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A survey of continuous frames for closed range operators in Hilbert $\mathcal{C}^*$-modules

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Abstract. Frame theory is a new and applicable part of harmonic analysis and plays an important role in many areas and fields. The current paper is concerned with the construction of continuous frames for some closed range operators in Hilbert modules.

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1. Introduction and preliminary

Frames are basis-like systems that span a vector space but allow for linear dependency, that can be used to obtain other desirable features unavailable with orthonormal bases. The aim of this frame theory, introduced by Duffin and Schaeffer in 1952, was to solve some problems related to nonharmonic Fourier series [7] and popularized later by Daubechieseven [6].

The continuous frames have been defined by Ali, Antoine and Gazeau [2] and later, independently by Kaiser[12]. these kinds frames are the first generalisation frames to measure space. Nowaday, frames have been used as a powerful alternative to Hilbert bases because of their redundancy and flexibility. They have also been a very useful tool in the characterization of function spaces and fields of applications, such coding and communications [24] signal processing [9]. Hilbert $\mathcal{C}^*$-module is an object like a Hilbert space except that the inner product is not scalar-valued, but takes its values in a $\mathcal{C}^*$-algebra of coefficients. These modules play an important role in the study of locally compact quantum groups and non-commutative geometry.

The main goal of this present paper is to study the invariance of continuous K-frames under some closed range operators in Hilbert $\mathcal{C}^*$-modules.

We first present a brief account of basic definitions of Hilbert C^*-modules and their frames, we refer the interested reader to [15, 10, 3, 11].

A left Hilbert $\mathcal{C}^*$-module over the unital $\mathcal{C}^*$-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module $\mathcal{H}$ endowed with an $\mathcal{A}$-valued inner product

$$\langle .\,,.\rangle \quad : \quad \mathcal{H}\times\mathcal{H} \quad \longrightarrow \quad \mathcal{A}$$

satisfying the following conditions:

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1. $\langle x, x\rangle \geq 0$, for all $x \in \mathcal{H}$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.
2. $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$, for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.
3. $\langle x, y\rangle = \langle y, x\rangle^*$, for all $x, y \in \mathcal{H}$.
4. $\mathcal{H}$ is complete with respect to the norm $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. We denote $\mathcal{L}(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $\mathcal{L}(\mathcal{H}, \mathcal{H})$ is abbreviated to $\mathcal{L}(\mathcal{H})$.

Example 1.1. Let us consider the following set

$$l^2(\mathcal{A}) = \{\{a_j\}_{j\in\mathbb{J}} \subseteq \mathcal{A} : \sum_{j\in\mathbb{J}} a_j a_j^* \textit{ converge in } ||\,.\,||_{\mathcal{A}}\}.$$

It is easy to see that $l^2(\mathcal{A})$ with pointwise operations and the inner product

$$\langle\{a_j\}, \{b_j\}\rangle = \sum_{j\in\mathbb{J}} a_j b_j^*,$$

is a Hilbert $\mathcal{C}^*$-module which is called the standard Hilbert $\mathcal{C}^*$-module over $\mathcal{A}$.

Throughout this paper, we suppose that $\mathcal{H}$ is a Hilbert $\mathcal{C}^*$-module and $\mathbb{J}$ a countable index set of $\mathbb{N}$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by $R(T)$ and $N(T)$ the range and the kernel subspaces of T respectively and I is the identity operator. As usual, for $E \subset \mathcal{H}$, the orthogonal projection on E is denoted by π_E . We also write $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, for the generalized range.

Following Muller in [19], if $T, G, \tilde{T}, \tilde{G}$ are mutually commuting in $\mathcal{L}(\mathcal{H})$ such that $T\tilde{T} + G\tilde{G} = I$, then

$$R^\infty(TG) = R^\infty(T) \cap R^\infty(G).$$

Definition 1.2. [8] Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and (Ω, μ) be a measure space. The mapping $F : \Omega \longrightarrow \mathcal{H}$ is called a continuous frame with respect to (Ω, μ), if

1. The map $: \omega \longrightarrow \langle x, F(\omega)\rangle$ is a measurable function on Ω, for all $x \in \mathcal{H}$.
2. There exist constants $\alpha, \beta > 0$ such that

$$\alpha\langle x, x\rangle \leq \int_\Omega \langle x, F(\omega)\rangle\langle F(\omega), x\rangle d\mu(\omega) \leq \beta\langle x, x\rangle,\ (x \in \mathcal{H}).$$

Definition 1.3. [8]Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and (Ω, μ) be a measure space. The mapping $F : \Omega \longrightarrow \mathcal{H}$ is called a continuous K- frame with respect to (Ω, μ), if

1. The map $: \omega \longrightarrow \langle x, F(\omega)\rangle$ is a measurable function on Ω, for all $x \in \mathcal{H}$.

