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On a divergence result for a subsequence of matrix transform means of Walsh-Fourier series

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Abstract. We investigate the subsequence $\sigma_{2^n}^T(f)$ of matrix transform means with respect to the Walsh system generated by nonincreasing and convex sequences $t_{k,n}$ determined by matrix T.

In particular, we prove for any $0 < p < 1/(1+\gamma)$ that there exists a martingale $f \in H_p(G)$ such that

$$\sup_{n\in\mathbb{N}} \left\| \sigma_{2^n}^T(f) \right\|_{weak-L_p} = \infty,$$

where γ depends on the sequences $t_{k,n}$.

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1. Definitions and notations

We follow the standard notions of dyadic analysis introduced by F. Schipp, W. R. Wade, P. Simon, and J. Pál [20] and others.

Let $\mathbb{P}$ be the set of positive natural numbers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let us denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition. Let be every subset open. The normalized Haar measure μ on $\mathbb{Z}_2$ is given in the way that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. $G := \mathop{\times}\limits_{k=0}^{\infty} \mathbb{Z}_2$, G is called the Walsh group. The elements of Walsh group G are sequences of numbers 0 and 1, that is $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\}$ $(k \in \mathbb{N})$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure μ is the product measure and the topology is the product topology. For another topology on the Walsh group see e.g. [7].

Dyadic intervals are defined in the usual way

$$I_0(x) := G,\ I_n(x) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}$$

for $x \in G, n \in \mathbb{P}$. They form a base for the neighbourhoods of G. Let $0 := (0 : i \in \mathbf{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbb{N}$.

Let $L_p(G)$ denote the usual Lebesgue spaces on G (with the corresponding norm $\|.\|_p$), where $1 \le p < \infty$.

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Next, we define

$$|x| := \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} \quad \text{for all } x \in G.$$

We introduce some concepts of Walsh-Fourier analysis. The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \ (x \in G, k \in \mathbb{N}).$$

The Walsh-Paley functions are the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed in the number system based 2, in the form

$$n = \sum_{k=0}^{\infty} n_k 2^k, \ n_k \in \{0, 1\} \ (k \in \mathbb{N}),$$

where only a finite number of n_k's different from zero. Let the order of $n \in \mathbb{P}$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$. Walsh-Paley functions are $w_0 := 1$ and for $n \in \mathbb{P}$

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = (-1)^{\sum_{k=0}^{|n|} n_k x_k}.$$

It is known [15] that the system $(w_n, n \in \mathbb{N})$ is the character system of $(G, +)$. The nth Fourier-coefficient, the nth partial sum of the Fourier series and the nth Dirichlet kernel are defined by

$$\hat{f}(n) := \int_G f w_n d\mu, \ S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k) w_k, \ D_n := \sum_{k=0}^{n-1} w_k, \ D_0 := 0.$$

Let $\{q_k : k \geq 0\}$ be a sequence of non-negative numbers. The nth Nörlund mean of the Walsh-Fourier series is defined by

$$t_n(f) := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k(f; x),$$

where $Q_n := \sum_{k=0}^{n-1} q_k \ (n \geq 1)$. It is always assumed that $q_0 > 0$ and

$$\lim_{n \to \infty} Q_n = \infty. \tag{1}$$

If $q_k = 1/k$, we get the so-called Nörlund logarithmic mean of the function f, denoted by $L_n(f)$.

Let $T := (t_{i,j})_{i,j=1}^{\infty}$ be a doubly infinite matrix of numbers. It is always supposed that matrix T is triangular. Let us define the nth matrix transform mean determined by the matrix T

$$\sigma_n^T(f) := \sum_{k=1}^{n} t_{k,n} S_k(f),$$

where $\{t_{k,n} : 1 \leq k \leq n,\ k \in \mathbb{P}\}$ is a finite sequence of non-negative numbers for each $n \in \mathbb{P}$.

It is a common generalization of the partial sum of Fourier series and several well-known (e.g. Fejér, Cesàro, Cesàro with varying parameters, Nörlund, weighted, Riesz) means.

The σ-algebra generated by the intervals $\{I_n(x) : x \in G\}$ will be denoted by $\mathbb{F}_n$ $(n \in \mathbb{N})$. Denote by $f := \{f^{(n)},\ n \in \mathbb{N}\}$ the martingale with respect to $\mathbb{F}_n$ (for details see, e.g., [21]).

