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Starcompact and related spaces on Pixley-Roy hyperspaces

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Abstract. In this paper, we study the relation between a space X satisfying certain generalized metric properties and the Pixley-Roy hyperspace $\mathtt{PR}[X]$ over X satisfying the same properties. We prove that if $\mathtt{PR}[X]$ is starcompact (resp., star-Lindelöf), then X is compact (resp., Lindelöf). However, there exists a compact space X such that $|X| = \omega$, but $\mathtt{PR}_n[X]$ for all $n \in \mathbb{N}$ and $\mathtt{PR}[X]$ are not starcompact spaces. Moreover, we show that $\mathtt{PR}[X]$ is strongly starcompact (resp., strongly star-Lindelöf) if and only if X is finite (resp., countable). By these results, we obtain that $\mathtt{PR}[X]$ is set strongly starcompact (resp., cosmic, set strongly star-Lindelöf) if and only if X is finite (resp., countable).

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1. Introduction

The generalized metric properties on Pixley-Roy hyperspaces have been studied by many authors ([2, 4–9, 12–15], for example). They considered several generalized metric properties and studied the relation between a space X satisfying such a property and its Pixley-Roy hyperspaces satisfying the same property.

In this paper, we study concepts such as compact, set strongly starcompact, strongly starcompact, starcompact, cosmic, Lindelöf, set strongly star-Lindelöf, strongly star-Lindelöf, star-Lindelöf on Pixley-Roy hyperspaces. We obtain some new results about Pixley-Roy hyperspaces.

Throughout this paper, all spaces are assumed to be at least T_1, $\mathbb{N}$ denotes the set of all positive integers, the first infinite ordinal is denoted by ω. Furthermore, if $\mathcal{P}$ is a family of subsets of a space X and $A \subset X$, then

$$\mathtt{St}(A, \mathcal{P}) = \bigcup\{P \in \mathcal{P} : P \cap A \neq \emptyset\}.$$

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2. Definitions

The *Pixley-Roy hyperspace* $\mathtt{PR}[X]$ over a space X, defined by C. Pixley and P. Roy in [7], is the set of all non-empty finite subsets of X with the topology generated by the sets of the form

$$[F, V] = \{G \in \mathtt{PR}[X] : F \subset G \subset V\},$$

where $F \in \mathtt{PR}[X]$ and V is an open subset in X containing F. For any space X, $\mathtt{PR}[X]$ is zero-dimensional, completely regular and hereditarily metacompact (see [15]).

For each $n \in \mathbb{N}$, let $\mathtt{PR}_n[X] = \{F \in \mathtt{PR}[X] : |F| \leq n\}$.

Remark 2.1 ([12], p. 305). For each $n \in \mathbb{N}$, $\mathtt{PR}_n[X]$ is a closed subspace of $\mathtt{PR}[X]$ and in particular, $\mathtt{PR}_1[X]$ is a closed discrete subspace of $\mathtt{PR}[X]$.

Remark 2.2 ([4], Remark 1.2). Every $\mathtt{PR}_m[X]$ is a closed subspace of $\mathtt{PR}_n[X]$ for each $m, n \in \mathbb{N}$, $m < n$.

For each $F \in \mathtt{PR}[X]$ and $A \subset X$, denote

$$[F, A] = \{H \in \mathtt{PR}[X] : F \subset H \subset A\}.$$

Definition 2.3. Let X be a space.

1. X is said to be *starcompact* (resp., *star-Lindelöf*) [3], if for every open cover $\mathcal{U}$ of X, there exists a finite (resp., countable) $\mathcal{V} \subset \mathcal{U}$ such that $\mathtt{St}(\bigcup \mathcal{V}, \mathcal{U}) = X$.

2. X is said to be *strongly starcompact* (resp., *strongly star-Lindelöf*) [3], if for every open cover $\mathcal{U}$ of X, there is a finite (resp., countable) subset A of X such that $\mathtt{St}(A, \mathcal{U}) = X$.

