Novi Sad J. Math. Vol. 55, No. 1, 2025, 161-181
https://doi.org/10.30755/NSJOM.15832

Fractional-order $\mathcal{SEIR}$ epidemic model with time delay and saturated incidence rate

Fatima Cherkaoui[1], Khalid Hilal[2], Abdelaziz Qaffou[3], Shahram Rezapour[456] and Mohammad Esmael Samei[78]

Abstract. In this study, we examine the qualitative behavior of a fractional $\mathcal{SEIR}$ model with nonlinear incidence rate function and a time delay, where the fractional derivative is defined in the sense of Caputo. The threshold parameter, $\mathcal{R}_0$ is obtained by using the method of next generation matrix and we give a complete study of existence of steady state. To establish the global stability of both the disease-free and endemic equilibria, we primarily apply the Lyapunov functional approach throughout the work. Finally, numerical simulations that exemplify our theoretical results are given.

AMS Mathematics Subject Classification (2010): 34A08; 34B16; 39A13

Key words and phrases: fractional epidemic model; SEIR epidemic model; Lyapunov function

1. Introduction

The goal of epidemiology is to understand how infectious diseases move across a community and the causes that contribute to their incidence. Many studies have been done on the dynamics of the $\mathcal{SIR}$ and $\mathcal{SIRS}$ epidemic models [1, 2, 5, 10, 12, 17, 26, 27]. But many diseases take time to develop inside their hosts before they spread to other people.

To further understand the importance of an incubation time in disease transmission, models other than the $\mathcal{SIR}$ or $\mathcal{SIRS}$ types must be examined. according to a compartmental approach, An individual who is susceptible to infection may initially experience a latent period (also known as exposed or class $\mathcal{E}$) after infection before becoming infectious. If the acquired immunity

[1]Laboratory LMACS, FST of Beni-Mellal, Sultan Moulay slimane University, Morocco, e-mail: cherkaoui2310@gmail.com

[2]Laboratory LMACS, FST of Beni-Mellal, Sultan Moulay slimane University, Morocco, e-mail: hilalkhalid2005@yahoo.fr

[3]Laboratory LMACS, FST of Beni-Mellal, Sultan Moulay slimane University, Morocco, e-mail: aziz.qaffou@gmail.com

[4]Department of Mathematics, Kyuing Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul, Republic of Korea, e-mail: rezapourshahram@yahoo.ca

[5]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran, e-mail: sh.rezapour@azaruniv.edu

[6]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, e-mail: sh.rezapour@mail.cmuh.org.tw

[7]Department of Mathematics, Faculty of Basic Science, Bu-Ali Sina University, Hamedan 65178-38695, Iran, e-mail: mesamei@basu.ac.ir; mesamei@gmail.com

[8]Corresponding author

is permanent or not, the resulting models are of the $\mathcal{SEIR}$ or $\mathcal{SEIRS}$ types, respectively.

Classical differentiation and integration are generalized to any order using fractional differentiation. This is very pertinent to modeling the spread of epidemics because it includes memory and non local effects by default. As a result, many authors [7, 11, 15, 16, 19, 20, 21, 28, 30, 31] have begun to study epidemic models using fractional differential equations. One of the effective and useful researches that have been presented recently, is the use of fractional derivative in epidemic model time delay to take into account the effects of delays that represent the incubation period, infection period, or immune-keeping period for many infectious diseases to reflect some biological facts of the disease [4, 8, 13, 14, 18, 23, 25, 29].

In 2020, Rezapour *et al.* provided a $\mathcal{SEIR}$ epidemic model (Fig. 1)

$$\begin{cases} \frac{\mathrm{d}\mathcal{S}}{\mathrm{d}\mathfrak{z}} = A\mu\mathcal{S} - (\beta_1\mathcal{E} + \beta_2\mathcal{I}), \\ \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\mathfrak{z}} = (\beta_1\mathcal{E} + \beta_2\mathcal{I})\,\mathcal{S} - (\lambda + \mu)\mathcal{E}, \\ \frac{\mathrm{d}\mathcal{I}}{\mathrm{d}\mathfrak{z}} = \lambda\mathcal{E} - (\gamma + \mu + \delta)\mathcal{I}, \\ \frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\mathfrak{z}} = \gamma\mathcal{I} - \mu\mathcal{R}, \end{cases}$$

for the spread of Covid-19 using the Caputo fractional derivative where $A = n \times N$, N is the total number of individuals and n is the birth rate, μ is the death rate of people, β_1, β_2 are the transmission rate of infection from $\mathcal{E}$ to $\mathcal{S}$, $\mathcal{I}$ to $\mathcal{S}$, respectively, λ is the transmission rate of people from $\mathcal{E}$ to $\mathcal{I}$, δ is the mortality rate due to the disease, and γ is the rate of recovery of infected people [21].

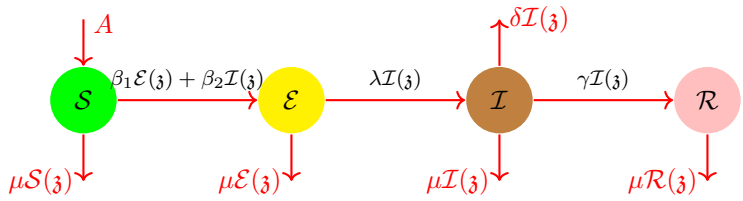


Figure 1: The diagram for the proposed $\mathcal{SEIR}$ model of Covid-19 was introduced by authors in [21].

Since applying varied incidence rates has the ability to change the system's behavior, it is well known that the dynamics of epidemic play a significant influence in the disease transmission process. Following incidence functions with or without delay are frequently employed in many epidemiological contexts in numerous epidemic models. The authors of [13], investigated the qualitative behavior of a class of fractional $\mathcal{SEIR}$ epidemic model with a more general incidence rate function and time delay to incorporate latent infected individuals, and and worked by a general incidence in the theoretical part. In the numerical part, they took a bilinear incidence $\beta\mathcal{SI}$ [13].

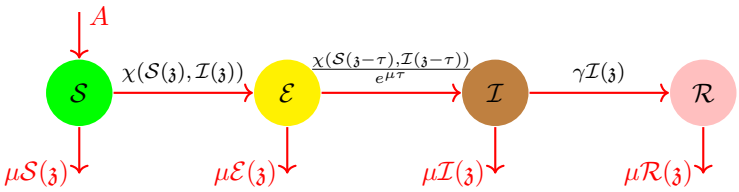


Figure 2: The transmission dynamics between individuals in different compartments.

In simpler terms, we were able to find out if a disease is stable or not using a mathematical method called Lyapunov function. We studied both the healthy and sick states of the disease to get a complete understanding.

In our work, we took a saturated incidence $\frac{\beta\mathcal{SI}}{1+\eta\mathcal{I}}$ which the other authors took the death rate of infected individuals caused by the disease but in our model we didn't. We didn't make the local stability of equilibria points. Ilhem *et al.* in [13], took the death rate of infected individuals caused by the disease but in our model we didn't. Also, we didn't make the local stability of equilibria points. The saturated incidence rate of the form

$$\wp(\mathcal{S},\mathcal{I}) = \frac{\beta\mathcal{SI}}{1+\eta\mathcal{I}},$$

which tends to a saturation level when $\mathcal{I}$ gets large, $\beta\mathcal{I}$ measures the infection force when the disease is entering a fully susceptible population, and $\frac{1}{1+\eta\mathcal{I}}$ measures the inhibition effect from the behavioral change of susceptible individuals when their number increases or from the crowding effect of the infective individuals. This way of measuring how fast a disease spreads is better than another way because it takes into account how people change their behavior and how many people are in one place. It also makes sure the amount of contact between people is not too high by choosing the right numbers.

