Existence of mild solutions for a coupled system of fractional evolution equations via Mönch's fixed point theorem in Banach spaces

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Abstract. The purpose of this paper is to investigate the existence of mild solutions for a nonlinear coupled system of fractional evolution equations with φ-Caputo fractional derivative in a Banach space. The proofs of our main results are based on Mönch's fixed point theorem and certain basic tools of Caputo fractional calculus combined with measure of noncompactness. Our theoretical results have been exploited by offering an interesting example.

1. Introduction

Fractional calculus, which is inherited and has the traits of infinite memory, expands on traditional integer calculus. For certain basic results in the theory of fractional calculus and fractional models, we advise the reader to read these monographs [21, 22, 26, 27]. In addition to the classical and fractional-order differential and integral operators, Almeida introduced in [7] the φ-Caputo fractional derivative, which is another type of a fractional derivative that is defined by using a strictly increasing function, when a specific exponent function is included in the kernel operator. According to this idea, for specific selections of $\varphi(t)$, a large class of well-known fractional derivatives, such as Caputo and Caputo-Hadamard, were found. Additionally, some intriguing information regarding the φ-Caputo fractional derivative initial value and boundary value problems may be found in [4, 8, 9, 15, 13, 14, 28]. The reader is urged to consult the references [5, 16, 17, 13, 20, 23] for further information on fixed point theory, which is a highly helpful tool in the theory of the existence of solutions to functional and differential equations.The existence and uniqueness of solutions for differential equations involving different types of fractional derivatives under diverse boundary conditions are of great interest to academics. The measure of noncompactness is another novel and successful approach to resolving

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the existence results for fractional differential equations. For instance, many researchers have obtained results of existence for nonlinear integral equations using the well-known Darbo fixed point theorem and the Mönch fixed point theory [1, 6, 12]. The research of interconnected systems with fractional differential equations, however, have grown to be fairly significant. These systems are used in several applications in applied science. However, studies of coupled evolution systems with other derivatives, such as Caputo derivatives, are relatively rare, if completely unavailable. Although slowly, there are some in a study of connected evolution systems, see for example [2, 3, 24]. As a result, the goal of this initiative is to introduce the growing pool of knowledge in this area of study. The authors of [28] investigated the presence of mild solutions to starting value problems for fractional semilinear evolution equations with compact and noncompact semigroups on a local and global scale, as well as their uniqueness, in order to be realistic in this study. They determine the structure of the basic solution from the Caputo fractional derivative-induced semigroup and the $\varphi-$function.

The aforementioned works have prompted us to prove certain results for a new coupled system of nonlinear fractional differential equations that have solutions using the φ-Caputo derivative in arbitrary Banach spaces. The associated nonlinear evolution problem systems of φ-Caputo differential equations with Banach space initial conditions have not been explored, as far as the authors are aware. The deficit will thereafter be eliminated. It's crucial to emphasize that the solutions presented in this work are unique and produce a number of novel results as exceptional factors for the proper parameter selection in the significant challenge. We pay special attention to the following problem.

$$(1.1) \quad \begin{cases} {}^{C}D_0^{q_1;\varphi}x(t) = A_1 x(t) + \phi_1(t, x(t), y(t)), \quad t \in I = [0,1], \\[2mm] {}^{C}D_0^{q_2;\varphi}y(t) = A_2 y(t) + \phi_2(t, x(t), y(t)), \quad t \in I, \\[2mm] x(0) = x_0, \\[2mm] y(0) = y_0. \end{cases}$$

Where ${}^{C}D_0^{q_i;\varphi}$ is the φ-Caputo fractional derivative of order $0 < q_i < 1$, $\phi_i :$ $[0, 1] \times X \times X \to X$, $i = 1, 2$ are a given functions satisfying some assumptions that will be specified later, X is a Banach space with norm $\| \cdot \|$ and $x_0, y_0 \in X$, A_1 and A_2 are two infinitesimal generators respectively of two semigroups $\{T_1(t)\}_{t \geq 0}$ and $\{T_2(t)\}_{t \geq 0}$ on X.

This paper has the following structure. In Section 2, we present the definitions and preliminaries we will use to support our main results. In Section 3, we demonstrate that coupled systems (1) have solutions. After that, in Section 4 we provide a specific example to illustrate our general results before concluding this study.

2. Preliminaries

In order to create the groundwork for further advancements, we start this section by introducing some necessary definition and basic results.