2. There exist constants $\alpha, \beta > 0$ such that

$$\alpha\langle K^*x, K^*x\rangle \leq \int_{\Omega}\langle x, F(\omega)\rangle\langle F(\omega), x\rangle d\mu(\omega) \leq \beta\langle x, x\rangle,\ (x \in \mathcal{H}).$$

Example 1.4. Let $K \in \mathcal{L}(\mathbb{C}^2)$ be defined as follows:

$$K = \begin{bmatrix} 0 & 0 \\ 1 & \sqrt{3}i \end{bmatrix}, \textit{where } i^2 = -1.$$

set

$$\begin{array}{rccc} F & : & \mathbb{R} & \longrightarrow & \mathbb{C}^2 \\ & & \omega & \longmapsto & \left(0; e^{-(2+i)\omega^2}\right). \end{array}$$

For $x = (x_1, x_2) \in \mathbb{C}^2$, we have

$$\int_{\mathbb{R}}\langle x, F(\omega)\rangle\langle F(\omega), x\rangle d\mu(\omega) = \frac{\sqrt{\pi}}{2} \mid x_2 \mid^2$$

and

$$\| K^*x \|^2 = 4 \mid x_2 \mid^2.$$

Consequently, we get

$$\frac{\sqrt{\pi}}{9} \| K^*x \|^2 \leq \int_{\mathbb{R}}\langle x, F(\omega)\rangle\langle F(\omega), x\rangle d\mu(\omega) \leq \| x \|^2.$$

Therefore, F is a continuous K-frame for $\mathbb{C}^2$.

In the next, we list some useful lemmas that we will need in this paper.

Lemma 1.5. *[20] Let $\mathcal{H}$ be Hilbert $\mathcal{A}$-module and $T \in \mathcal{L}(\mathcal{H})$. Then*

$$\langle Tx, Tx\rangle \leq \| T \|^2 \langle x, x\rangle, \textit{ for all } x \in \mathcal{H}.$$

Lemma 1.6. *([27]) Assume that $T, G \in \mathcal{L}(\mathcal{H})$ such that $R(G)$ is closed. Then the following statements are equivalent:*

1. *$R(T) \subseteq R(G)$;*
2. *$\lambda\langle T^*x, T^*x\rangle \leq \langle G^*x, G^*x\rangle$, for some $\lambda > 0$.*

Lemma 1.7. *[26] Let (Ω, μ) be a measure space, X and Y two Banach spaces, $g \in \mathcal{B}(X, Y)$ be a bounded linear operator and $f : \Omega \longrightarrow Y$ a measurable function. Then*

$$g\int_{\Omega} f d\mu = \int_{\Omega} g f d\mu.$$

The closeness of range of operators is an attractive problem which appears in operator theory, especially, in the theory of Fredholm operators and generalized inverses.

Theorem 1.8. *[23] Suppose that $T, G \in \mathcal{L}(\mathcal{H})$ are operators with closed range such that $TG = GT$. Then $R(TG)$ is closed.*

Moreover, we pay attention that the concept of the conorme $\gamma(T)$ plays a fundamental role in the perturbation theory of Fredholm operators.

Definition 1.9. [16] The conorme of $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$\gamma(T) := \inf\{\| Tx \|,\ x \in \mathcal{H},\ dist(x, N(T)) = 1\}$$

Formally, we set $\gamma(0) := \infty$. Notice that $\gamma(T) > 0$ if and only if $R(T)$ is closed.