The next part of this paper is structured like this: We will explain some basic things about fractional calculus in Section 2. In Section 3, we show how we made a model and prove that there is only one solution. We also study whether there are any fixed situations where there are no infecting diseases or whether diseases are always present, depending on how fast they spread. In Section 4, we use a method called the Lyapunov function to show that both natural and disease-infected conditions will stay stable. In the fifth section, we use computer calculations to show our ideas.

2. Preliminaries

In this section, we present the definition of Caputo fractional-order derivative, and some useful lemmas are recalled for next analysis.

The fractional integral and Caputo fractional derivative of order σ for a function $\wp(\mathfrak{z})$ are defined as

$$\mathbb{I}^{\sigma}\wp(\mathfrak{z}) = \frac{1}{\Gamma(\sigma)}\int_0^{\mathfrak{z}} (\mathfrak{z}-r)^{\sigma-1}\wp(r)\,\mathrm{d}r, \qquad \sigma > 0,$$

$$\mathbb{D}^{\sigma}\wp(\mathfrak{z}) = \frac{1}{\Gamma(n-\sigma)}\int_0^{\mathfrak{z}} \frac{\wp^{(n)}(r)}{(\mathfrak{z}-r)^{\sigma-n+1}}\,\mathrm{d}r, \qquad \wp(\mathfrak{z}) \in \mathcal{C}^n([0,\infty)),\ n-1 \le \sigma < n,$$

for $\mathfrak{z} \ge 0$, respectively, where n is a positive integer and $\Gamma(\cdot)$ is the gamma function, $\Gamma(\sigma) = \int_0^{\infty} s^{\sigma-1}e^{-s}\,\mathrm{d}s$ [20]. Furthermore, when $0 < \sigma < 1$,

$$\mathbb{D}^{\sigma}\wp(\mathfrak{z}) = \frac{1}{\Gamma(1-\sigma)}\int_0^{\mathfrak{z}} \frac{\wp'(r)}{(\mathfrak{z}-r)^{\sigma}}\,\mathrm{d}r.$$

The Mittag-Leffler type function with one parameter is defined as follows [20]

$$E_{\sigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\sigma+1)}, \qquad \sigma > 0,\ z \in \mathbb{C}.$$

Theorem 2.1 ([6]). *Let $\sigma \in (0,1]$, $\Omega \subset \mathbb{R}^n$ a domain and $\wp : [\tau_0,\infty)\times\Omega \to \mathbb{R}^n$ be a function satisfying the Lipschitz condition on $\mathfrak{z}$ and consider the equation $\mathbb{D}^{\sigma}\wp(\mathfrak{z}) = \chi(\mathfrak{z},\wp(\mathfrak{z}))$, for $\mathfrak{z} > \tau_0$, with the initial condition $\wp(\tau_0) = \wp_0 \in \Omega$. Then the above system has a unique maximal solution.*

3. The model

We call the number of people in a place at a certain time "total population size", $\mathfrak{z}$, by $\aleph(\mathfrak{z})$ and assume that $\aleph(\mathfrak{z})$ is divided into four compartments at time $\mathfrak{z}$, which are:

$\mathcal{S}(\mathfrak{z})$, Susceptible individuals;

$\mathcal{E}(\mathfrak{z})$, Exposed individuals;

$\mathcal{I}(\mathfrak{z})$, Infected individuals;

$\mathcal{R}(\mathfrak{z})$, Recovered individuals.

The susceptible group $\mathcal{S}$ includes individuals at risk of infection through close contact with infected individuals. Contact classes $\mathcal{E}$ are detected but not infectious yet. Infected group $\mathcal{I}$ includes individuals who already have the disease and can transmit the disease to susceptible subjects. The recovered group $\mathcal{R}$ includes individuals who have been infected and are now healthy. The diagram of the model is given in Figure 2 which in it

$$\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})) = \frac{\beta\mathcal{S}(\mathfrak{z})\mathcal{I}(\mathfrak{z})}{1+\eta\mathcal{I}(\mathfrak{z})}.$$

The spreading dynamic of the epidemic is then governed by the following frac-

tional system

$$
\begin{cases}
\mathbb{D}^{\sigma}\mathcal{S}(\mathfrak{z}) = A - \dfrac{\beta\mathcal{S}(\mathfrak{z})\mathcal{I}(\mathfrak{z})}{1+\eta\mathcal{I}(\mathfrak{z})} - \mu\mathcal{S}(\mathfrak{z}), \\
\mathbb{D}^{\sigma}\mathcal{E}(\mathfrak{z}) = \dfrac{\beta\mathcal{S}(\mathfrak{z})\mathcal{I}(\mathfrak{z})}{1+\eta\mathcal{I}(\mathfrak{z})} - \beta e^{-\mu\tau}\dfrac{\mathcal{S}(\mathfrak{z}-\tau)\mathcal{I}(\mathfrak{z}-\tau)}{1+\eta\mathcal{I}(\mathfrak{z}-\tau)} - \mu\mathcal{E}(\mathfrak{z}), \\
\mathbb{D}^{\sigma}\mathcal{I}(\mathfrak{z}) = \beta e^{-\mu\tau}\dfrac{\mathcal{S}(\mathfrak{z}-\tau)\mathcal{I}(\mathfrak{z}-\tau)}{1+\eta\mathcal{I}(\mathfrak{z}-\tau)} - \gamma\mathcal{I}(\mathfrak{z}) - \mu\mathcal{I}(\mathfrak{z}), \\
\mathbb{D}^{\sigma}\mathcal{R}(\mathfrak{z}) = \gamma\mathcal{I}(\mathfrak{z}) - \mu\mathcal{R}(\mathfrak{z}),
\end{cases} \tag{3.1}
$$

with initial conditions

$$
\mathcal{S}(\upsilon) = \hbar_1(\upsilon), \quad \mathcal{E}(\upsilon) = \hbar_2(\upsilon), \quad \mathcal{I}(\upsilon) = \hbar_3(\upsilon), \quad \mathcal{R}(\upsilon) = \hbar_4(\upsilon), \tag{3.2}
$$

where τ is time delay and $\hbar_i \in C([-\tau, 0], \mathbb{R}^+)$, $i = 1, 2, 3, 4$. The parameters used in (3.1) are explained in Table 1. In this model, τ represents the incubation

Table 1: The parameters description used in model

The physical interpretation	Parameter	Values
Recruitment rate	A	10^{-5}
Effective contact rate	β	1.2
Natural death rate	μ	10^{-5}
Measure of inhibition	η	10^{-2}
Recovery rate	γ	0.1

period and $e^{-\mu\tau}$ is the probability that an individual survives the latent period $[\mathfrak{z}-\tau, \mathfrak{z}]$, since the number of susceptible individuals that become exposed at time $\mathfrak{z}-\tau$ are

$$
\beta\frac{\mathcal{S}(\mathfrak{z}-\tau)\mathcal{I}(\mathfrak{z}-\tau)}{1+\eta\mathcal{I}(\mathfrak{z}-\tau)}.
$$

There will then be

$$
e^{-\mu\tau}\beta\frac{\mathcal{S}(\mathfrak{z}-\tau)\mathcal{I}(\mathfrak{z}-\tau)}{1+\eta\mathcal{I}(\mathfrak{z}-\tau)},
$$

individuals surviving in the latent period τ and becoming infective at time $\mathfrak{z}$.