Assume that $X = \mathcal{C}(I; R)$ is a Banach space of all continuous functions from I into $\mathbb{R}$ with the norm:

$$\|y\|_\infty = \sup\{\|y(t)\|,\ t \in I\}.$$

Let $L^1(I)$ the space of Bochner-integrable functions $y : I \to X$, with the norm:

$$\|y\|_1 = \int_0^1 \|y(t)\| \mathrm{d}t.$$

Then several essential characteristics properties of the Kuratowski measure of noncompactness are defined.

Definition 1. *[11] The Kuratowski measure of noncompactness ν defined on bounded set B of Banach space X is:*

$$\nu(B) := \inf\{\varepsilon > 0 : B = \bigcup_{m=1}^{n} B_m \text{ and } diam(B_m) \le \varepsilon \text{ for } m = 1, 2, \ldots, n\}.$$

The Kuratowski measure of noncompactness has the following well-known properties.

Lemma 1. *[11] The Kuratowski measure of noncompactness satisfies the following properties.:*

1. $\Omega_1 \subset \Omega_2 \Rightarrow \nu(\Omega_1) \le \nu(\Omega_2)$;

2. $\nu(\Omega_1) = \nu(\overline{\Omega_1}) = \nu\left(\overline{conv}(\Omega_1)\right)$;

3. $\nu(\Omega_1) = 0$ if and only if Ω_1 is relatively compact;

4. $\nu(\lambda \Omega_1) = |\lambda| \nu(\Omega_1)$ where $\lambda \in \mathbb{R}$;

5. $\nu(\Omega_1 \cup \Omega_2) = \max\{\nu(\Omega_1),\ \nu(\Omega_2)\}$;

6. $\nu(\Omega_1 + \Omega_2) \le \nu(\Omega_1) + \nu(\Omega_2)$;

7. $\nu(\Omega_1 + x) = \nu(\Omega_1)$ for any $x \in X$.

Lemma 2. *[19] Let $\mathcal{V} \subset \mathcal{C}(I,\ X)$ be a bounded and equicontinuous subset. Then, the function $t \to \nu(\mathcal{V}(t))$ is continuous on I:*

$$\nu_\mathcal{C}(\mathcal{V}) = \max_{t \in I} \nu(\mathcal{V}(t))\ ,$$

and

$$\nu\left(\int_I x(s) \mathrm{d}s\right) \le \int_I \nu(\mathcal{V}(s)) \mathrm{d}s,\ where\ \mathcal{V}(s) = \{x(s) : x \in \mathcal{V}, s \in I\}.$$

Definition 2. *[29] A function $f : [0,1] \times X \to X$ is said to satisfy the Carathéodory conditions, if the following holds:*

- *The map $t \to f(t,u)$ is measurable for $u \in X$;*

- *The map $u \to f(t,u)$ is continuous for each $t \in [0,1]$.*

The following is a beneficial fixed point result for our objectives:

Theorem 1. *[25] Let $\mathcal{D}$ be a bounded, closed and convex subset of a Banach space, such that $0 \in \mathcal{D}$, and let N be a continuous mapping of $\mathcal{D}$ into itself. If the implication*

$$(2.1) \qquad V = \overline{conv}(\mathcal{N}(V)), with \ V = \mathcal{N}(V) \cup \{0\} \Rightarrow \nu(V) = 0,$$

holds for every subset $V \subset \mathcal{D}$, then $\mathcal{N}$ has a fixed point.

We now present some results and properties from the theory of fractional calculus.

Definition 3. *[10](φ-Riemann-Liouville fractional integral)*
Let $q > 0$, h be an integrable function defined on $[a,b]$ and $\varphi \in \mathcal{C}^n(I,\mathbb{R})$, such that $\varphi'(t) > 0$ for all $t \in [a,b]$.
The φ-Riemann-Liouville fractional integral operator at order q of a function h is given by

$$I_a^{q;\varphi}h(t) = \frac{1}{\Gamma(q)} \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{q-1}h(s)ds.$$

Definition 4. *[10](φ-Riemann-Liouville fractional derivative)*
Let $n \in \mathbb{N}$, $h, \varphi \in \mathcal{C}^n([a,b])$ be two functions such that $\varphi'(t) > 0$, for all $t \in [a,b]$.
φ-Riemann-Liouville fractional derivative at order q of a function h is given by

$$D_a^{q;\varphi}h(t) = \left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n (I_a^{n-q;\varphi}h(t))$$

$$= \frac{1}{\Gamma(n-q)}\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{n-q-1}h(s)ds,$$

where $n = [q]+1$ and $[\alpha]$ denotes the integer part of q.