Example 1.10. Let $T \in \mathcal{L}(\mathbb{C}^2)$ be defined as follows

$$T \ : \ \begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ (x_1, x_2) & \longmapsto & (x_1, x_1). \end{array}.$$

For $x = (x_1, x_2) \in \mathbb{C}^2$, we have

$$\| Tx \| = \sqrt{2} \mid x_1 \mid$$

hence

$$dist(x, N(T)) = \mid x_1 \mid .$$

Then, $\gamma(T) = \sqrt{2}$.

Recall from [25], that the Moore-Penrose inverse of an operator $T \in \mathcal{L}(\mathcal{H})$ with closed range is a unique operator T^+ such that

$$TT^+(x) = x, \quad for\, all\, x \in R(T).$$

Example 1.11. [13] For each $T \in \mathcal{L}(\mathcal{H})$ with closed range, we have

$$T^\dagger = \int_0^{+\infty} T^* e^{-T^*T}\, dt.$$

Now, we list below some useful properties related to Moore-Penrose inverse.

Proposition 1.12. [*25*] *Let $T \in \mathcal{L}(\mathcal{H})$ be a closed range. Then*

1. $TT^\dagger = P_{R(T)},\ T^\dagger T = P_{R(T^*)}$;
2. $R\left(T^\dagger\right) = R(T^*) = N(T)^\perp$;
3. $N\left(T^\dagger\right) = N(T^*) = R(T)^\perp$;
4. $\left(T^\dagger\right)^* = (T^*)^\dagger$.

Theorem 1.13. [*18*] *Let $T \in \mathcal{L}(\mathcal{H})$ be closed range and $G \in \mathcal{L}(\mathcal{H})$ be an arbitrary operator which commutes with T. Then G commutes with $T^\dagger$.*

EP matrix, as an extension of normal matrix, was introduced by Schwerdtfeger [22] and has been extended by Campbell and Meyer[5] to operators with closed range on a Hilbert space and so by Sharifi for Hilbert $\mathcal{A}$-modules [23].

Definition 1.14. [23] We shall say that $T \in \mathcal{L}(\mathcal{H})$ is an EP operator if $R(T)$ is closed and $R(T) = R(T^*)$.

Example 1.15. Let $T \in \mathcal{L}\left(l^2(\mathbb{C})\right)$ be defined as follows:

$$T\left((x_j)_{j\geq 1}\right) = \left((y_j)_{j\geq 1}\right),$$

where

$$y_j = \begin{cases} x_1 - x_3 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \\ x_j & \text{if } j \geq 3 \end{cases}$$

By some straightforward computations, we obtain that T is an EP-operator.

The concept of regularity has greatly benefited from the work of Mbekhta, Ouahab [17] and Rakocevič [21].

Definition 1.16. [17] An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be semi-regular if $R(T)$ is closed and $N(T) \subset R(T^n)$, for every $n \geq 1$.

Clearly, all injective operators with closed range are semi-regular. Some examples of semi-regular operators may be found in [14].

Theorem 1.17. [4] *Let $T \in \mathcal{L}(\mathcal{H})$ be semi-regular and $G \in \mathcal{L}(\mathcal{H})$ such that $TGT = T$. Then*

$$T^n G^n T^n = T^n, \; for\, all\, n \geq 1.$$

Theorem 1.18. *[19] Let $T, G, \tilde{T}, \tilde{G}$ be mutually commuting operators such that $T\tilde{T} + G\tilde{G} = I$. Then, TG is semi-regular if and only if both T and G are semi-regular.*

Next, we collect some useful properties of semi-regular operators and we set

$$\mathbb{D}(0, \gamma(T)) = \{\lambda \in \mathbb{C} :\mid \lambda \mid < \gamma(T)\}.$$

Proposition 1.19. ([1]) *Let T be semi-regular and $\lambda \in \mathbb{D}(0, \gamma(T))$. Then*

1. *T^n is semi-regular;* $(\forall n \geq 1)$
2. *$R^\infty(T) = T(R^\infty(T))$;*
3. *$T - \lambda I$ is semi-regular;*
4. *$R^\infty(T) \subset R^\infty(T - \lambda I)$.*

Recall that the semi-regular resolvent of $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$reg(T) = \{\lambda \in \mathbb{C} : T - \lambda I \; is \; semi - regular\}.$$

In the next result, we show that the subspaces $R^\infty(T - \lambda I)$ are constant as λ ranges through a component Ω of $reg(T)$.