The first and third equations don't depend on $\mathcal{E}$ and $\mathcal{R}$, so the model can be reduced to just these equations:

$$
\begin{cases}
\mathbb{D}^{\sigma}\mathcal{S}(\mathfrak{z}) = A - \dfrac{\beta\mathcal{S}(\mathfrak{z})\mathcal{I}(\mathfrak{z})}{1+\eta\mathcal{I}(\mathfrak{z})} - \mu\mathcal{S}(\mathfrak{z}), \\
\mathbb{D}^{\sigma}\mathcal{I}(\mathfrak{z}) = \beta e^{-\mu\tau}\dfrac{\mathcal{S}(\mathfrak{z}-\tau)\mathcal{I}(\mathfrak{z}-\tau)}{1+\eta\mathcal{I}(\mathfrak{z}-\tau)} - \gamma\mathcal{I}(\mathfrak{z}) - \mu\mathcal{I}(\mathfrak{z}).
\end{cases} \tag{3.3}
$$

Lemma 3.1. *The system* (3.3)*, for any given initial condition* (3.2) *admits a unique positive solution which is global and remains nonnegative.*

Proof. The right-hand side of system (3.3) satisfies the Lipschitz condition on $\mathfrak{X} = (\mathcal{S}, \mathcal{I})$, therefore the system (3.3) with initial condition (3.2) has a unique maximal solution. In addition, we have

$$\mathbb{D}^{\sigma}\mathcal{S}\Big|_{\mathcal{S}=0} = A \geq 0, \qquad \mathbb{D}^{\sigma}\mathcal{I}\Big|_{\mathcal{I}=0} = 0 \geq 0.$$

Then $\mathcal{S}(\mathfrak{z})$ and $\mathcal{I}(\mathfrak{z})$ are positive solution. Next, we give the global existence of the solution, we have $\mathcal{S}(\mathfrak{z}) + \mathcal{I}(\mathfrak{z}) \leq \aleph(\mathfrak{z})$. Since $\mathbb{D}^{\sigma}\aleph(\mathfrak{z}) = A - \mu\aleph(\mathfrak{z})$. Then

$$\aleph(\mathfrak{z}) = \left(\aleph_0 - \frac{A}{\mu}\right)\mathcal{E}_{\sigma}(-\mu\mathfrak{z}^{\sigma}) + \frac{A}{\mu}.$$

Therefore

$$\mathcal{S}(\mathfrak{z}) + \mathcal{I}(\mathfrak{z}) \leq \left(\aleph_0 - \frac{A}{\mu}\right)\mathcal{E}_{\sigma}\left(-\mu\mathfrak{z}^{\sigma}\right) + \frac{A}{\mu},$$

where $\aleph_0 = \mathcal{S}(0) + \mathcal{I}(0)$. Thus

$$\lim_{\mathfrak{z}\to\infty} \sup(\mathcal{S}(\mathfrak{z}) + \mathcal{I}(\mathfrak{z})) \leq \frac{1}{\mu}A.$$

Indeed, the solution is global and bounded. Hence,

$$\Omega = \left\{(\mathcal{S}, \mathcal{I}) \in \mathbb{R}^2_+ \,:\, 0 \leq \mu\left(\mathcal{S} + \mathcal{I}\right) \leq A\right\},$$

is a positive attracting set for system (3.3). □

3.1. Steady states

To figure out unchanging conditions in the system (3.3), we solve this set of equations:

$$A - \beta\frac{\mathcal{S}\mathcal{I}}{1 + \eta\mathcal{I}} - \mu\mathcal{S} = 0, \tag{3.4}$$

$$\beta e^{-\mu\tau}\frac{\mathcal{S}\mathcal{I}}{1 + \eta\mathcal{I}} - \gamma\mathcal{I} - \mu\mathcal{I} = 0. \tag{3.5}$$

The system (3.3) has a free steady state $\mathcal{E}^0 = (\mathcal{S}^0, 0)$ with $\mathcal{S}^0 = \frac{A}{\mu}$. Before obtaining the endemic steady state, we determine the basic reproduction number denoted by $\mathcal{R}_0$ by using the method of next generation matrix. Let $\mathfrak{X} = (\mathcal{S}, \mathcal{I})$, then the system (3.3) can be written as $\mathbb{D}^{\sigma}\mathfrak{X} = \Upsilon(\mathfrak{X}) - \Lambda(\mathfrak{X})$, where

$$\Upsilon(\mathfrak{X}) = \begin{pmatrix} 0 \\ \\ \beta e^{-\mu\tau}\dfrac{\mathcal{S}(\mathfrak{z}-\tau)\mathcal{I}(\mathfrak{z}-\tau)}{1 + \eta\mathcal{I}(\mathfrak{z}-\tau)} \end{pmatrix},$$

$$\Lambda(\mathfrak{X}) = \begin{pmatrix} A - \beta\dfrac{\mathcal{S}(\mathfrak{z})\mathcal{I}(\mathfrak{z})}{1 + \eta\mathcal{I}(\mathfrak{z})} - \mu\mathcal{S}(\mathfrak{z}) \\ -\gamma\mathcal{I}(\mathfrak{z}) - \mu\mathcal{I}(\mathfrak{z}), \end{pmatrix}.$$

We represent the Jacobian matrix for the matrices $\Upsilon(\mathfrak{X})$ and $\Lambda(\mathfrak{X})$ at the free point as the following form

$$\widetilde{\Upsilon} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{S}^0 \end{pmatrix}, \qquad \widetilde{\Lambda} = \begin{pmatrix} -\mu & -\beta\mathcal{S}^0 \\ 0 & -\mu-\gamma \end{pmatrix}.$$

Therefore we define $\mathcal{R}_0$ as the spectral radius of the next generation matrix $-\widetilde{\Upsilon}\widetilde{\Lambda}^{-1}$, which is given by

$$\mathcal{R}_0 = \frac{\beta A e^{-\mu\tau}}{\mu(\gamma+\mu)}. \tag{3.6}$$

Let prove the existence and uniqueness for endemic equilibrium denoted by $\mathcal{E}^* = (\mathcal{S}^*, \mathcal{I}^*)$. Using (3.4) and (3.5) it is the solution for

$$A - \mu\mathcal{S} = \beta\frac{\mathcal{SI}}{1+\eta\mathcal{I}} = e^{\mu\tau}(\gamma+\mu)\mathcal{I}. \tag{3.7}$$

Then we obtain the following relationship:

$$\mathcal{S} = \frac{1}{\mu}(A - e^{\mu\tau}(\gamma+\mu)\mathcal{I}).$$

Clearly, $\mathcal{S}$ exists if and only if $\mathcal{I} < \frac{1}{\gamma+\mu}Ae^{-\mu\tau}$. Where $\mathcal{I}$ is the solution of the equation $G(\mathcal{I}) = 0$, with

$$G(\mathcal{I}) = \frac{1}{1+\eta\mathcal{I}}\beta e^{-\mu\tau}\mathcal{S} - \gamma - \mu.$$

Then we have

$$\lim_{\mathcal{I}\to 0} G(\mathcal{I}) = \frac{\beta e^{-\mu\tau}A}{\mu} - \gamma - \mu = (\gamma+\mu)\left(\mathcal{R}_0 - 1\right).$$

If $\mathcal{R}_0 > 1$ we obtain $\lim_{\mathcal{I}\to 0} G(\mathcal{I}) > 0$. On the other hand if $\mathcal{I} = \frac{1}{\gamma+\mu}Ae^{-\mu\tau}$, then we have $G(\mathcal{I}) = -(\gamma+\mu) < 0$. Furthermore, the function G is bijective. The classical mean value theorem implies that that there exists a unique positive root $\mathcal{I}^*$, which is the solution of the equation $G(\mathcal{I}) = 0$ and satisfies the following equation $0 < \mathcal{I}^* \leq \frac{1}{\gamma+\mu}Ae^{-\mu\tau}$.