Definition 5. *[10] Let $n \in \mathbb{N}$, $h, \varphi \in \mathcal{C}^n([a,b])$ be two functions such that $\varphi'(t) > 0$, for all $t \in [a,b]$.*
The φ-Caputo fractional derivative at order q of a function h is given by

$$^C D_a^{q;\varphi}h(t) = (I_a^{n-q;\varphi}h_\varphi^{[n]})(t)$$

$$= \frac{1}{\Gamma(n-q)} \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{n-q-1}h_\varphi^{[n]}(s)ds,$$

where $n = [q]+1$, for $\alpha \notin \mathbb{N}$. And $h_\varphi^{[n]}(t) = \left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n h(t)$ on $[a,b]$.

We note that if $h \in \mathcal{C}^n([a,b])$, the φ-Caputo fractional derivative of order q of h is determined as

$$^C D_a^{q;\varphi} h(t) = D_a^{q;\varphi} \left(h(t) - \sum_{k=0}^{n-1} \frac{h_\varphi^{[k]}(a^+)}{k!} (\varphi(t) - \varphi(a))^k \right).$$

Theorem 2. *[10]*

$$I_a^{q;\varphi} {}^C D_a^{q;\varphi} h(t) = h(t) - \sum_{k=0}^{n-1} \frac{h_\varphi^{[k]}(a^+)}{k!} (\varphi(t) - \varphi(a))^k.$$

In particular, given $q \in (0,1)$ we have:

$$I_a^{q;\varphi} {}^C D_a^{q;\varphi} h(t) = h(t) - h(a).$$

3. Existence and uniquness of mild solutions

Let's start with the following problem:

(3.1)
$$\begin{cases} {}^C D_0^{q;\varphi} x(t) = Ax(t) + h(t, x(t)), & t \in I = [0,1], \\ x(0) = x_0, \end{cases}$$

where $0 < q < 1$, A is the infinitesimal generator of a C_0-semigroup of uniformly bounded linear operators $\{T(t)\}_{t \geq 0}$ on X, $x_0 \in X$ and $f : I \times X \to X$ is function.

Proposition 1. *The problem* (3.1) *is equivalent to the following integral equation*

$$x(t) = x_0 + I_0^{q;\varphi} Ax(t) + I_0^{q;\varphi} h(t, x(t)).$$

Proof. Taking the $\varphi-$Riemann-Liouville fractional integral at order q to the first equation of (3.1) and using Theorem (2), we get:

$$x(t) - x(0) = I_0^{q;\varphi} Ax(t) + I_0^{q;\varphi} h(t, x(t)).$$

Applying (3.1)'s boundary condition, it results in:

$$x(t) = x_0 + I_0^{q;\varphi} Ax(t) + I_0^{q;\varphi} h(t, x(t)).$$

The converse follows by direct computation which completes the proof. $\qquad \square$

Definition 6. *[18] A function x from I to X is said to be an integral solution to* (3.1) *if and only if:*

1. *x is a continuous function from I to X,*

2. *$I^{q;\varphi} x(t) \in D(A)$, for all $t \in I$,*

3.

(3.2)
$$x(t) = x_0 + A I^{q;\varphi} x(t) + I^{q;\varphi} h(t, x(t)).$$

Setting

$$\phi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\varphi_q(\theta^{-\frac{1}{q}}),$$

with:

$$\varphi_q(\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\theta^{-qk-1}\frac{\Gamma(qk+1)}{k!}\sin(k\pi q) , \ \theta \in (0, \ \infty).$$

Definition 7. *[28] A function $x \in \mathcal{C}([0,1], X)$ is said to be a mild solution of (3.1) if it satisfies:*

$$x(t) = S_\varphi^q(t,0)x_0 + \int_0^t (\varphi(t) - \varphi(s))^{q-1}T_\varphi^q(t,s)h(s,x(s))\varphi'(s)ds, \ t \in [0,1],$$

where

$$S_\varphi^q(t,s)x = \int_0^\infty \phi_q(\theta)T((\varphi(t) - \varphi(s))^q\theta)xd\theta,$$

and

$$T_\varphi^q(t,s)x = q\int_0^\infty \theta\phi_q(\theta)T((\varphi(t) - \varphi(s))^q\theta)xd\theta.$$