3. X is said to be *set starcompact* (resp., *set star-Lindeöf*) [11], if for each nonempty subset A of X and each collection $\mathcal{U}$ of open sets in X such that $\overline{A} \subset \bigcup \mathcal{U}$, there is a finite (resp., countable) subset $\mathcal{V}$ of $\mathcal{U}$ such that $A \subset \mathtt{St}(\bigcup \mathcal{V}, \mathcal{U})$.

4. X is said to be *set strongly starcompact* (resp., *set strongly star-Lindeöf*) [11], if for each nonempty subset A of X and each collection $\mathcal{U}$ of open sets in X such that $\overline{A} \subset \bigcup \mathcal{U}$, there is a finite (resp., countable) subset F of A such that $A \subset \mathtt{St}(F, \mathcal{U})$.

5. X is said to be *cosmic* [1], if X has a countable network.

compact $\to$ countable compact $\to$ set strongly starcompact $\to$ set starcompact
set strongly starcompact $\downarrow$ strongly starcompact; set starcompact $\downarrow$ starcompact
strongly starcompact $\to$ starcompact

cosmic $\to$ Lindelöf $\to$ set strongly star-Lindelöf $\to$ set star-Lindelöf
set strongly star-Lindelöf $\downarrow$ strongly star-Lindelöf; set star-Lindelöf $\downarrow$ star-Lindelöf
strongly star-Lindelöf $\to$ star-Lindelöf

Remark 2.4. The property of being starcompact (resp., strongly starcompact) is also called *1-starcompact* (resp., *strongly 1-starcompact*) in [10].

3. Main results

Theorem 3.1. *Let X is a space. If $\mathtt{PR}[X]$ is starcompact, then X is compact.*

Proof. Suppose that $\mathcal{U}$ is an open cover of X. Then, for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. If we put

$$\mathfrak{U} = \Big\{[F, \textstyle\bigcup_{x\in F} U_x] : F \in \mathtt{PR}[X]\Big\},$$

then it is clear that $\mathfrak{U}$ is an open cover of $\mathtt{PR}[X]$. Because $\mathtt{PR}[X]$ is starcompact, there exists a finite subfamily $\mathfrak{V}$ of $\mathfrak{U}$ such that

$$\mathtt{St}(\textstyle\bigcup \mathfrak{V}, \mathfrak{U}) = \mathtt{PR}[X].$$

Put

$$\mathfrak{V} = \Big\{[F_i, \textstyle\bigcup_{x\in F_i} U_x] : i \leq n\Big\};$$
$$\mathcal{V} = \{U_x : x \in F_i, i \leq n\}.$$

Then, it is clear that $\mathcal{V}$ is a finite subfamily of $\mathcal{U}$. Next, we need only to prove that $\mathcal{V}$ is a cover of X. Indeed, for each $y \in X$, since $\{y\} \in \mathtt{PR}[X]$, $\{y\} \in \mathtt{St}(\bigcup \mathfrak{V}, \mathfrak{U})$. Hence, there exist $F \in \mathtt{PR}[X]$ and $i \leq n$ such that

$$\{y\} \in [F, \textstyle\bigcup_{x\in F} U_x] \text{ and } [F, \textstyle\bigcup_{x\in F} U_x] \cap [F_i, \textstyle\bigcup_{x\in F_i} U_x] \neq \emptyset.$$

This implies that $F = \{y\}$ and there exists $H \in \mathtt{PR}[X]$ such that

$$F \subset H \subset \textstyle\bigcup_{x\in F} U_x \text{ and } F_i \subset H \subset \textstyle\bigcup_{x\in F_i} U_x.$$

Hence, $F \subset \bigcup_{x\in F_i} U_x$. This shows that there exists $U_x \in \mathcal{V}$ such that $y \in U_x$. Therefore, $y \in \bigcup \mathcal{V}$. This implies that $X \subset \bigcup \mathcal{V}$. Thus, $\mathcal{V}$ is a cover of X. □

Example 3.2. There exists a compact space X such that $|X| = \omega$, but $\mathtt{PR}_n[X]$ for all $n \in \mathbb{N}$ and $\mathtt{PR}[X]$ are not starcompact spaces.