Theorem 3.2. *If $\mathcal{R}_0 > 1$, then system* (3.3) *has a unique endemic steady state $\mathcal{E}^* = (\mathcal{S}^*, \mathcal{I}^*)$ such that*

$$0 < \mathcal{I}^* \leq \frac{1}{\gamma+\mu}Ae^{-\mu\tau}.$$

4. Global stability

We begin our investigation with the free steady state $\mathcal{E}^0$. Let the Lyapunov function be defined by

$$\begin{aligned}\Lambda_1(\mathfrak{z}) = \mathcal{S}(\mathfrak{z}) - \mathcal{S}^0 - \mathcal{I}^{\sigma}_{[0,\mathfrak{z}]}\left[\frac{\chi(\mathcal{S}^0, \mathcal{S}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}), \mathcal{I}(\mathfrak{z}))}\mathbb{D}^{\sigma}\mathcal{S}(\mathfrak{z})\right] \\ + e^{\mu\tau}\mathcal{I}(\mathfrak{z}) + \mathcal{I}^{\sigma}_{[\mathfrak{z}-\tau,\mathfrak{z}]}\left[\chi(\mathcal{S}(\mathfrak{z}), \mathcal{I}(\mathfrak{z}))\right],\end{aligned} \tag{4.1}$$

with

$$\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))=\frac{\beta\mathcal{S}(\mathfrak{z})\mathcal{I}(\mathfrak{z})}{1+\eta\mathcal{I}(\mathfrak{z})}.$$

Obviously, Λ_1 is non-negative defined function at $\mathcal{E}^0$. We have

$$\begin{aligned}\mathbb{D}^\sigma\Lambda_1(\mathfrak{z})&=\left(1-\frac{\chi(\mathcal{S}^0,\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)\mathbb{D}^\sigma\mathcal{S}(\mathfrak{z})+e^{\mu\tau}\mathbb{D}^\sigma\mathcal{I}(\mathfrak{z})\\&\quad+\chi\big(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\big)-\chi\big(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau)\big),\\&=A\left(1-\frac{\chi(\mathcal{S}^0,\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)-\mu\left(\mathcal{S}(\mathfrak{z})-\mathcal{S}^0\right)\\&\quad+\chi\left(\mathcal{S}^0,\mathcal{I}(\mathfrak{z})\right)-e^{\mu\tau}(\gamma+\mu)\mathcal{I}(\mathfrak{z}),\\&=\left(\mathcal{S}(\mathfrak{z})-\mathcal{S}^0\right)\left(\frac{A}{\mathcal{S}(\mathfrak{z})}-\mu\right)+\mathcal{I}(\mathfrak{z})\left(\frac{\beta\mathcal{S}^0}{1+\eta\mathcal{I}(\mathfrak{z})}-e^{\mu\tau}(\gamma+\mu)\right),\\&=-\frac{\mu}{\mathcal{S}(\mathfrak{z})}\left(\mathcal{S}(\mathfrak{z})-\mathcal{S}^0\right)^2+\mathcal{I}(\mathfrak{z})e^{\mu\tau}(\gamma+\mu)\left(\frac{\mathcal{R}_0}{1+\eta\mathcal{I}(\mathfrak{z})}-1\right).\end{aligned}$$

Since the function $\mathcal{I}\longmapsto\frac{1}{1+\eta\mathcal{I}}$ is decreasing, we have $\frac{1}{1+\eta\mathcal{I}}\leq 1$. Then we obtain

$$\mathbb{D}^\sigma\Lambda_1(\mathfrak{z})\leq-\frac{\mu}{\mathcal{S}(\mathfrak{z})}\left(\mathcal{S}(\mathfrak{z})-\mathcal{S}^0\right)^2+\mathcal{I}(\mathfrak{z})e^{\mu\tau}(\gamma+\mu)\left(\mathcal{R}_0-1\right).$$

If $\mathcal{R}_0<1$, then $\mathbb{D}^\sigma\Lambda_1(\mathfrak{z})\leq 0$. So, the Lyapunov-Lasalle asymptotic stability implies that $\mathcal{E}^0$ is globally asymptotically stable.

Theorem 4.1. *The disease-free equilibrium $\mathcal{E}^0$ is globally asymptotically stable, whenever $\mathcal{R}_0<1$.*

The second step of this section is to prove the endemic point $\mathcal{E}_1$ is asymptotically globally stable. To do that, we consider the Lyapunov function as form

$$\begin{aligned}\Lambda_2(\mathfrak{z})&=\mathcal{S}(\mathfrak{z})-\mathcal{S}^*-\int_{\mathcal{S}^*}^{\mathcal{S}(\mathfrak{z})}\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(r,\mathcal{I}(\mathfrak{z})\right)}\,\mathrm{d}r\\&\quad+e^{\mu\tau}\left(\mathcal{I}(\mathfrak{z})-\mathcal{I}^*-\int_{\mathcal{I}^*}^{\mathcal{I}(\mathfrak{z})}\frac{\chi\left(\mathcal{S},\mathcal{I}^*\right)}{\chi(\mathcal{S},r)}\,\mathrm{d}r\right)\\&\quad+\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)\mathbb{I}^\sigma_{[\mathfrak{z}-\tau,\mathfrak{z}]}\left(\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}-1-\ln\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{D}^{\sigma}\Lambda_2(\mathfrak{z}) &= \left(1-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\mathbb{D}^{\sigma}\mathcal{S}(\mathfrak{z})\\ &\quad+e^{\mu\tau}\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\mathbb{D}^{\sigma}\mathcal{I}(\mathfrak{z})\\ &\quad+\left(\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}-\frac{\chi\left(\mathcal{S}\left(\mathfrak{z}-\tau\right),\mathcal{I}(\mathfrak{z}-\tau)\right)}{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}\right.\\ &\quad\left.+\ln\frac{\chi\left(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau)\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\chi\left(\mathcal{S}^*,\mathcal{I}^*\right).\end{aligned}\tag{4.2}$$

As

$$\begin{aligned}&\left(1-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\mathbb{D}^{\sigma}\mathcal{S}(\mathfrak{z})\\ &\quad=\left(1-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)\left(A-\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))-\mu\mathcal{S}(\mathfrak{z})\right),\end{aligned}$$

and at equilibrium state: $A=\chi(\mathcal{S}^*,\mathcal{I}^*)+\mu\mathcal{S}^*$. Thus

$$\begin{aligned}\left(1-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\mathbb{D}^{\sigma}\mathcal{S}(\mathfrak{z}) &= \mu\left(1-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\left(\mathcal{S}^*-\mathcal{S}\right)\\ &\quad+\left(1-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\left(1-\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}\right)\chi\left(\mathcal{S}^*,\mathcal{I}^*\right).\end{aligned}\tag{4.3}$$

Since

$$\begin{aligned}e^{\mu\tau}\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\mathbb{D}^{\sigma}\mathcal{I}(\mathfrak{z}) &= e^{\mu\tau}\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\\ &\quad\times\left(e^{-\mu\tau}\chi\left(\mathcal{S}\left(\mathfrak{z}-\tau\right),\mathcal{I}(\mathfrak{z}-\tau)\right)-\gamma\mathcal{I}(\mathfrak{z})-\mu\mathcal{I}(\mathfrak{z})\right),\end{aligned}$$

and at equilibrium state, we have $\frac{\chi(\mathcal{S}^*,\mathcal{I}^*)}{\mathcal{I}^*}=e^{\mu\tau}(\gamma+\mu)$. That is,

$$\begin{aligned}&e^{\mu\tau}\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\mathbb{D}^{\sigma}\mathcal{I}(\mathfrak{z})\\ &\quad=\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\\ &\qquad\times\left(\chi\left(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau)\right)-\frac{\mathcal{I}}{\mathcal{I}^*}\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)\right),\\ &\quad=\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\left(\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}-\frac{\mathcal{I}}{\mathcal{I}^*}\right)\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)\\ &\qquad+\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)\\ &\qquad\times\left(\frac{\chi\left(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau)\right)}{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}\right).\end{aligned}\tag{4.4}$$