Proposition 2. *[28] The operators S_φ^q and T_φ^q have the following properties:*

- *For any fixed $t \geq s \geq 0$, $S_\varphi^q(t,s)$ and $T_\varphi^q(t,s)$ are bounded linear operators with $\|S_\varphi^q(t,s)x\| \leq M\|x\|$ and $\|T_\varphi^q(t,s)x\| \leq \frac{qM}{\Gamma(1+q)}\|x\| = \frac{M}{\Gamma(q)}\|x\|$, with $M > 0$ and $x \in X$.*

- *The operators $S_\varphi^q(t,s)$ and $T_\varphi^q(t,s)$ are strongly continuous for all $t \geq s \geq 0$. And for every $x \in X$, $0 \leq s \leq t_1' < t_2' \leq 1$ we have:*
  $\|S_\varphi^q(t_2',s)x - S_\varphi^q(t_1',s)x\| \to 0$ *and* $\|T_\varphi^q(t_2',s)x - T_\varphi^q(t_1',s)x\| \to 0$,
  as $t_1' \to t_2'$.

Theorem 3. *[28] If the third properties of Definition (6) hold, then the problem (3.1) has a mild solution:*

$$x(t) = \int_0^\infty \phi_q(\theta)T((\varphi(t) - \varphi(0))^q\theta)x_0d\theta$$

(3.3)

$$+ q\int_0^t\int_0^\infty \theta\phi_q(\theta)(\varphi(t) - \varphi(s))^{q-1}T((\varphi(t) - \varphi(0))^q\theta)h(s, \ x(s))\varphi'(s)d\theta ds.$$

For any $x \in X$, wa define the operators $S_\varphi^q(t,s)$ and $T_\varphi^q(t,s)$ by:

$$S_\varphi^q(t,s)x = \int_0^\infty \phi_q(\theta)T((\varphi(t) - \varphi(s))^q\theta)xd\theta,$$

and

$$T_\varphi^q(t,s)x = q\int_0^\infty \theta\phi_q(\theta)T((\varphi(t) - \varphi(s))^q\theta)xd\theta,$$

with $0 \leq s \leq t \leq 1$.
Now, we consider the following space:

$$\widetilde{X} = \{(x(t), y(t)) : (x, y) \in \mathcal{C}(I, X) \times \mathcal{C}(I, X)\},$$

is a Banach space equipped with norm $||.||_{\widetilde{X}}$ defined by:

$$||(x, y)||_{\widetilde{X}} = ||x||_{\infty} + ||y||_{\infty}.$$

First, we define what we mean by a boundary value problem solution (1.1).

Definition 8. *A pair of measurable functions* $(x, y) \in \mathcal{C}(I, X) \times \mathcal{C}(I, X)$ *is said to be a solution of problem* (1.1) *if those satisfy equations* (1.1) *on* I, *and the initial conditions.*

Before starting and proving the main results, we introduce the following hypothesis.
(H$_1$): The functions $\phi_i : [0, 1] \times X \rightarrow X$ for $i = 1, 2$ satisfy Carathéodory conditions.
(H$_2$): There exist $\nu_{\phi_i} \in L^{\infty}(I, \mathbb{R}_+)$ with $i = 1, 2$ and a functions $\xi_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ continuous nondecreasing, such that:
$$\|\phi_i(t, x, y)\| \leq \nu_{\phi_i}(t)\xi_i(||x|| + ||y||) \text{ , for a.e. } t \in I \text{ and each } x, y \in X.$$
(H$_3$): For each bounded set $\mathcal{D} \subset X \times X$ and $t \in I$, the following inequality holds:
$$\nu(\phi_i(t, \mathcal{D})) \leq \nu_{\phi_i}(t)\nu(\mathcal{D}) \text{ , } i = 1, 2.$$

In the following, for computational convenience, we put:

$$\mathcal{L}_i := \frac{(\varphi(1) - \varphi(0))^{q_i}}{\Gamma(1 + q_i)} \text{ and } \nu_{\phi_i}^* = \sup_{t \in I} \nu_{\phi_i}(t).$$

Theorem 4. *Assume that the hypotheses* (H$_1$) $-$ (H$_3$) *are satisfied. If*

$$(3.4) \qquad \max\{\mathcal{L}_1 \nu_{\phi_1}^*, \ \mathcal{L}_2 \nu_{\phi_2}^*\} < 1,$$

then the problem (1.1) *has at least one mild solution defined on* I.