We say that this martingale belongs to the Hardy martingale spaces $H_p(G)$ for $0 < p < \infty$ if

$$\|f\|_{H_p} := \|f^*\|_p, \text{ where } f^* := \sup_{n\in\mathbb{N}} f^{(n)}.$$

When $f \in L_1(G)$, the maximal functions are also given by

$$M(f;x) := \sup_{n\in\mathbb{N}} \left(\mu(I_n(x)) \left| \int_{I_n(x)} f(u) d\mu(u) \right| \right).$$

If $f \in L_1(G)$, then the sequence $F = \{S_{2^n}(f) : n \in \mathbb{N}\}$ is a martingale and $F^* = M(f)$.

If $f = \{f^{(n)} : n \in \mathbb{N}\}$ is a martingale, then the Walsh-Fourier coefficients must be defined in a slightly different manner:

$$\hat{f}(n) := \lim_{k\to\infty} \int_G f^{(k)}(x) w_n(x) d\mu(x).$$

A bounded measurable function a is p-atom if there exists an interval I, such that

$$\operatorname{supp}(a) \subset I,\ \int_I a d\mu = 0 \text{ and } \|a\|_\infty \leq \mu(I)^{1/p}.$$

2. Historical overview

Weisz [21] proved that for every $f \in H_p(G)$ there exists an absolute constant c_p, such that

$$\|\sigma_{2^n}(f)\|_{H_p} \leq c_p \|f\|_{H_p},\ n \in \mathbb{N},\ p > 0.$$

Móricz and Siddiqi [18] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of $L_p(G)$ functions in norm. Similar approximation properties for matrix transform mean summability methods can be found in [4, 5]. Fridli, Manchanda, and Siddiqi [9] improved and extended the results of Móricz and Siddiqi [18] to martingale Hardy spaces. The case when $\{q_k = 1/k : k \in N\}$ was excluded, since the methods are not applicable to Nörlund logarithmic means. In [10] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the Lebesgue space $L_1(G)$. In particular, they proved that there exists a function f in the space L_1, such that

$$\sup_{n\in\mathbb{N}} \|L_n\|_p = \infty.$$

In [3] it was proved that there exists a martingale $f \in H_p$, $(0 < p < 1)$ such that

$$\sup_{n\in\mathbb{N}} \|L_{2^n}\|_p = \infty.$$

A counterexample for $p = 1$ was proved in [19], however, Goginava [12] proved that

$$\sup_{n\in\mathbb{N}} \|L_{2^n}\|_1 \le c\|f\|_{H_1},\ n \in \mathbb{N}.$$

The convergence of subsequences of Nörlund logarithmic means in martingale Hardy spaces for Walsh-Paley system was investigated by Goginava [14] and for the character system of the group of 2-adic integers by Memić [17].

Matrix transforms means are common generalizations of several well-known summation methods. It follows by simple consideration that the Nörlund means, the Fejér (or the $(C,1)$) and the (C,α) means are special cases of the matrix transform summation method introduced above. For matrix transforms means with respect to trigonometric system see e.g. results of Chandra [8] and Leindler [16], to Walsh system see paper of Blyumin [6].

3. Auxiliary results

To prove Theorem 4.1 we need the following results.

The martingale Hardy space $H_p(G)$ has an atomic characterization (see Weisz [21, 22]):

Lemma 3.1. *A martingale $f = \{f^{(n)} : n \in \mathbb{N}\}$ is in H_p $(0 < p \le 1)$ if and only if there exists a sequence $\{a_k : k \in \mathbb{N}\}$ of p-atoms and a sequence $\{\mu_k : k \in \mathbb{N}\}$ of real numbers such that for every $n \in \mathbb{N}$:*

$$\sum_{k=0}^{\infty} \mu_k S_{2^n}(a_k) = f^{(n)},\ \text{where} \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{2}$$

Moreover, the following two-sided inequality holds

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of f of the form (2).