Substituting the equations (4.3) and (4.4) into (4.2) we get

$$\begin{aligned}
\mathbb{D}^{\sigma}\Lambda_2(\mathfrak{z}) &= \mu\left(1-\frac{\chi(\mathcal{S}^*,\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)(\mathcal{S}^*-\mathcal{S})\\
&\quad+\left(1-\frac{\chi(\mathcal{S}^*,\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)\left(1-\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}^*,\mathcal{I}^*)}\right)\chi(\mathcal{S}^*,\mathcal{I}^*)\\
&\quad+\left(1-\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)\left(\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}-\frac{I}{\mathcal{I}^*}\right)\chi(\mathcal{S}^*,\mathcal{I}^*)\\
&\quad+\left(1-\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)\chi(\mathcal{S}^*,\mathcal{I}^*)\left(\frac{\chi(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau))}{\chi(\mathcal{S}^*,\mathcal{I}^*)}\right.\\
&\quad\left.-\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}\right)+\chi(\mathcal{S}^*,\mathcal{I}^*)\left(\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}^*,\mathcal{I}^*)}\right.\\
&\quad\left.-\frac{\chi(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau))}{\chi(\mathcal{S}^*,\mathcal{I}^*)}+\ln\frac{\chi(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)\\
&=\mu\left(1-\frac{\chi(\mathcal{S}^*,\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)(\mathcal{S}^*-\mathcal{S})\\
&\quad+\left(1-\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)\left(\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}-\frac{\mathcal{I}}{\mathcal{I}^*}\right)\chi(\mathcal{S}^*,\mathcal{I}^*)\\
&\quad+\chi(\mathcal{S}^*,\mathcal{I}^*)\left[2-\frac{\chi(\mathcal{S}^*,\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}+\frac{\chi(\mathcal{S}^*,\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}^*,\mathcal{I}^*)}\right.\\
&\quad-\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}-\frac{\chi(\mathcal{S},\mathcal{I}^*)\,\chi(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))\,\chi(\mathcal{S}^*,\mathcal{I}^*)}\\
&\quad\left.+\ln\frac{\chi(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right]\\
&=\mu\left(1-\frac{\chi(\mathcal{S}^*,\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)(\mathcal{S}^*-\mathcal{S})\\
&\quad+\chi(\mathcal{S}^*,\mathcal{I}^*)\left(1-\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)\left(\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}-\frac{\mathcal{I}}{\mathcal{I}^*}\right)\\
&\quad+\chi(\mathcal{S}^*,\mathcal{I}^*)\left[-\frac{\chi(\mathcal{S}^*,\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}+\frac{\chi(\mathcal{S}^*,\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}^*,\mathcal{I}^*)}\right.\\
&\quad-\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}-\frac{\chi(\mathcal{S},\mathcal{I}^*)\,\chi(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))\,\chi(\mathcal{S}^*,\mathcal{I}^*)}\\
&\quad+\ln\frac{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)\,\chi(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))\,\chi(\mathcal{S}^*,\mathcal{I}^*)}\\
&\quad\left.+\ln\frac{\chi(\mathcal{S}^*,\mathcal{I}^*)}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}+\frac{\chi(\mathcal{S}^*,\mathcal{I}^*)}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}-\frac{\chi(\mathcal{S}^*,\mathcal{I}^*)}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*)}\right].
\end{aligned}$$

Let $\phi(\mathfrak{z})=\mathfrak{z}-(1+\ln\mathfrak{z})$. Note that $\phi:\mathbb{R}^+\to\mathbb{R}^+$ has a strict global minimum at $\mathfrak{z}=1$. Thus we get

$$\mathbb{D}^{\sigma}\Lambda_2(\mathfrak{z})=\mu\left(1-\frac{\chi(\mathcal{S}^*,\mathcal{I}(\mathfrak{z}))}{\chi(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z}))}\right)(\mathcal{S}^*-\mathcal{S})$$

$$
\begin{aligned}
&+\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\left(\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}-\frac{\mathcal{I}}{\mathcal{I}^*}\right)\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)\\
&+\Bigg[-\phi\left(\frac{\chi\left(\mathcal{S},\mathcal{I}^*\right)\chi\left(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau)\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}\right)\\
&-\phi\left(\frac{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}\right)+\frac{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\\
&+\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}\Bigg]\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)\\
&=\mu\left(1-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\left(\mathcal{S}^*-S\right)\\
&+\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\left(\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}-\frac{\mathcal{I}}{\mathcal{I}^*}\right)\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)\\
&+\Bigg[-\phi\left(\frac{\chi\left(\mathcal{S},\mathcal{I}^*\right)\chi\left(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau)\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}\right)\\
&-\phi\left(\frac{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}\right)+\Bigg(\frac{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}{\chi\left(\mathcal{S},\mathcal{I}^*\right)}\\
&-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\Bigg)\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}\right)\Bigg]\chi\left(\mathcal{S}^*,\mathcal{I}^*\right).
\end{aligned}
$$

Since

$$
\frac{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}{\chi\left(\mathcal{S},\mathcal{I}^*\right)}-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}=0,
$$

we obtain

$$
\begin{aligned}
\mathbb{D}^{\sigma}\Lambda_2(\mathfrak{z})&=\mu\left(1-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\left(\mathcal{S}^*-\mathcal{S}\right)\\
&+\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\left(\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}-\frac{\mathcal{I}}{\mathcal{I}^*}\right)\\
&+\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)\Bigg[-\phi\left(\frac{\chi\left(\mathcal{S},\mathcal{I}^*\right)\chi\left(\mathcal{S}(\mathfrak{z}-\tau),\mathcal{I}(\mathfrak{z}-\tau)\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}\right)\\
&-\phi\left(\frac{\chi\left(\mathcal{S}^*,\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}\right)\Bigg].
\end{aligned}
$$

We have

$$
\mu\left(1-\frac{\chi\left(\mathcal{S}^*,\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\left(\mathcal{S}^*-\mathcal{S}\right)=-\frac{1}{\mathcal{S}(\mathfrak{z})}\left(\mathcal{S}(\mathfrak{z})-\mathcal{S}^*\right)^2\leq 0,
$$

and

$$
\begin{aligned}
&\left(1-\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}\right)\left(\frac{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}(\mathfrak{z})\right)}{\chi\left(\mathcal{S}(\mathfrak{z}),\mathcal{I}^*\right)}-\frac{\mathcal{I}}{\mathcal{I}^*}\right)\\
&\qquad=\left(\frac{\mathcal{I}}{\mathcal{I}^*}-\frac{1+\eta\mathcal{I}(\mathfrak{z})}{1+\eta\mathcal{I}^*}\right)\left(\frac{1+\eta\mathcal{I}^*}{1+\eta\mathcal{I}(\mathfrak{z})}-1\right)\leq 0.
\end{aligned}
$$

Hence we can conclude that $\mathbb{D}^{\sigma}\Lambda_2(\mathfrak{z}) \leq 0$ and by the Lasalle invariance principle, the endemic steady state $\mathcal{E}^*$ is globally asymptotically stable. We give the next theorem.

Theorem 4.2. *If $\mathcal{R}_0 > 1$, the endemic steady state $\mathcal{E}^*$ is globally asymptotically stable.*

5. Study numerical simulation

To demonstrate the theoretical findings in the earlier sections, in order to solve the fractional-order ODEs, we offer a numerical approach. For this purpose, we follow the modified Adams-Bashforth-Moulton method (ABMM𝕄), which has proven to be an effective tool for solving differential equations with numerical fractional-order delay [3, 9, 10, 22, 24].