Proof. We define the operators $\mathcal{H}_i : \widetilde{X} \rightarrow \mathcal{C}(I, X), i = 1, 2$ by:

$$(\mathcal{H}_1(x, y))(t) = S_{\varphi}^{q_1}(t, 0)x_0$$
$$(3.5) \qquad + \int_0^t \varphi'(s)(\varphi(t) - \varphi(s))^{q_1 - 1} T_{\varphi}^{q_1}(t, 0)\phi_1(s, x(s), y(s))ds, \quad t \in I,$$

and

$$(\mathcal{H}_2(x, y))(t) = R_{\varphi}^{q_2}(t, 0)y_0$$
$$(3.6) \qquad + \int_0^t \varphi'(s)(\varphi(t) - \varphi(s))^{q_2 - 1} W_{\varphi}^{q_2}(t, 0)\phi_2(s, x(s), y(s))ds, \quad t \in I.$$

Where

$$S_\varphi^q(t,s)x = \int_0^\infty \phi_q(\theta)T((\varphi(t) - \varphi(s))^q\theta)xd\theta,$$

$$T_\varphi^q(t,s)x = q\int_0^\infty \theta\phi_q(\theta)T((\varphi(t) - \varphi(s))^q\theta)xd\theta,$$

$$R_\varphi^q(t,s)y = \int_0^\infty \phi_q(\theta)S((\varphi(t) - \varphi(s))^q\theta)yd\theta,$$

and

$$W_\varphi^q(t,s)y = q\int_0^\infty \theta\phi_q(\theta)S((\varphi(t) - \varphi(s))^q\theta)yd\theta.$$

Consider the continuous operator $\mathcal{H} : \widetilde{X} \to \widetilde{X}$ defined by:

(3.7) $$(\mathcal{H}(x,y))(t) = \big((\mathcal{H}_1(x,y))(t), \ (\mathcal{H}_2(x,y))(t)\big).$$

With (H_1) and (H_2) it is evident that $\mathcal{H}$ is definded. Then, the fractional integral equation (3.3) can be used to represent as the following operator equation:

(3.8) $$(x,y) = \mathcal{H}(x,y).$$

Thus, the existence of solutions for problem (1.1) is equivalent to the existence of a fixed point for operator $\mathcal{H}$ which satisfies operator equation (3.8). Define a bounded closed convex set:

$$B_\mathcal{R} = \{(x,y) \in \tilde{X} \ : \ \|(x,y)\|_{\widetilde{X}} \leq \mathcal{R}\},$$

with $\mathcal{R} > 0$, such that:

$$M\left(\|x_0\| + \|y_0\| + \sum_{i=1}^2 \mathcal{L}_i\nu_{\phi_i}^*\xi_i(R)\right) \leq \mathcal{R}.$$

We divided the proof into four steps to satisfy Mönch's fixed point theorem's assumptions.

Step 1: The operator $\mathcal{H}$ maps the set $B_\mathcal{R}$ into itself.

Let $(x,y) \in B_\mathcal{R}$. Then, for all $t \in I$ we have:

$$\|(\mathcal{H}_1(x,\ y))(t)\| \leq M\left(\|x_0\| + \int_0^t \frac{\varphi'(s)(\varphi(t) - \varphi(s))^{q_1-1}}{\Gamma(q_1)}\|\phi_1(s,x(s),y(s))\|\mathrm{d}s\right),$$

and

$$\|(\mathcal{H}_2(x,\ y))(t)\| \leq M\left(\|y_0\| + \int_0^t \frac{\varphi'(s)(\varphi(t) - \varphi(s))^{q_2-1}}{\Gamma(q_2)}\|\phi_2(s,x(s),y(s))\|\mathrm{d}s\right).$$

By using (H_2). For any $t \in I$, we have:

$$\|\phi_i(t,x(t),y(t))\| \leq \nu_{\phi_i}^*\xi_i(\|x(t)\| + \|y(t)\|), \ \ i = 1,2.$$

Hence,

$$\|(\mathcal{H}_1(x,y))\|_\infty \leq M\left(\|x_0\| + \mathcal{L}_1\nu_{\phi_1}^*\xi_1(\mathcal{R})\right),$$

and

$$\|(\mathcal{H}_2(x, y))\|_\infty \leq M \left(\|y_0\| + \mathcal{L}_2 \nu^*_{\phi_2} \xi_2(\mathcal{R}) \right).$$

This implies that:

$$\|\mathcal{H}(x, y)\|_{\widetilde{X}} \leq \|(\mathcal{H}_1(x, y))\|_\infty + \|(\mathcal{H}_2(x, y))\|_\infty$$

$$\leq M \left(\|x_0\| + \|y_0\| + \sum_{i=1}^2 \mathcal{L}_i \nu^*_{\phi_i} \xi_i(R) \right) \leq \mathcal{R}.$$

This proves that $\mathcal{H}$ transforms the ball $B_\mathcal{R}$ into itself.