Proof. Assume that

$$X = \{x_0\} \cup \{x_k : k \in \mathbb{N}\},$$

where every x_k and x_0 are different from each other. The set X endowed with the following topology: each x_k is isolated; a basic neighborhood of x_0 has the form $\{x_0\} \cup \{x_k : k \geq m\}$ for some $m \in \mathbb{N}$.

(1) It is obvious that X is compact.

(2) $\mathtt{PR}[X]$ is not starcompart. Indeed, we take an open cover of $\mathtt{PR}[X]$ as follows:

$$\begin{aligned}\mathfrak{U} &= \Big\{[\{x_0\}, X]\Big\} \cup \Big\{[\{x_k\}, \{x_k\}] : k \in \mathbb{N}\Big\} \cup \Big\{[A, X] : A \in \mathtt{PR}[X] \setminus \mathtt{PR}_1[X]\Big\} \\ &= \Big\{[\{x_0\}, X]\Big\} \cup \Big\{\{\{x_k\}\} : k \in \mathbb{N}\Big\} \cup \Big\{[A, X] : A \in \mathtt{PR}[X] \setminus \mathtt{PR}_1[X]\Big\}.\end{aligned}$$

Obviously, $\{x_k\} \notin [\{x_0\}, X]$ and $\{x_k\} \notin [A, X]$ for each $k \in \mathbb{N}$ and for each $A \in \mathtt{PR}[X] \setminus \mathtt{PR}_1[X]$. Now, suppose that $\mathfrak{V}$ is a finite subfamily of $\mathfrak{U}$. Then,

$$|\{k \in \mathbb{N} : \{x_k\} \in \textstyle\bigcup \mathfrak{V}\}| < \omega.$$

This implies that there exists $k_0 \in \mathbb{N}$ such that $\{x_{k_0}\} \notin \bigcup \mathfrak{V}$. Let $\{x_{k_0}\} \in \mathtt{St}(\bigcup \mathfrak{V}, \mathfrak{U})$. Then, there exists $\mathcal{A} \in \mathfrak{U}$ such that

$$(\bigcup \mathfrak{V}) \cap \mathcal{A} \neq \emptyset,\ \{x_{k_0}\} \in \mathcal{A}.$$

Since $\{x_{k_0}\} \notin [\{x_0\}, X]$ and $\{x_{k_0}\} \notin [A, X]$ for each $A \in \mathtt{PR}[X] \setminus \mathtt{PR}_1[X]$, $\mathcal{A} = \{\{x_{k_0}\}\}$. Thus, $\{x_{k_0}\} \in \bigcup \mathfrak{V}$, which is a contradiction. Therefore, $\mathtt{St}(\bigcup \mathcal{V}, \mathcal{B}) \neq \mathtt{PR}[X]$. Hence, $\mathtt{PR}[X]$ is not starcompact.

(3) $\mathtt{PR}_n[X]$ is not starcompact for all $n \in \mathbb{N}$.

Indeed, for each $n \in \mathbb{N}$, we take an open cover $\mathtt{PR}_n[X]$ as follows:

$$\begin{aligned}\mathfrak{U} = &\Big\{[\{x_0\}, X] \cap \mathtt{PR}_n[X]\Big\} \cup \Big\{[\{x_k\}, \{x_k\}] \cap \mathtt{PR}_n[X] : k \in \mathbb{N}\Big\} \\ &\qquad \cup \Big\{[A, X] \cap \mathtt{PR}_n[X] : A \in \mathtt{PR}_n[X] \setminus \mathtt{PR}_1[X]\Big\} \\ = &\Big\{[\{x_0\}, X] \cap \mathtt{PR}_n[X]\Big\} \cup \Big\{\{\{x_k\}\} : k \in \mathbb{N}\Big\} \\ &\qquad \cup \Big\{[A, X] \cap \mathtt{PR}_n[X] : A \in \mathtt{PR}_n[X] \setminus \mathtt{PR}_1[X]\Big\}.\end{aligned}$$