Let us highlight the main features of the modified ABMM𝕄 algorithm. Consider the following fractional-order delay differential equation

$$\begin{cases} \mathbb{D}^{\sigma}\wp(\mathfrak{z}) = \chi(\mathfrak{z}, \wp(\mathfrak{z}), \wp(\mathfrak{z}-\tau)), & \mathfrak{z} \in [0,T],\ \sigma \in (0,1], \\ \wp(\mathfrak{z}) = \phi(\mathfrak{z}), & \mathfrak{z} \in [-\tau, 0], \end{cases} \tag{5.1}$$

where continuous function $\chi : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz and

$$\wp(\mathfrak{z}) = \begin{bmatrix} \wp_1(\mathfrak{z}) & \wp_2(\mathfrak{z}) & \dots & \wp_n(\mathfrak{z}) \end{bmatrix}^T.$$

This initial value problem is equivalent to the following Volterra-integral equation

$$\wp(\mathfrak{z}) = \phi(0) + \int_0^{\mathfrak{z}} \frac{(\mathfrak{z}-r)^{\sigma-1}}{\Gamma(\sigma)}\, \chi(r, \wp(r), \wp(r-\tau))\, \mathrm{d}r. \tag{5.2}$$

For given mesh points,

$$\mathcal{T} = \left\{ \mathfrak{z}_{-k}, \mathfrak{z}_{-k+1}, \dots, \mathfrak{z}_{-1}, \mathfrak{z}_0, \mathfrak{z}_1, \dots, \mathfrak{z}_n \right\},$$

with step-size $h = \frac{1}{k}\tau$ such that $\mathfrak{z}_{-k} = -\tau$, $\mathfrak{z}_0 = 0$ and $\mathfrak{z}_n = T$. Let

$$\begin{aligned} \wp_h(\mathfrak{z}_j) &= \phi(\mathfrak{z}_j), & j &= -k, -k+1, \dots, -1, 0, \\ \wp_h(\mathfrak{z}_j - \tau) &= \wp_h(\mathfrak{z}_j h - kh), & j &= 0, 1, \dots, n. \end{aligned}$$

Suppose

$$\wp_h(\mathfrak{z}_j) \approx \phi(\mathfrak{z}_j), \qquad (j = -k, -k+1, \dots, -1, 0, 1, \dots, n).$$

Then the numerical scheme for the integral equation (5.2) is given by

$$\wp_h(\mathfrak{z}_{n+1}) = \phi(0) + \int_0^{\mathfrak{z}_{n+1}} \frac{(\mathfrak{z}_{n+1}-r)^{\sigma-1}}{\Gamma(\alpha)}\, \chi\left(r, \wp(r), \wp(r-\tau)\right)\, \mathrm{d}r. \tag{5.3}$$

To find the value of the the right side of integral (5.3) with the formula for trapezoidal quadrature, in which the nodes $\mathfrak{z}_j$, $(j = 0, 1, \ldots, n+1)$ are considered with respect to the weight function $(\mathfrak{z}_{n+1} - r)^{\sigma-1}$. One can be written by the use of the standard technique of quadrature theory

$$\begin{aligned}&\int_0^{\mathfrak{z}_{n+1}} (\mathfrak{z}_{n+1} - r)^{\sigma-1} \chi\left(r, \wp(r), \wp(r-\tau)\right) \mathrm{d}r \\ &\approx \frac{h^\sigma}{\sigma(\sigma+1)} \sum_{j=0}^{n+1} a_{j,n+1} \chi\left(\mathfrak{z}_j, \wp_h(\mathfrak{z}_j), \wp_h(\mathfrak{z}_j - \tau)\right), \end{aligned} \tag{5.4}$$

where $a_{j,n+1}$ are given by

$$a_{j,n+1} = n^{\sigma+1} - (n-\sigma)(n+1)\sigma,$$

whenever $j = 0$;

$$a_{j,n+1} = (n-j+2)^{\sigma+1} - (n-\sigma)(n+1)^\sigma - 2(n-j+1)^{\sigma+1},$$

whenever $1 \leq j \leq n$, and $a_{j,n+1} = 1$ whenever $j = n+1$. Therefore, the numerical scheme of the fractional-order delay differential equation (5.1) can be formulated as

$$\begin{aligned}\wp_h(\mathfrak{z}_{n+1}) &= \phi(0) + \frac{h^\sigma}{\Gamma(\sigma+2)} \chi(\mathfrak{z}_{n+1}, \wp_h(\mathfrak{z}_{n+1}), \wp_h(\mathfrak{z}_{n+1} - \tau)) \\ &\quad + \frac{h^\sigma}{\Gamma(\sigma+2)} \sum_{j=0}^{n+1} a_{j,n+1} \chi(\mathfrak{z}_j, \wp_h(\mathfrak{z}_j), \wp_h(\mathfrak{z}_j - \tau)) \\ &= \phi(0) + \frac{h^\sigma}{\Gamma(\sigma+2)} \chi(\mathfrak{z}_{n+1}, \wp_h(\mathfrak{z}_{n+1}), \wp_h(\mathfrak{z}_{n+1-k})) \\ &\quad + \frac{h^\sigma}{\Gamma(\sigma+2)} \sum_{j=0}^{n+1} a_{j,n+1} \chi(\mathfrak{z}_j, \wp_h(\mathfrak{z}_j), \wp_h(\mathfrak{z}_{j-k})). \end{aligned} \tag{5.5}$$

By substituting $\wp_h(\mathfrak{z}_{n+1})$ in (5.5) by an approximation $\wp_h^{\mathcal{P}}(\mathfrak{z}_{n+1})$, called predictor, which is evaluated by Product rectangle rule in equation (5.3) to evaluate predictor term

$$\begin{aligned}\wp_h^{\mathcal{P}}(\mathfrak{z}_{n+1}) &= \phi(0) + \frac{1}{\Gamma(\sigma)} \sum_{j=0}^{n+1} b_{j,n+1} \chi\left(\mathfrak{z}_j, \wp_h(\mathfrak{z}_j), \wp_h(\mathfrak{z}_j - \tau)\right) \\ &= \phi(0) + \frac{1}{\Gamma(\sigma)} \sum_{j=0}^{n+1} b_{j,n+1} \chi\left(\mathfrak{z}_j, \wp_h(\mathfrak{z}_j), \wp_h(\mathfrak{z}_{j-k})\right), \end{aligned} \tag{5.6}$$

where $b_{j,n+1}$ is given by

$$b_{j,n+1} = \frac{h^\sigma}{\sigma}\left((n+1-j)^\sigma - (n-j)^\sigma\right). \tag{5.7}$$

Now, we write the numerical scheme (5.6) for the model (3.1):

$$\begin{aligned}
\mathcal{S}(\mathfrak{z}_{n+1}) &= \mathcal{S}(0) + \sum_{j=1}^{n} \frac{b_{j,n+1}}{\Gamma(\sigma)} \left(A - \mu \mathcal{S}(\mathfrak{z}_j) - \frac{\beta \mathcal{S}(\mathfrak{z}_j)\mathcal{I}(\mathfrak{z}_j)}{1+\eta \mathcal{I}(\mathfrak{z}_j)} \right) \\
\mathcal{E}(\mathfrak{z}_{n+1}) &= \mathcal{E}(0) + \sum_{j=1}^{n} \frac{b_{j,n+1}}{\Gamma(\sigma)} \left(\frac{\beta \mathcal{S}(\mathfrak{z}_j)\mathcal{I}(\mathfrak{z}_j)}{1+\eta \mathcal{I}(\mathfrak{z}_j)} \right. \\
&\quad \left. - e^{-\mu k} \frac{\beta \mathcal{S}(\mathfrak{z}_{j-k})\mathcal{I}(\mathfrak{z}_{j-k})}{1+\eta \mathcal{I}(\mathfrak{z}_{j-k})} - \mu \mathcal{E}(\mathfrak{z}_j) \right) \\
\mathcal{I}(\mathfrak{z}_{n+1}) &= \mathcal{I}(0) + \sum_{j=1}^{n} \frac{b_{j,n+1}}{\Gamma(\sigma)} \left(e^{-\mu k} \frac{\beta \mathcal{S}(\mathfrak{z}_{j-k})\mathcal{I}(\mathfrak{z}_{j-k})}{1+\eta \mathcal{I}(\mathfrak{z}_{j-k})} \right. \\
&\quad \left. - \gamma \mathcal{I}(\mathfrak{z}_j) - \mu \mathcal{I}(\mathfrak{z}_j) \right) \\
\mathcal{R}(\mathfrak{z}_{n+1}) &= \mathcal{R}(0) + \sum_{j=1}^{n} \frac{b_{j,n+1}}{\Gamma(\sigma)} \left(\gamma \mathcal{I}(\mathfrak{z}_j) - \mu \mathcal{R}(\mathfrak{z}_j) \right).
\end{aligned}$$