Theorem 1.20. *[1] Let $T \in \mathcal{L}(\mathcal{H})$ and Ω be a connected component of $reg(T)$. If $\lambda_0 \in \Omega$, then $R^\infty(T - \lambda I) = R^\infty(T - \lambda_0 I)$, for every $\lambda \in \Omega$.*

2. Main results

Consider $T, K \in \mathcal{L}(\mathcal{H})$ with closed range such that $TK = KT$ and we fix such notations:

$$d(T) = T^{\dagger *}$$

$$p_\lambda(T) = T^n - \lambda T^{n-1}, \ \ (\lambda \in \mathbb{C}\ ,\ n \geq 1)\,.$$

In this section, we start with the following result

Lemma 2.1. *Let $T \in \mathcal{L}(\mathcal{H})$. Then*

$$R(d(T)) = R(T)\,.$$

Proof. It follows from Proposition 1.1, that

$$R(d(T)) = R\left(T^{\dagger *}\right) = \left(N\left(T^{\dagger}\right)\right)^{\perp} = \left(N\left(T^{*}\right)\right)^{\perp} = R(T)\,.$$

This completes the proof. □

Theorem 2.2. *Let $K \in \mathcal{L}(\mathcal{H})$ be a closed range. Then the following are equivalent:*

1. *F is a continuous K-frame for $\mathcal{H}$;*
2. *F is a continuous $d(K)$-frame for $\mathcal{H}$.*

Proof. $1 \Rightarrow 2$: The first, there exist $\alpha, \beta > 0$ such that

$$\alpha\langle K^* x, K^* x\rangle \leq \int_\Omega \langle x, F(\omega)\rangle\langle F(\omega), x\rangle d\mu(\omega) \leq \beta\langle x, x\rangle$$

By Lemma 1.2 and Lemma 2.1, there exists $\xi > 0$ such that

$$\xi\langle d(K)^* x, d(K)^* x\rangle \leq \langle K^* x, K^* x\rangle.$$

This implies

$$\alpha\xi\langle d(K)^* x, d(K)^* x\rangle \leq \int_\Omega \langle x, F(\omega)\rangle\langle F(\omega), x\rangle d\mu(\omega) \leq \beta\langle x, x\rangle.$$

Then, F is a continuous $d(K)$-frame for $\mathcal{H}$.
$2 \Rightarrow 1$: Obvious. □

Corollary 2.3. *Assume that $K \in \mathcal{L}(\mathcal{H})$ is a closed range and F is a continuous K-frame for $\mathcal{H}$. Then F is a continuous $\pi_{R(K)}$-frame for $\mathcal{H}$.*

Proof. It follows from Theorem 2.1, that there exist $\alpha, \beta > 0$ such that

$$\alpha\langle d(K)^* x, d(K)^* x\rangle \leq \int_\Omega \langle x, F(\omega)\rangle\langle F_\omega, x\rangle d\mu(\omega) \leq \beta\langle x, x\rangle,\ (x \in \mathcal{H})\,.$$

By Proposition 1,1, we get

$$d(K)K^* = \left(KK^\dagger\right)^* = \left(\pi_{R(K)}\right)^* = \pi_{R(K)}.$$

Thus

$$\langle \pi^*_{R(K)}x, \pi^*_{R(K)}x\rangle \leq \| K \|^2 \langle d(K)^* x, d(K)^* x\rangle.$$

Then

$$\alpha \| K \|^{-2} \langle \pi^*_{R(K)}x, \pi^*_{R(K)}x\rangle \leq \int_\Omega \langle x, F(\omega)\rangle\langle F(\omega), x\rangle d\mu(\omega) \leq \beta\langle x, x\rangle. \ (x \in \mathcal{H}).$$

This completes the proof. □

Proposition 2.4. *Let U be a unitary operator and F be a continuous-K-frame for $\mathcal{H}$. Then UF is a continuous $(Ud(K)U^*)$-frame for $\mathcal{H}$.*

