_1 and ξ_2 in Eq. (3.43), we attain

$$h^s(\xi_2, P\xi_1) = \omega\phi_4\nabla_{\xi_2}\xi_1 + nh^s(\xi_2, \xi_1). \tag{3.44}$$

Further, from Eqs. (3.43) and (3.44), we obtain

$$h^s(\xi_1, P\xi_2) - h^s(\xi_2, P\xi_1) = \omega\phi_4[\xi_1, \xi_2]. \tag{3.45}$$

Hence, the proof follows from Eqs. (3.36), (3.39), (3.42) and (3.45). □

Theorem 3.14. *For a golden slant lightlike submanifold K of a golden semi-Riemannian manifold $\bar{K}$, the distribution $PRad(TK)$ is integrable, if and only if,*

(i) $\phi_2(\nabla_{\xi_1}P\xi_2) = \phi_2(\nabla_{\xi_2}P\xi_1)$ *and* $\phi_4(\nabla_{\xi_1}f\phi_4\xi_2) = \phi_4(\nabla_{\xi_2}f\phi_4\xi_1)$,

(ii) $h^l(\xi_1, P\xi_2) = h^l(\xi_2, P\xi_1)$ *and* $h^s(\xi_1, P\xi_2) = h^s(\xi_2, P\xi_1)$,

for $\xi_1, \xi_2 \in \Gamma(PRad(TK))$.

Proof. For $\xi_1, \xi_2 \in \Gamma(PRad(TK))$, from Eq. (3.7), we have

$$\phi_2(\nabla_{\xi_1}P\xi_2) = P\phi_1\nabla_{\xi_1}\xi_2. \tag{3.46}$$

On interchanging the role of ξ_1 and ξ_2, Eq. (3.46) becomes

$$\phi_2(\nabla_{\xi_2}P\xi_1) = P\phi_1\nabla_{\xi_2}\xi_1. \tag{3.47}$$

Then, from Eqs. (3.46) and (3.47), we obtain

$$\phi_2(\nabla_{\xi_1}P\xi_2) - \phi_2(\nabla_{\xi_2}P\xi_1) = P\phi_1[\xi_1, \xi_2]. \tag{3.48}$$

Further, employing Eq. (3.10), we acquire

$$\phi_4(\nabla_{\xi_1}f\phi_4\xi_2) = f\phi_4\nabla_{\xi_1}\xi_2 + th^s(\xi_1, \xi_2) \tag{3.49}$$

and interchanging the role of ξ_1 and ξ_2 in Eq. (3.49), we get

$$\phi_4(\nabla_{\xi_2}f\phi_4\xi_1) = f\phi_4\nabla_{\xi_2}\xi_1 + th^s(\xi_2, \xi_1). \tag{3.50}$$

Then, using Eqs. (3.49) and (3.50), we obtain

$$\phi_4(\nabla_{\xi_1}f\phi_4\xi_2) - \phi_4(\nabla_{\xi_2}f\phi_4\xi_1) = f\phi_4[\xi_1, \xi_2]. \tag{3.51}$$

Next, from Eq. (3.9), we attain

$$h^l(\xi_1, P\xi_2) = \omega\phi_3\nabla_{\xi_1}\xi_2. \tag{3.52}$$

On interchanging the role of ξ_1 and ξ_2, Eq. (3.52) yields

$$h^l(\xi_2, P\xi_1) = \omega\phi_3\nabla_{\xi_2}\xi_1. \tag{3.53}$$

Following Eqs. (3.52) and (3.53), we obtain

$$h^l(\xi_1, P\xi_2) - h^l(\xi_2, P\xi_1) = \omega\phi_3[\xi_1, \xi_2]. \tag{3.54}$$

Finally, using Eq. (3.11), we derive

$$h^s(\xi_1, P\xi_2) = \omega\phi_4\nabla_{\xi_1}\xi_2 + nh^s(\xi_1, \xi_2) \tag{3.55}$$

and interchanging the role of ξ_1 and ξ_2 in Eq. (3.55), we get

$$h^s(\xi_2, P\xi_1) = \omega\phi_4\nabla_{\xi_2}\xi_1 + nh^s(\xi_2, \xi_1). \tag{3.56}$$