We consider initial values, which are given in Table 1,

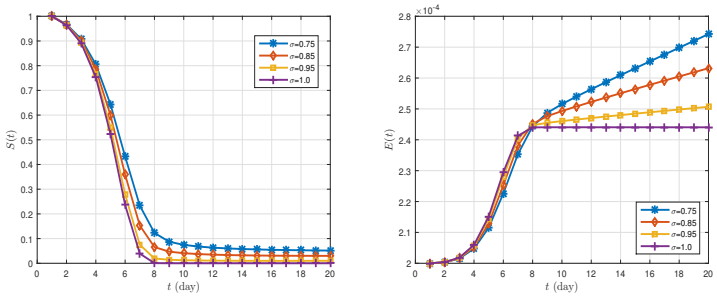
$$\mathcal{S}(0) = 1, \quad \mathcal{E}(0) = 0.0002, \quad \mathcal{I}(0) = 0.03, \quad \mathcal{R}(0) = 0.$$

In the first case, we check the results of the proposed model by changing the order of the derivative. And then in the second case, we check the state of the presented model by changing effective contact rate.

Case I. In Figures 3a, 3b, 4a and 4b, we have plotted the results of model (3.1) for

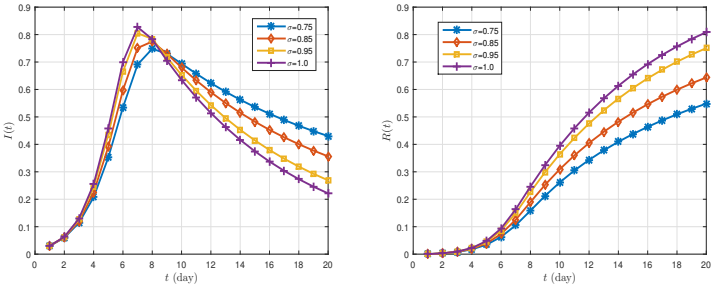
$$\sigma \in \Big\{ 0.75, 0.85, 0.95, 1.00 \Big\}.$$

Also, Tables 2, 3, 4 and 5 show these results. As can be seen, when σ gets closer to 1, the groups $\mathcal{S}$, $\mathcal{E}$, $\mathcal{I}$ decrease but $\mathcal{R}$ increase.



(a) $\mathcal{S}(\mathfrak{z})$ (b) $\mathcal{E}(\mathfrak{z})$

Figure 3: $\mathcal{S}(\mathfrak{z})$ and $\mathcal{E}(\mathfrak{z})$ on period τ whenever $\sigma = 0.75, 0.85, 0.95, 1$.



(a) $\mathcal{I}(\mathfrak{z})$ (b) $\mathcal{R}(\mathfrak{z})$

Figure 4: Dynamics of $\mathcal{I}(\mathfrak{z})$ and $\mathcal{R}(\mathfrak{z})$ on period τ for $\sigma = 0.75, 0.85, 0.95, 1$.

Table 2: Obtained results of $\mathcal{S}(\mathfrak{z})$ on period $\tau = 20$ days for different fractional orders $\sigma = 0.75, 0.85, 0.95, 1.00$.

$\mathfrak{z}$	$\mathcal{S}(\mathfrak{z})$			
	$\boldsymbol{\sigma = 0.75}$	$\boldsymbol{\sigma = 0.85}$	$\boldsymbol{\sigma = 0.95}$	$\boldsymbol{\sigma = 1.00}$
1	1.00000	1.00000	1.00000	1.00000
2	0.96728	0.96602	0.96470	0.96401
3	0.90806	0.90172	0.89484	0.89119
4	0.80531	0.78601	0.76452	0.75287
5	0.64361	0.59970	0.55021	0.52320
⋮	⋮	⋮	⋮	⋮
16	0.05467	0.03093	0.00989	0.00002
17	0.05358	0.03054	0.00989	0.00002
18	0.05271	0.03028	0.00994	0.00002
19	0.05201	0.03012	0.01003	0.00003
20	0.05145	0.03005	0.01015	0.00003

Table 3: Obtained results of $\mathcal{E}(\mathfrak{z})$ on period $\tau = 20$ days for different fractional orders $\sigma = 0.75, 0.85, 0.95, 1.00$.

$\mathfrak{z}$	$\mathcal{E}(\mathfrak{z})$			
	$\boldsymbol{\sigma = 0.75}$	$\boldsymbol{\sigma = 0.85}$	$\boldsymbol{\sigma = 0.95}$	$\boldsymbol{\sigma = 1.00}$
1	0.00020	0.00020	0.00020	0.00020
2	0.00020	0.00020	0.00020	0.00020
3	0.00020	0.00020	0.00020	0.00020
4	0.00021	0.00021	0.00021	0.00021
5	0.00021	0.00021	0.00021	0.00022
⋮	⋮	⋮	⋮	⋮
16	0.00027	0.00026	0.00025	0.00024
17	0.00027	0.00026	0.00025	0.00024
18	0.00027	0.00026	0.00025	0.00024
19	0.00027	0.00026	0.00025	0.00024
20	0.00027	0.00026	0.00025	0.00024

Case II. In Figures 5a, 5b, 6a and 6b, we have plotted the results of model (3.1)

Table 4: Obtained results of $\mathcal{I}(\mathfrak{z})$ on period $\tau = 20$ days for different fractional orders $\sigma = 0.75, 0.85, 0.95, 1.00$.

$\mathfrak{z}$	$\mathcal{I}(\mathfrak{z})$			
	$\sigma = 0.75$	$\sigma = 0.85$	$\sigma = 0.95$	$\sigma = 1.00$
1	0.03000	0.03000	0.03000	0.03000
2	0.05999	0.06115	0.06236	0.06299
3	0.11410	0.11989	0.12617	0.12951
4	0.20734	0.22484	0.24432	0.25487
5	0.35199	0.39113	0.43511	0.45905
⋮	⋮	⋮	⋮	⋮
16	0.51145	0.45287	0.37950	0.33736
17	0.48876	0.42558	0.34777	0.30363
18	0.46776	0.40052	0.31897	0.27327
19	0.44829	0.37748	0.29282	0.24595
20	0.43018	0.35625	0.26905	0.22136

Table 5: Obtained results of $\mathcal{R}(\mathfrak{z})$ on period $\tau = 20$ days for different fractional orders $\sigma = 0.75, 0.85, 0.95, 1.00$.

$\mathfrak{z}$	$\mathcal{R}(\mathfrak{z})$			
	$\sigma = 0.75$	$\sigma = 0.85$	$\sigma = 0.95$	$\sigma = 1.00$
1	0.00000	0.00000	0.00000	0.00000
2	0.00273	0.00283	0.00294	0.00300
3	0.00784	0.00839	0.00898	0.00930
4	0.01735	0.01914	0.02115	0.02225
5	0.03439	0.03915	0.04466	0.04774
⋮	⋮	⋮	⋮	⋮
16	0.46382	0.54614	0.64056	0.69257
17	0.48759	0.57382	0.67229	0.72630
18	0.50946	0.59914	0.70104	0.75666
19	0.52963	0.62234	0.72710	0.78397
20	0.54829	0.64364	0.75074	0.80856

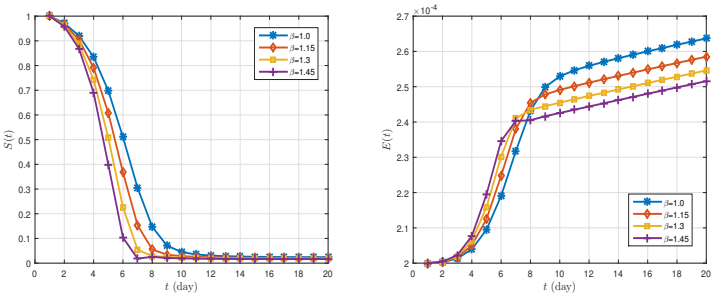
for

$$\beta \in \Big\{1.0, 1.15, 1.3, 1.45\Big\}.$$

Also, Tables 6, 7, 8 and 9 show these results. As can be seen, when β gets closer to 1, $\mathcal{S}$ is decreasing, but $\mathcal{R}$ is increasing.