Lemma 3.2 (Paley's Lemma [20], p. 7.)**.** *For $n \in \mathbb{N}$*

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}$$

It is also well-known, that

$$D_n = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k} = w_n \sum_{k=0}^{\infty} n_k (D_{2^{k+1}} - D_{2^k}). \tag{3}$$

Lemma 3.3. *For every $n \in \mathbb{P}$, $\{t_{k,n} : 1 \le k \le n\}$ be a finite, convex and non-decreasing sequence of non-negative numbers and $x \in I_2(e_0 + e_1)$. Then for any increasing sequence of positive integers α_k the following inequality holds*

$$\left|\sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} t_{j,2^{2\alpha_k+1}} D_j(x)\right| \ge t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}} - \frac{3}{2} t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}.$$

Proof. Using Lemma 3.2 and (3) we get that if $x \in I_2(e_0 + e_1)$ then

$$D_j(x) = \begin{cases} -w_j, & \text{if } j > 1 \text{ is an odd number,} \\ 0, & \text{if } j \text{ is an even number.} \end{cases} \tag{4}$$

From $w_{2k+1}(x) = w_{2k}(x) w_1(x)$ and from (4) we obtain

$$\begin{aligned} \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} t_{j,2^{2\alpha_k+1}} D_j(x) &= - \sum_{j=2^{2\alpha_k-1}}^{2^{2\alpha_k}-1} t_{2j+1,2^{2\alpha_k+1}} w_{2j+1}(x) \\ &= -w_1(x) \sum_{j=2^{2\alpha_k-1}}^{2^{2\alpha_k}-1} t_{2j+1,2^{2\alpha_k+1}} w_{2j}(x) \end{aligned}$$

Since $w_{4k+2}(x) = w_{4k}(x) w_2(x)$, and if $x \in I_2(e_0 + e_1)$, then $w_2(x) = -1$, so

$$\begin{aligned} &\left|\sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}-1} t_{j,2^{2\alpha_k+1}} D_j(x)\right| \\ &= \left|\sum_{j=2^{2\alpha_k-1}}^{2^{2\alpha_k}-1} t_{2j+1,2^{2\alpha_k+1}} w_{2j}(x)\right| \\ &= \Bigg| t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}} w_{2^{2\alpha_k+1}-4}(x) + t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}} w_{2^{2\alpha_k+1}-2}(x) \\ &\quad + \sum_{j=2^{2\alpha_k-1}}^{2^{2\alpha_k}-3} t_{2j+1,2^{2\alpha_k+1}} w_{2j}(x) \Bigg| \\ &= \Bigg| \left(t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}} - t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}\right) w_{2^{2\alpha_k+1}-2}(x) \\ &\quad + \sum_{j=2^{2\alpha_k-2}}^{2^{2\alpha_k-1}-2} \left(t_{4j+1,2^{2\alpha_k+1}} w_{4j}(x) + t_{4j+3,2^{2\alpha_k+1}} w_{4j+2}(x)\right) \Bigg| \\ &\ge \left| t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}} - t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}} \right| \\ &\quad - \sum_{j=2^{2\alpha_k-2}}^{2^{2\alpha_k-1}-2} \left| t_{4j+3,2^{2\alpha_k+1}} - t_{4j+1,2^{2\alpha_k+1}} \right| =: (*) \end{aligned}$$

From convexity of $t_{k,n}$ we get

$$2t_{k,n} \le t_{k-2,n} + t_{k+2,n} \Leftrightarrow t_{k,n} - t_{k-2,n} \le \frac{1}{2}(t_{k+2,n} - t_{k-2,n}),$$

so

$$\sum_{j=2^{2\alpha_k-2}}^{2^{2\alpha_k-1}-2}(t_{4j+3,2^{2\alpha_k+1}}-t_{4j+1,2^{2\alpha_k+1}}) \le \frac{1}{2}\sum_{j=2^{2\alpha_k-2}}^{2^{2\alpha_k-1}-2}(t_{4j+5,2^{2\alpha_k+1}}-t_{4j+1,2^{2\alpha_k+1}})$$
$$=\frac{1}{2}(t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}-t_{2^{2\alpha_k}+1,2^{2\alpha_k+1}}),$$

that is why

$$(*) \ge t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}}-t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}-\frac{1}{2}t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}+\frac{1}{2}t_{2^{2\alpha_k}+1,2^{2\alpha_k+1}}$$
$$\ge t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}}-\frac{3}{2}t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}$$