Proof. Firstly, there exist $\alpha, \beta > 0$ such that

$$\alpha\langle K^*U^*x, K^*U^*x\rangle \leq \int_\Omega \langle x, UF(\omega)\rangle\langle UF(\omega), x\rangle d\mu(\omega) \leq \beta\langle U^*x, U^*x\rangle, \ (x \in \mathcal{H}).$$

Direct computations show that

$$(UT^*U^*)^\dagger = UT^{*\dagger}U^*.$$

This implies

$$d(UTU^*) = (UTU^*)^{*\dagger} = Ud(T)U^*.$$

By Lemma 1.2, we obtain

$$\langle (UKU^*)^* x, (UKU^*)^* x\rangle \leq \| U \|^2 \langle K^*U^*x, K^*U^*x\rangle \leq \langle K^*U^*x, K^*U^*x\rangle.$$

By Lemma 1.5, we get

$$\langle U^*x, U^*x\rangle \leq \| U \|^2 \langle x, x\rangle \leq \langle x, x\rangle.$$

Therefore

$$\alpha\langle (UKU^*)^* x, (UKU^*)^* x\rangle \leq \int_\Omega \langle x, UF(\omega)\rangle\langle UF(\omega), x\rangle d\mu(\omega) \leq \beta\langle x, x\rangle.$$

By Theorem 2.1, we deduce that UF is a continuous-$Ud(K)U^*$-frame for $\mathcal{H}$. □

Lemma 2.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be EP. Then $d(T)$ is EP too.*

Proof. Clearly, we have

$$d(T)^* = d(T^*)$$

By Lemma 2.1, we get

$$R(d(T^*)) = R(T^*).$$

Since T is EP. Then

$$R(d(T)^*) = R(T) = R(d(T)).$$

Therefore, $d(T)$ is EP. □

In the next Theorem, for a given appropriate operator T, we intend to construct some continuous K-frame for $R(T)$

Theorem 2.6. *Let F be a continuous K-frame for $\mathcal{H}$ and T be an EP-operator such that $KT^* = T^*K$. Then, $d(T)F$ is a continuous K-frame for $R(T)$.*

Proof. For $x \in R(d(T)) = R(T)$, we have

$$x = d(T)d(T)^{\dagger}x$$

hence

$$K(x) = (Kd(T))\left(d(T)^{\dagger}x\right).$$

By Theorem 1.2, we get

$$Kd(T) = d(T)K$$

Since

$$d(T)^{\dagger} \in R(T)$$

By Lemma 1.2, there exists $\alpha' > 0$ such that

$$\xi\langle K^*x, K^*x\rangle \le \langle (d(T)K)^* x, (d(T)K)^* x\rangle.$$

Since F is a continuous K-frame for $\mathcal{H}$, there exist $\alpha, \beta > 0$ such that

$$\begin{aligned}\alpha\langle (d(T)K)^* x, (d(T)K)^* x\rangle &\le \int_{\Omega} \langle x, d(T)F(\omega)\rangle.\langle d(T)F(\omega), x\rangle d\mu(\omega)\\ &\le \beta\langle d(T)^* x, d(T)^* x\rangle.\end{aligned}$$

By Lemma 1.1, we obtain

$$\langle d(T)^* x, d(T)^* x\rangle \le \| T^{\dagger} \|^2 \langle x, x\rangle.$$

Therefore

$$\alpha\xi\langle K^*x, K^*x\rangle \le \int_{\Omega} \langle x, d(T)F(\omega)\rangle.\langle d(T)F(\omega), x\rangle d\mu(\omega) \le \beta \| T^{\dagger} \|^2 \langle x, x\rangle.$$

This completes the proof. □

Example 2.7. Let $K, T \in \mathcal{L}\left(\mathbb{C}^2\right)$ such that

$$K = \begin{bmatrix} 1 - e^{\frac{2\pi i}{3}} & 1 \\ 0 & 0 \end{bmatrix},\ T = \begin{bmatrix} 1 & 0 \\ 1 & e^{\frac{-2\pi i}{3}} \end{bmatrix}.$$

where $i^2 = -1$. An easy computation shows that

$$T^{\dagger} = T^2.$$

Consequently, T is an EP operator and simple computations show that

$$d(T) = \begin{bmatrix} 1 & e^{\frac{i\pi}{3}} \\ 0 & e^{\frac{4i\pi}{3}} \end{bmatrix},$$

set

$$F : \mathbb{R} \longrightarrow \mathbb{C}^2$$
$$\omega \longmapsto \left(e^{-\frac{(1+i)\,\omega^2}{2}} ; 0 \right).$$