Then, it is obvious that $\{x_k\} \notin [\{x_0, X\}] \cap \mathtt{PR}_n[X]$ and $\{x_k\} \notin [A, V] \cap \mathtt{PR}_n[X]$ for each $k \in \mathbb{N}$ and for each $A \in \mathtt{PR}_n[X] \setminus \mathtt{PR}_1[X]$. Now, if $\mathfrak{V}$ is a finite subfamily of $\mathfrak{U}$, then by the proof of (2), there exists $k_0 \in \mathbb{N}$ such that $\{x_{k_0}\} \notin \mathtt{St}(\bigcup \mathfrak{V}, \mathfrak{U})$. Hence, $\mathtt{St}(\bigcup \mathfrak{V}, \mathfrak{U}) \neq \mathtt{PR}_n[X]$. This shows that $\mathtt{PR}_n[X]$ is not starcompact. □

Theorem 3.3. *Let X be a space. Then, the following statements are equivalent:*

1. *$\mathtt{PR}[X]$ is compact;*
2. *$\mathtt{PR}[X]$ is countably compact;*
3. *$\mathtt{PR}[X]$ is set strongly starcompact;*
4. *$\mathtt{PR}[X]$ is strongly starcompact;*
5. *X is finite.*

Proof. (1) ⇔ (2) ⇔ (3) ⇔ (4) by [10, Theorem 2.8] and $\mathtt{PR}[X]$ is a metacompact space. On the other hand, (1) ⇔ (5) by [13, Remark 3.5]. □

Theorem 3.4. *Let X is a space. If $\mathtt{PR}[X]$ is star-Lindelöf, then X is Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. Then, for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. If we put

$$\mathfrak{U} = \Big\{[F, \bigcup_{x \in F} U_x] : F \in \mathtt{PR}[X]\Big\},$$

then it is obvious that $\mathfrak{U}$ is an open cover of $\mathtt{PR}[X]$. Since $\mathtt{PR}[X]$ is star-Lindelöf, there exists a countable subfamily $\mathfrak{V}$ of $\mathfrak{U}$ such that

$$\mathtt{St}(\bigcup \mathfrak{V}, \mathfrak{U}) = \mathtt{PR}[X].$$

Put

$$\mathfrak{V} = \left\{[F_i, \bigcup_{x\in F_i} U_x] : i \in \mathbb{N}\right\};$$
$$\mathcal{V} = \{U_x : x \in F_i, i \in \mathbb{N}\}.$$

Obviously, $\mathcal{V}$ is a countable subfamily of $\mathcal{U}$. On the other hand, similar to the proof of Theorem 3.1, we claim that $\mathcal{V}$ is a cover of X. Therefore, X is Lindelöf. □

Since set starcompact (resp., set star-Lindelöf) ⇒ starcompact (resp., star-Lindelöf) and by Theorems 3.1 and 3.4, we obtain the following corollary.

Corollary 3.5. *Let X is a space. If $\mathtt{PR}[X]$ is set starcompact (resp., set star-Lindelöf), then X is compact (resp., Lindelöf).*

Question 1. If X is Lindelöf, then are $\mathtt{PR}[X]$ and $\mathtt{PR}_n[X]$ for some $n \in \mathbb{N}$ star-Lindelöf?

Theorem 3.6. *Let X be a space. Then, the following statements are equivalent:*

1. *$\mathtt{PR}[X]$ is cosmic;*
2. *$\mathtt{PR}[X]$ is Lindelöf;*
3. *$\mathtt{PR}[X]$ is set strongly star-Lindelöf;*
4. *$\mathtt{PR}[X]$ is strongly star-Lindelöf;*
5. *X is countable.*

Proof. (1) ⇒ (2) ⇔ (5) ⇒ (1) is obvious. Moreover, since $\mathtt{PR}[X]$ is a metacompact space, $\mathtt{PR}[X]$ is a metaLindelöf space. It follows from [11, Theorem 2.9] that (2) ⇔ (3) ⇔ (4). □

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