These complete the proof of Lemma 3.3. □

4. The main result

Theorem 4.1. *Let* $0 \le \gamma \le 1$ *and* β *be any nonnegative real number. For every* $n \in \mathbb{P}$*, let* $\{t_{k,2^n} : 1 \le k \le 2^n\}$ *be a convex, non-decreasing and finite sequence of non-negative numbers such that*

$$\sum_{k=1}^{2^n} t_{k,2^n} = 1 \tag{5}$$

and

$$t_{2^n-1,2^n}-\frac{3}{2}t_{2^n-3,2^n} \ge \frac{c_1}{2^{n\gamma}n^{\beta}}. \tag{6}$$

are satisfied for some positive constant c_1*. Then, for any* $0 < p < 1/(1+\gamma)$ *there exists a martingale* $f \in H_p(G)$ *such that*

$$\sup_{n\in\mathbb{N}}\left\|\sigma^T_{2^n}(f)\right\|_{weak-L_p}=\infty$$

Proof. Let $\{\alpha_k : k \in \mathbb{N}\}$ be an increasing sequence of positive integers such that

$$\sum_{k=0}^{\infty}\alpha_k^{-p/2}<\infty, \tag{7}$$

$$\sum_{i=0}^{k-1}\frac{2^{2\alpha_i/p}}{\sqrt{\alpha_i}} \le c_2\frac{2^{2\alpha_{k-1}/p}}{\sqrt{\alpha_{k-1}}} \tag{8}$$

for some constant c_2 and

$$c_2\frac{2^{2\alpha_{k-1}/p}}{\sqrt{\alpha_{k-1}}} \le c_1\frac{2^{2\alpha_k(1/p-1-\gamma)-\gamma}}{\alpha_k(2\alpha_k+1)^{\beta}}. \tag{9}$$

Let us notice, that (6) implies inequality
(10)
$$\frac{c_1 2^{2\alpha_k(1/p-1-\gamma)-\gamma}}{\alpha_k(2\alpha_k+1)^\beta} \leq \frac{2^{2\alpha_k(1/p-1)-1}}{\alpha_k}\left(t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}} - \frac{3}{2}t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}\right).$$

Let
$$f^{(n)} := \sum_{k;2\alpha_k<n} \lambda_k a_k,$$

where
$$\lambda_k := \frac{1}{\sqrt{\alpha_k}} \quad \text{and} \quad a_k := 2^{2\alpha_k(1/p-1)}(D_{2^{2\alpha_k+1}} - D_{2^{2\alpha_k}}).$$

Inequality (7) and Lemma 1 yield that $f \in H_p(G)$.

Let us introduce the following notation

$$\sigma^T_{2^{2\alpha_k+1}}(f) = \sum_{j=1}^{2^{2\alpha_k}-1} t_{j,2^{2\alpha_k+1}} S_j(f) + \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} t_{j,2^{2\alpha_k+1}} S_j(f)$$
$$=: I + II.$$

Based on the technique of paper [13], using conditions (5) and (8) we obtain

$$|I| \leq c_2 \frac{2^{2\alpha_{k-1/p}}}{\sqrt{\alpha_{k-1}}}$$

and if $2^{2\alpha_k} \leq j < 2^{2\alpha_k+1}$, then

$$S_j(f) = \sum_{i=0}^{k-1} \frac{2^{2\alpha_i(1/p-1)}}{\sqrt{\alpha_i}}(D_{2^{2\alpha_i+1}} - D_{2^{2\alpha_i}}) + \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}}(D_j - D_{2^{2\alpha_k}}).$$

Let $x \in I_2(e_0+e_1)$. Since $2 \leq 2\alpha_k$ for every $k \in \mathbb{N}$, in this case $D_{2^{2\alpha_k}} = 0$. It means that

$$S_j(f) = \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} D_j,$$

hence based on Lemma 3.3

$$|II| = \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} \left| \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} t_{j,2^{2\alpha_k+1}} D_j \right|$$

$$\geq \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} \left(t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}} - \frac{3}{2}t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}\right)$$

Then for $x \in I_2(e_0+e_1)$, using inequalities (9), (10) and condition (6) we get

$$\begin{aligned}
\left|\sigma^T_{2^{2\alpha_k+1}}(f)\right| \geq& |II| - |I| \\
\geq& \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}}\left(t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}} - \frac{3}{2}t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}\right) - c_2\frac{2^{2\alpha_{k-1}/p}}{\sqrt{\alpha_{k-1}}} \\
\geq& \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}}\left(t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}} - \frac{3}{2}t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}\right) \\
&- \frac{2^{2\alpha_k(1/p-1)-1}}{\alpha_k}\left(t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}} - \frac{3}{2}t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}\right) \\
\geq& \frac{2^{2\alpha_k(1/p-1)-1}}{\sqrt{\alpha_k}}\left(t_{2^{2\alpha_k+1}-1,2^{2\alpha_k+1}} - \frac{3}{2}t_{2^{2\alpha_k+1}-3,2^{2\alpha_k+1}}\right) \\
\geq& \frac{2^{2\alpha_k(1/p-1)-1}}{\sqrt{\alpha_k}}\frac{c_1}{2^{(2\alpha_k+1)\gamma}(2\alpha_k+1)^\beta} \\
\geq& c_3\frac{2^{2\alpha_k(1/p-1-\gamma)}}{\alpha_k^{\beta+1/2}}
\end{aligned}$$