For $x = (x_1, x_2) \in \mathbb{C}^2$, we have

$$\int_{\mathbb{R}} | \langle x, F(\omega) \rangle |^2 \, d\mu(\omega) = \int_{\mathbb{R}} | \langle x, d(T) F(\omega) \rangle |^2 \, d\mu(\omega) = \sqrt{\pi} \, | x_1 |^2$$

and

$$\| K^* x \|^2 = 4 \, | x_1 |^2 .$$

Then

$$\frac{1}{4} \| K^* x \|^2 \leq \int_{\mathbb{R}} | \langle x, F(\omega) \rangle |^2 \, d\mu(\omega) \leq \sqrt{\pi} \, \| x \|^2 .$$

Therefore, F and $d(T) F$ are both a continuous K-frame for $\mathbb{C}^2$.

In what follows, we assume that T is a semi-regular operator and Ω is a connected component of $reg(T)$ and we agree to use the following notations:

$$\mathcal{R}_0 = R^\infty (T - \lambda_0 I), \ (\lambda_0 \in \Omega).$$

Proposition 2.8. *Let $T \in \mathcal{L}(\mathcal{H})$ be semi-regular and $\lambda \in \mathbb{D}(0, \gamma(T))$. Then*

$$R^\infty (p_\lambda (T)) = R^\infty (T).$$

Moreover, $p_\lambda(T)$ is semi-regular.

Proof. By Proposition 1.2, we have

$$R^\infty (p_\lambda (T)) = R^\infty (T).$$

Thus

$$\begin{aligned} R^\infty (p_\lambda (T)) &= R^\infty \left(T^{n-1}\right) \cap R^\infty (T - \lambda I) \\ &= R^\infty (T) \cap R^\infty (T - \lambda I) \\ &= R^\infty (T). \end{aligned}$$

It follows from Proposition 1.2 and Theorem 1.4, that $p_\lambda(T)$ is semi-regular. This completes the proof. □

Theorem 2.9. *Let T be semi-regular and F be a continuous K-frame for $\mathcal{H}$. Then $p_\lambda(T) F$ is a continuous K-frame for $R^\infty(T)$, for every $| \lambda | < \gamma(T)$.*

Proof. Let $x \in R^{\infty}(T)$, there exists $y \in R^{\infty}(T)$ such that

$$x = p_{\lambda}(T)\, y$$

hence

$$K(x) = (p_{\lambda}(T)\, K)(y).$$

It follows from Theorem 1.1, that $R(p_{\lambda}(T)\, K)$ is closed.
By Lemma 1.2, there is exists $\xi > 0$ such that

$$\xi \langle K^{*}x, K^{*}x\rangle \leq \langle (p_{\lambda}(T)\, K)^{*}\, x, (p_{\lambda}(T)\, K)^{*}\, x\rangle.$$

By Lemma 1.1, we get

$$\langle p_{\lambda}(T)^{*}\, x, p_{\lambda}(T)^{*}\, x\rangle \leq || \, p_{\lambda}(T)\, ||^{2}\, \langle x, x\rangle.$$

Since F is a continuous K-frame for $\mathcal{H}$ with frame bounds α, β, we obtain

$$\begin{aligned}\alpha\xi \langle K^{*}x, K^{*}x\rangle &\leq \int_{\Omega} \langle x, p_{\lambda}(T)\, F(\omega)\rangle \langle p_{\lambda}(T)\, F(\omega), x\rangle d\mu(\omega)\\ &\leq \beta\, || \, p_{\lambda}(T)\, ||^{2}\, \langle x, x\rangle.\end{aligned}$$

This completes the proof. □

Proposition 2.10. *Let T be semi-regular and F be a continuous K-frame for $\mathcal{H}$. Then, $(T - \lambda I)\, F$ is a continuous K-frame for $\mathcal{R}_0$, for all $\lambda \in \Omega$.*

Proof. Result from Theorem 2.3 and Theorem 1.5. □

In the next Theorem, we focus on constructing some continuous K-frames induced by some idempotent operators.