Then from Eqs. (3.55) and (3.56), we have

$$h^s(\xi_1, P\xi_2) - h^s(\xi_2, P\xi_1) = \omega\phi_4[\xi_1, \xi_2] \tag{3.57}$$

and hence, the proof follows from Eqs. (3.48), (3.51), (3.54) and (3.57). □

Theorem 3.15. *Consider K, a golden slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{K}$. Then $Rad(TK)$ defines a totally geodesic foliation in K, if and only if,*

$$\begin{aligned} g(\nabla_{\xi_1} f\phi_4 Z, P\xi_2) &= g(A_{\omega\phi_3 Z}\xi_1, P\xi_2) + g(A_{\omega\phi_4 Z}\xi_1, P\xi_2) + \bar{g}(h^l(\xi_1, f\phi_2 Z), \xi_2) \\ &\quad + \bar{g}(\nabla^l_{\xi_1}\omega\phi_3 Z, \xi_2) + \bar{g}(h^l(\xi_1, f\phi_4 Z), \xi_2) + \bar{g}(D^l(\xi_1, \omega\phi_4 Z), \xi_2), \end{aligned}$$

for $\xi_1, \xi_2 \in \Gamma(Rad(TK))$ and $Z \in \Gamma(S(TK))$.

Proof. Let K be a golden slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{K}$. To prove that $Rad(TK)$ defines a totally geodesic foliation in K, it is sufficient to show that $\nabla_{\xi_1}\xi_2 \in Rad(TK)$, for $\xi_1, \xi_2 \in \Gamma(Rad(TK))$. Since $\bar{\nabla}$ is a metric connection, therefore using Eq. (2.10), for $\xi_1, \xi_2 \in \Gamma(Rad(TK))$ and $Z \in \Gamma(S(TK))$, we obtain

$$\begin{aligned} g(\nabla_{\xi_1}\xi_2, Z) &= \bar{g}(P\bar{\nabla}_{\xi_1}\xi_2, PZ) - \bar{g}(P\bar{\nabla}_{\xi_1}\xi_2, Z) \\ &= -\bar{g}(\bar{\nabla}_{\xi_1} PZ, P\xi_2) + \bar{g}(\bar{\nabla}_{\xi_1} PZ, \xi_2). \end{aligned}$$

Further, employing Eqs. (2.2)-(2.4) and (3.5), we derive

$$\begin{aligned} g(\nabla_{\xi_1}\xi_2, Z) = &- g(\nabla_{\xi_1} f\phi_2 Z, P\xi_2) + g(A_{\omega\phi_3 Z}\xi_1, P\xi_2) - g(\nabla_{\xi_1} f\phi_4 Z, P\xi_2) \\ &+ g(A_{\omega\phi_4 Z}\xi_1, P\xi_2) + \bar{g}(h^l(\xi_1, f\phi_2 Z), \xi_2) + \bar{g}(\nabla^l_{\xi_1}\omega\phi_3 Z, \xi_2) \\ &+ \bar{g}(h^l(\xi_1, f\phi_4 Z), \xi_2) + \bar{g}(D^l(\xi_1, \omega\phi_4 Z), \xi_2), \end{aligned}$$

which gives the proof. □

Theorem 3.16. *Assume that K is a golden slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{K}$. Then D defines a totally geodesic foliation in K, if and only if,*

(i) $g(\nabla_{Z_1} PN, Z_2) = g(\nabla_{Z_1} PN, fZ_2) + \bar{g}(h^s(Z_1, PN), \omega Z_2)$,

(ii) $g(A_{PY} Z_1, fZ_2) = g(A_{PY} Z_1, Z_2) + \bar{g}(D^s(Z_1, PY), \omega Z_2)$,

for $Z_1, Z_2 \in \Gamma(D)$, $Y \in \Gamma(Pltr(TK))$ and $N \in \Gamma(ltr(TK))$.