At the end, with the standard technique we obtain

$$\begin{aligned}
&\left\|\sigma^T_{2^{2\alpha_k+1}}(f)\right\|_{weak-L_p} \\
&\geq c_3\frac{2^{2\alpha_k(1/p-1-\gamma)}}{\alpha_k^{\beta+1/2}}\mu\left\{x \in G : \left|\sigma^T_{2^{2\alpha_k+1}}(f)\right| \geq c_3\frac{2^{2\alpha_k(1/p-1-\gamma)}}{\alpha_k^{\beta+1/2}}\right\}^{1/p} \\
&\geq c_3\frac{2^{2\alpha_k(1/p-1-\gamma)}}{\alpha_k^{\beta+1/2}}\mu\left\{x \in I_2(e_0+e_1) : \left|\sigma^T_{2^{2\alpha_k+1}}(f)\right| \geq c_3\frac{2^{2\alpha_k(1/p-1-\gamma)}}{\alpha_k^{\beta+1/2}}\right\}^{1/p} \\
&\geq c_3\frac{2^{2\alpha_k(1/p-1-\gamma)}}{\alpha_k^{\beta+1/2}}\mu(I_2(e_0+e_1))^{1/p} \\
&\geq c_4\frac{2^{2\alpha_k(1/p-1-\gamma)}}{\alpha_k^{\beta+1/2}} \to \infty, \text{ if } k \to \infty.
\end{aligned}$$

This completes the proof of our Theorem 4.1. □

Remark 4.2. From Theorem 1. of paper of Gát and Goginava [11] follows for every $|f| \in H_p$, where $p > 0$ that

$$\left\|\sup_{n\in\mathbb{N}}|\sigma^{\alpha_n}_{2^n}(f)|\right\|_p \leq c_p\||f|\|_{H_p} \tag{11}$$

and $\sigma^{\alpha_n}_{2^n}$, the so-called Cesàro mean with varying parameters is a special cases of $\sigma^T_{2^n}$ mean (see Example 5.1). Theorem 4.1 implies that the operator $\sigma^{\alpha_n}_{2^n}$ is not bounded from H_p to L_p, when $0 < p < 1/2$ and $0 < \alpha_n < 1$. Or if we choose $0 < \alpha_n < 1$ and $\lim_{n\to\infty}\alpha_n = 0$, the statement will be true for $0 < p < 1$ (see also Example 5.1). So it means that the condition $|f| \in H_p$ for (11) in these cases cannot be replaced by the condition $f \in H_p$.

5. Examples

Theorem 4.1 is a generalization of the main result of paper [1]. In their article, Baramidze, Persson, Tangrand and Tephnadze proved analogous statement for Nörlund means. (For the Nörlund logarithmic mean case see [2].) Hereinafter we give two examples (a notable and an elementary one) for sequence $t_{k,2^n}$ for which $\sigma^T_{2^n}$ is not a Nörlund mean, but we can apply our main theorem to that.

Example 5.1. *Let* $k, n \in \mathbb{N},\ 1 \le k \le 2^n$ *and*

$$t_{k,2^n} := \frac{A^{\alpha_n - 1}_{2^n - k}}{A^{\alpha_n}_{2^n - 1}},$$

where

$$A^{\alpha}_n := \frac{(1+\alpha)\dots(n+\alpha)}{n!}$$

for any $\alpha \neq -1, -2, \dots$ *and we suppose that* $0 < \alpha_n < 1$.

It is known (see [23]), that $\sum_{k=1}^{n} A^{\alpha-1}_{n-k} = A^{\alpha}_{n-1}$, *so* $\sum_{k=1}^{2^n} t_{k,2^n} = 1$.