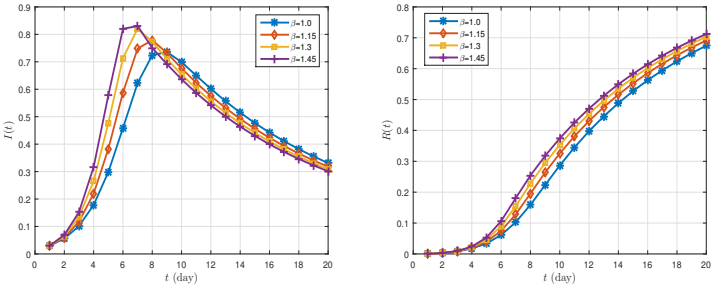
6. Conclusion

This study showed a new way to use math to understand diseases spreading, called the $\mathcal{SEIR}$ model. It also used a type of math called Caputo fractional derivative and added a delay factor. We have figured out where the system can work and where it can't. We also looked at points where it stays balanced and checked if they are solid. The feasibility region of the system and equilibrium points have been calculated, and the stability of the equilibrium points has been investigated. We have proven that there is one answer to the model using fixed point theory. The existence of a unique solution for the model by using fixed point theory has been proved. The next generation matrix method was



(a) $\mathcal{S}(\mathfrak{z})$ (b) $\mathcal{E}(\mathfrak{z})$

Figure 5: $\mathcal{S}(\mathfrak{z})$ and $\mathcal{E}(\mathfrak{z})$ on period τ whenever $\beta = 1.0, 1.15, 1.3, 1.45$.



(a) $\mathcal{I}(\mathfrak{z})$ (b) $\mathcal{R}(\mathfrak{z})$

Figure 6: Dynamics of $\mathcal{I}(\mathfrak{z})$ and $\mathcal{R}(\mathfrak{z})$ on period τ for $\beta = 1.0, 1.15, 1.3, 1.45$.

Table 6: Obtained results of $\mathcal{S}(\mathfrak{z})$ on period $\tau = 20$ days for different effective contact rate $\beta = 1.0, 1.15, 1.3, 1.45$.

$\mathfrak{z}$	$\mathcal{S}(\mathfrak{z})$			
	$\boldsymbol{\beta = 1.0}$	$\boldsymbol{\beta = 1.15}$	$\boldsymbol{\beta = 1.3}$	$\boldsymbol{\beta = 1.45}$
1	1.00000	1.00000	1.00000	1.00000
2	0.97114	0.96681	0.96248	0.95815
3	0.92034	0.90403	0.88663	0.86816
4	0.83409	0.79104	0.74292	0.68979
5	0.69809	0.60811	0.50767	0.39935
⋮	⋮	⋮	⋮	⋮
16	0.02503	0.02113	0.01844	0.01643
17	0.02460	0.02094	0.01836	0.01639
18	0.02434	0.02085	0.01835	0.01642
19	0.02420	0.02084	0.01840	0.01650
20	0.02417	0.02090	0.01850	0.01662

used to estimate an answer to the model. Using the method of next generation matrix, an approximate answer to the model has been calculated.

Table 7: Obtained results of $\mathcal{E}(\mathfrak{z})$ on period $\tau = 20$ days for different effective contact rate $\beta = 1.0, 1.15, 1.3, 1.45$.

$\mathfrak{z}$	$\mathcal{E}(\mathfrak{z})$			
	$\boldsymbol{\beta = 1.0}$	$\boldsymbol{\beta = 1.15}$	$\boldsymbol{\beta = 1.3}$	$\boldsymbol{\beta = 1.45}$
1	0.00020	0.00020	0.00020	0.00020
2	0.00020	0.00020	0.00020	0.00020
3	0.00020	0.00020	0.00020	0.00020
4	0.00020	0.00021	0.00021	0.00021
5	0.00021	0.00021	0.00022	0.00022
⋮	⋮	⋮	⋮	⋮
16	0.00026	0.00026	0.00025	0.00025
17	0.00026	0.00026	0.00025	0.00025
18	0.00026	0.00026	0.00025	0.00025
19	0.00026	0.00026	0.00025	0.00025
20	0.00026	0.00026	0.00026	0.00025

Table 8: Obtained results of $\mathcal{I}(\mathfrak{z})$ on period $\tau = 20$ days for different effective contact rate $\beta = 1.0, 1.15, 1.3, 1.45$.

$\mathfrak{z}$	$\mathcal{I}(\mathfrak{z})$			
	$\boldsymbol{\beta = 1.0}$	$\boldsymbol{\beta = 1.15}$	$\boldsymbol{\beta = 1.3}$	$\boldsymbol{\beta = 1.45}$
1	0.03000	0.03000	0.03000	0.03000
2	0.05597	0.06030	0.06463	0.06896
3	0.10154	0.11743	0.13441	0.15246
4	0.17837	0.21950	0.26559	0.31658
5	0.29794	0.38213	0.47620	0.57757
⋮	⋮	⋮	⋮	⋮
16	0.44228	0.42306	0.40947	0.39941
17	0.41068	0.39306	0.38065	0.37146
18	0.38181	0.36567	0.35433	0.34594
19	0.35541	0.34063	0.33025	0.32258
20	0.33124	0.31769	0.30819	0.30118

Table 9: Obtained results of $\mathcal{R}(\mathfrak{z})$ on period $\tau = 20$ days for different effective contact rate $\beta = 1.0, 1.15, 1.3, 1.45$.

$\mathfrak{z}$	$\mathcal{R}(\mathfrak{z})$			
	$\boldsymbol{\beta = 1.0}$	$\boldsymbol{\beta = 1.15}$	$\boldsymbol{\beta = 1.3}$	$\boldsymbol{\beta = 1.45}$
1	0.00000	0.00000	0.00000	0.00000
2	0.00289	0.00289	0.00289	0.00289
3	0.00813	0.00854	0.00896	0.00938
4	0.01753	0.01945	0.02148	0.02362
5	0.03396	0.03975	0.04612	0.05305
⋮	⋮	⋮	⋮	⋮
16	0.56263	0.58576	0.60203	0.61412
17	0.59466	0.61594	0.63094	0.64209
18	0.62379	0.64342	0.65727	0.66759
19	0.65032	0.66847	0.68130	0.69086
20	0.67453	0.69134	0.70325	0.71214

To guess which parts of the world don't have any diseases, they used computer programs with information from real life. To predict the disease-free in

the world, the numerical simulations based on real data have been provided. In the numerical part, we looked at using the fractional-order derivative instead of the integer-order for better results and we compared these results with real data and put them in Tables 2, 3, 4, 5, and Charts 3a, 3b, 4a, 4b. The study found that using a fractional-order model gave better modeling results. In the number part and case II, we looked at using the different values of effective contact rate β for better results and we compared these results with real data and put them in Tables 6, 7, 8, 9, and Charts 5a, 5b, 6a, 6b.

Declarations

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

Funding

Not applicable.

Authors' contributions

All authors have equal contributions. All authors read and approved the final manuscript.

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Received by the editors April 19, 2023
First published online September 25, 2023