Proof. Consider K, a golden slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{K}$. To prove that D defines a totally geodesic foliation in K, it is sufficient to show that $\nabla_{Z_1} Z_2 \in D$ for $Z_1, Z_2 \in \Gamma(D)$. Since $\bar{\nabla}$ is a metric connection then using Eqs. (2.9), (2.10), for $Z_1, Z_2 \in \Gamma(D)$ and $N \in \Gamma(ltr(TK))$, we obtain

$$\begin{aligned} \bar{g}(\nabla_{Z_1} Z_2, N) &= \bar{g}(P\bar{\nabla}_{Z_1} Z_2, PN) - \bar{g}(P\bar{\nabla}_{Z_1} Z_2, N) \\ &= -\bar{g}(\bar{\nabla}_{Z_1} PN, PZ_2) + \bar{g}(\bar{\nabla}_{Z_1} PN, Z_2). \end{aligned} \tag{3.58}$$

Further, employing Eqs. (2.2) in Eq. (3.58), we acquire

$$\bar{g}(\nabla_{Z_1} Z_2, N) = -g(\nabla_{Z_1} PN, fZ_2) - \bar{g}(h^s(Z_1, PN), \omega Z_2) + g(\nabla_{Z_1} PN, Z_2).$$

Next, using Eqs. (2.9) and (2.10), for $Z_1, Z_2 \in \Gamma(D)$ and $Y \in \Gamma(Pltr(TK))$, we get

$$\begin{aligned} g(\nabla_{Z_1} Z_2, Y) &= \bar{g}(P\bar{\nabla}_{Z_1} Z_2, PY) - \bar{g}(P\bar{\nabla}_{Z_1} Z_2, Y) \\ &= -\bar{g}(\bar{\nabla}_{Z_1} PY, PZ_2) + \bar{g}(\bar{\nabla}_{Z_1} PY, Z_2), \end{aligned}$$

which on using Eq. (2.3) yields

$$g(\nabla_{Z_1} Z_2, Y) = g(A_{PY} Z_1, fZ_2) - \bar{g}(D^s(Z_1, PY), \omega Z_2) - g(A_{PY} Z_1, Z_2).$$

Hence, the proof follows. □

4. Minimal golden slant lightlike submanifolds

The general definition of a minimal lightlike submanifold of a semi-Riemannian manifold was given by Bejan and Duggal [2] as follows:

Definition 4.1. A lightlike submanifold $(K, g, S(TK))$ isometrically immersed in a semi-Riemannian manifold $(\bar{K}, \bar{g})$ is said to be a minimal lightlike submanifold if the following conditions hold:

(i) $h^s(\xi_1, \xi_2) = 0, \ \forall\ \xi_1, \xi_2 \in \Gamma(Rad(TK))$.

(ii) $trace\ h|_{S(TK)} = 0$.

Theorem 4.2. *Consider a totally umbilical golden slant lightlike submanifold K of a golden semi-Riemannian manifold $\bar{K}$. Then K is minimal, if and only if, K is totally geodesic.*

Proof. Firstly, we assume that K is minimal for $\xi_1, \xi_2 \in \Gamma(Rad(TK))$ which implies that $h^s(\xi_1, \xi_2) = 0$. Further, using the hypothesis that K is totally umbilical, we have $h^l(\xi_1, \xi_2) = H^l g(\xi_1, \xi_2) = 0$ for $\xi_1, \xi_2 \in \Gamma(Rad(TK))$. Now, for an orthonormal basis $\{e_1, e_2, ..., e_{m-r}\}$ of $S(TK)$ and as K is totally umbilical, we obtain

$$\begin{aligned} trace\ h(e_i, e_i) &= \sum_{i=1}^{m-r} \epsilon_i \{h^l(e_i, e_i) + h^s(e_i, e_i)\} \\ &= (m-r)\{H^l + H^s\}. \end{aligned}$$

From the definition of minimal lightlike submanifold, we have $trace\ h|_{S(TK)} = 0$ and employing Eq. (2.1), we get $H^l = 0$ and $H^s = 0$, which implies that K is totally geodesic. Also the converse is trivial and hence the result holds. □

Theorem 4.3. *A proper totally umbilical golden slant lightlike submanifold K of a golden semi-Riemannian manifold $\bar{K}$ is minimal, if and only if, $trace\ A_{W_l}|_D = 0$ and $trace\ A^*_{\xi_j}|_D = 0$, for $W_l \in \Gamma(S(TK^\perp))$ and $\xi_i \in \Gamma(Rad(TK))$, where $l \in \{1, 2, ..., t\}$ and $i \in \{1, 2, ..., r\}$.*