A simple calculation shows, that for $1 \le k < 2^n$ *inequality* $A^{\alpha_n-1}_{2^n-k} < A^{\alpha_n-1}_{2^n-k-1}$ *holds. It follows that for any fixed* n, *the sequence* $t_{k,2^n}$ *is increasing. Similarly, for* $1 \le k < 2^n - 1$, *the inequality* $A^{\alpha_n-1}_{2^n-k} + A^{\alpha_n-1}_{2^n-k-2} > 2A^{\alpha_n-1}_{2^n-k-1}$ *is valid, which means, that for any fixed* n, *the sequence* $t_{k,2^n}$ *is convex.*

The condition (6) needs $t_{2^n-1,2^n} - \frac{3}{2}t_{2^n-3,2^n} > 0$, *that is* $A^{\alpha_n-1}_1 = \alpha_1 > \frac{3}{2}A^{\alpha_n-1}_3 = \frac{\alpha_1(1+\alpha_2)(2+\alpha_3)}{4}$. *It will be true if* $4 > (1+\alpha_2)(2+\alpha_3)$, *for example if* $\alpha_2 := \alpha_3 := \frac{1}{2}$. *(It is not a strict limitation, since the situation* $n \to \infty$ *is interesting for us.) Since* $1 < A^{\alpha_n}_{2^n-1} < 2^n$ *holds, so choosing* $\gamma := 1,\ \beta := 0, c_1 := \alpha_1 - \frac{\alpha_1(1+\alpha_2)(2+\alpha_3)}{4}$, *condition (6) is fulfilled. Or, for example if* $\lim_{n\to\infty} \alpha_n = 0$, *then* $\lim_{n\to\infty} A^{\alpha_n}_{2^n-1} = 1$, *so we can choose* $\gamma := 0$.

On the other side easy to see that Cesàro means with varying parameters (in general) are not Nörlund means. (Otherwise q_k *would be* $A^{\alpha_n}_k$, *which depends not only on* k, *but also on* n.*)*

Example 5.2. *Let* $2 \le n \in \mathbb{N}$ *and*

$$t_{k,2^n} := \begin{cases} \frac{1}{2^{n+2}}, & \text{if } 1 \le k \le 2^n - 2, \\ \frac{1}{4}, & \text{if } k = 2^n - 1, \\ \frac{1}{2} + \frac{1}{2^{n+1}}, & \text{if } k = 2^n. \end{cases}$$

(Since in Theorem 4.1 we take the supremum for all $n \in \mathbb{P}$ *and the value of the supremum is infinity, therefore it is not interesting, what happens for small* n*'s.) Then*

$$\sum_{k=1}^{2^n} t_{k,2^n} = (2^n - 2)\frac{1}{2^{n+2}} + \frac{1}{4} + \left(\frac{1}{2} + \frac{1}{2^{n+1}}\right) = 1.$$

Easy to see, that $t_{k,2^n}$ is non-decreasing for every $2 \le n \in \mathbb{N}$.
Since

$$t_{k,2^n} + t_{k+2,2^n} = 2t_{k+1,2^n}, \text{ if } 1 \le k \le 2^n - 4 \text{ and } 3 \le n,$$
$$t_{2^n-3,2^n} + t_{2^n-1,2^n} = \frac{1}{2^{n+2}} + \frac{1}{4} > \frac{1}{2^{n+2}} + \frac{1}{2^{n+2}} = 2t_{2^n-2,2^n}, \text{ if } 2 \le n,$$
$$t_{2^n-2,2^n} + t_{2^n,2^n} = \frac{1}{2^{n+2}} + \frac{1}{2} + \frac{1}{2^{n+1}} > \frac{1}{2} = 2t_{2^n-1,2^n}, \text{ if } 2 \le n,$$

sequence $t_{k,n}$ is convex for every $3 \le n \in \mathbb{N}$.

On the other hand if $2 \le n$, then

$$t_{2^n-1,2^n} - \frac{3}{2}t_{2^n-3,2^n} = \frac{1}{4} - \frac{3}{2} \cdot \frac{1}{2^{n+2}} = \frac{2^{n+1}-3}{2^{n+3}} > \frac{2^{n+1}-2^n}{2^{n+3}} = \frac{1}{8}.$$

It means, that choosing $\gamma := \beta := 0$ and $c_1 := \frac{1}{8}$ sequence $t_{k,n}$ is suitable for Theorem 4.1.

If this concrete $\sigma^T_{2^n}$ were a Nörlund mean, next equality would hold

$$\sum_{k=1}^{2^n} t_{k,2^n} S_k = \frac{1}{Q_{2^n}} \sum_{k=1}^{2^n} q_{2^n-k} S_k.$$

It follows, that in this case $\frac{1}{4} = t_{2^n-1,2^n} = \frac{q_1}{Q_{2^n}}$, but it contradicts to (1).

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