Theorem 2.11. *Let K be semi-regular and F be a continuous K-frame for $\mathcal{H}$. If L is such that $KLK = K$, then $(K^{n}L^{n})\, F$ is a continuous K^{n}-frame for $\mathcal{H}$, $(n \geq 2)$.*

Proof. The first, there exist $\alpha, \beta > 0$ such that

$$\alpha \langle K^{*}x, K^{*}x\rangle \leq \int_{\Omega} \langle x, F(\omega)\rangle \langle F(\omega), x\rangle d\mu(\omega) \leq \beta \langle x, x\rangle, \;\; for\ every\ x \in \mathcal{H}.$$

By Lemma 1.1, we have, for each $n \geq 2$

$$\langle K^{n*}x, K^{n*}x\rangle \leq || \, K^{n-1}\, ||^{2}\, \langle K^{*}x, K^{*}x\rangle.$$

This implies

$$\alpha\, || \, K^{n-1}\, ||^{-2}\, \langle K^{n*}x, K^{n*}x\rangle \leq \int_{\Omega} \langle x, F(\omega)\rangle \langle F(\omega), x\rangle d\mu(\omega) \leq \beta \langle x, x\rangle, \;\; for\ every\ x \in \mathcal{H}.$$

Using Theorem 1.3, we have

$$K^{n}L^{n}K^{n} = K^{n},$$

Thus

$$\alpha\langle K^{n*}x, K^{n*}x\rangle \leq \int_\Omega \langle x, (K^n L^n)\, F(\omega)\rangle\langle (K^n L^n)\, F(\omega), x\rangle d\mu(\omega) \leq \beta' \langle x, x\rangle.$$

where, $\beta' = \beta \mid\mid K^n L^n \mid\mid^2$.
Then, $(K^n L^n)\, F$ is a continuous K^n-frame for $\mathcal{H}$, for every $n \geq 2$. □

Remark 2.12. For every $T \in \mathcal{L}(\mathcal{H})$, the above Theorem is still valid for $n = 1$.

Example 2.13. Assume that $a \in \mathbb{R}$ and $K, L \in \mathcal{L}\left(\mathbb{C}^2\right)$ defined by:

$$K = \begin{pmatrix} e^{ia} & e^{-ia} \\ 0 & 0 \end{pmatrix}, \; L = \begin{pmatrix} e^{-ia} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{array}{rcccc} F & : & \mathbb{R} & \longrightarrow & \mathbb{C}^2 \\ & & \omega & \longmapsto & \left(ie^{-\omega^2}, 0\right). \end{array}$$

where $i^2 = -1$.
For $x = (x_1, x_2) \in \mathbb{C}^2$ and $n \geq 2$, we have

$$\mid\mid (K^n)^* x \mid\mid = \mid\mid K^* x \mid\mid^2 = 2 \mid x_1 \mid^2 \;\; \textit{and} \;\; \int_\mathbb{R} \mid \langle x, F(\omega)\rangle \mid^2 d\mu(\omega) = \sqrt{\frac{\pi}{2}} \mid x_1 \mid^2 .$$

Then

$$\frac{1}{2} \mid\mid K^* x \mid\mid^2 \leq \int_\mathbb{R} \mid \langle x, F(\omega)\rangle \mid^2 d\mu(\omega) \leq \sqrt{\pi} \mid\mid x \mid\mid^2 .$$

Since K is semi-regular and $KLK = K$. An easy computation shows that

$$K^n L^n = \begin{pmatrix} e^{nia} & 0 \\ 0 & 0 \end{pmatrix}.$$

We deduce that

$$\frac{1}{2} \mid\mid (K^n)^* x \mid\mid^2 \leq \int_\mathbb{R} \mid \langle x, (K^n L^n)\, F(\omega)\rangle \mid^2 d\mu(\omega) \leq \sqrt{\pi} \mid\mid x \mid\mid^2 .$$

Therefore, $(K^n L^n)\, F$ is a continuous K^n-frames for $\mathcal{H}$.

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