Proof. In view of definition of golden slant lightlike submanifold, K is minimal if and only if

$$h^s(\xi_i, \xi_j) = 0$$

and

$$\sum_{i=1}^{r} h(P\xi_i, P\xi_i) + \sum_{i=1}^{r} h(PN_i, PN_i) + \sum_{j=1}^{q} h(e_j, e_j) = 0,$$

where $\{\xi_i\}_{i=1}^r$, $\{N_i\}_{i=1}^r$ and $\{e_j\}_{j=1}^q$, respectively, are the bases of $Rad(TK)$, $ltr(TK)$ and D. Since K is a proper totally umbilical golden slant lightlike submanifold, we note that $h(\xi_i, \xi_j) = h(P\xi_i, P\xi_i) = h(PN_i, PN_i) = 0$, which gives that $h^s = 0$ on $Rad(TK)$. Therefore, K is minimal, if and only if, $\sum_{j=1}^q h(e_j, e_j) = 0$, where

$$\sum_{j=1}^{q} h(e_j, e_j) = \sum_{j=1}^{q} \left\{ \frac{1}{r} \sum_{i=1}^{r} \bar{g}(h^l(e_j, e_j), \xi_i) N_i \right\} + \sum_{j=1}^{q} \left\{ \frac{1}{t} \sum_{l=1}^{t} \bar{g}(h^s(e_j, e_j), W_l) W_l \right\}.$$

Then, using Eqs. (2.5) and (2.7), above equation becomes

$$\sum_{j=1}^{q} h(e_j, e_j) = \sum_{j=1}^{q} \left\{ \frac{1}{r} \sum_{i=1}^{r} g(A^*_{\xi_i} e_j, e_j) N_i + \frac{1}{t} \sum_{l=1}^{t} g(A_{W_l} e_j, e_j) W_l \right\}.$$

Hence, the result follows. □

Definition 4.4. [6] A lightlike submanifold K of a semi-Riemannian manifold $\bar{K}$ is called an irrotational lightlike submanifold, if and only if, $\bar{\nabla}_Y \xi \in \Gamma(TK)$, for $Y \in \Gamma(TK)$ and $\xi \in \Gamma(Rad(TK))$.

Theorem 4.5. *Consider K, an irrotational golden slant lightlike submanifold of a golden semi-Riemannian manifold $\bar{K}$. Then K is minimal, if and only if, for $\xi_j \in \Gamma(Rad(TK))$ and $W_l \in \Gamma(S(TK)^\perp)$, one has*

$$trace\ A^*_{\xi_j}|_{S(TK)} = 0 \quad and \quad trace\ A_{W_l}|_{S(TK)} = 0,$$

where $l \in \{1, 2, ..., t\}$ and $j \in \{1, 2, ..., r\}$.

Proof. By the hypothesis, we have $h^s = 0$ on $Rad(TK)$. Moreover, one has

$$trace\ h|_{S(TK)} = \sum_{i=1}^{r} h(P\xi_i, P\xi_i) + \sum_{i=1}^{r} h(PN_i, PN_i) + \sum_{z=1}^{q} h(e_z, e_z).$$

Then, using Eqs. (2.5) and (2.7), we obtain

(4.1)
$$\sum_{i=1}^{r} h(P\xi_i, P\xi_i) = \sum_{i=1}^{r} \left\{ \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} P\xi_i, P\xi_i) N_j + \frac{1}{t} \sum_{l=1}^{t} g(A_{W_l} P\xi_i, P\xi_i) W_l \right\},$$

(4.2)
$$\sum_{i=1}^{r} h(PN_i, PN_i) = \sum_{i=1}^{r} \left\{ \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} PN_i, PN_i) N_j + \frac{1}{t} \sum_{l=1}^{t} g(A_{W_l} e_i, e_i) W_l \right\}$$

and

$$\sum_{z=1}^{q} h(e_z, e_z) = \sum_{z=1}^{q} \left\{ \frac{1}{r} \sum_{j=1}^{r} g(A^*_{\xi_j} e_z, e_z) N_j + \frac{1}{t} \sum_{l=1}^{t} g(A_{W_l} e_z, e_z) W_l \right\}. \tag{4.3}$$

Thus, we conclude that *trace* $h|_{S(TK)} = 0$, if and only if, *trace* $A^*_{\xi_j} = 0$ and *trace* $A_{W_l} = 0$. Hence, the result follows from Eqs. (4.1)–(4.3). □

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