Step 2: We prove that $\mathcal{H}$ is continuous.

We consider a sequence $\{u_n = (x_n, y_n)\} \in B_\mathcal{R}$, such that $u_n \to u = (x, y)$ as $n \to \infty$. We need to show that $\|\mathcal{H}u_n - \mathcal{H}u\| \to 0$ as $n \to \infty$. It is easy to see that $\phi_1(s, x_n(s), y_n(s)) \to \phi_1(s, x(s), y(s))$ as $n \to +\infty$, due to the Carathéodory continuity of ϕ_1.

On the other hand, taking (H$_2$) into consideration, we get the following inequality:

$$\varphi'(s)(\varphi(t) - \varphi(s))^{q-1} \|\phi_1(s, x_n(s), y_n(s)) - \phi_1(s, x(s), y(s))\|$$

$$\leq 2\nu^*_{\phi_1} \xi_1(\mathcal{R}) \varphi'(s)(\varphi(t) - \varphi(s))^{q-1}.$$

We notice that since the function $s \mapsto 2\nu^*_{\phi_1} \xi_1(\mathcal{R}) \varphi'(s)(\varphi(t) - \varphi(s))^{q-1}$ is Lebesgue integrable over $[0, t]$. This fact together with the Lebesgue dominated convergence theorem implies that:

$$\lim_{n \to +\infty} \int_0^t \frac{\varphi'(s)(\varphi(t) - \varphi(s))^{q_1-1}}{\Gamma(q_1)} \|\phi_1(s, x_n(s), y_n(s)) - \phi_1(s, x(s), y(s))\| \mathrm{d}s = 0.$$

It follows that:

$$\lim_{n \to +\infty} \|(\mathcal{H}_1(x_n, y_n))(t) - (\mathcal{H}_1(x, y))(t)\| = 0,$$

for any $t \in I$, we get that:

$$\lim_{n \to +\infty} \|(\mathcal{H}_1(x_n, y_n)) - (\mathcal{H}_1(x, y))\| = 0,$$

which implies the continuity of the operator $\mathcal{H}_1$. Similarly, we obtain:

$$\lim_{n \to +\infty} \|(\mathcal{H}_2(x_n, y_n)) - (\mathcal{H}_2(x, y))\| = 0.$$

Then:

$$\lim_{n \to +\infty} \|(\mathcal{H}(x_n, y_n)) - (\mathcal{H}(x, y))\| = 0,$$

thus the operator $\mathcal{H}$ is continuous.

Step 3: The operator $\mathcal{H}$ is equicontinuous.

For any $0 < t_1' < t_2' < 1$ and $(x, y) \in B_{\mathcal{R}}$, we get:

$$\|(\mathcal{H}_i(x,y))(t_2') - (\mathcal{H}_i(x,y))(t_1')\| \leq M \int_0^{t_1'} \frac{\varphi'(s)[(\varphi(t_1') - \varphi(s))^{q_i-1} - (\varphi(t_2') - \varphi(s))^{q_i-1}]}{\Gamma(q_i)}$$

$$\times \|\phi_i(s, x(s), y(s))\| ds$$

$$+ M \int_{t_1'}^{t_2'} \frac{\varphi'(s)(\varphi(t_2') - \varphi(s))^{q_i-1}}{\Gamma(q_i)} \|\phi_i(s, x(s), y(s))\| ds$$

$$\leq \frac{M\nu_{\phi_i}^* \xi_i(\mathcal{R})}{\Gamma(q_i+1)}[(\varphi(t_1') - \varphi(0))^{q_i} + 2(\varphi(t_2') - \varphi(t_1'))^{q_i}$$

$$- (\varphi(t_2') - \varphi(0))^{q_i}]$$

$$\leq \frac{2M\nu_{\phi_i}^* \xi_i(\mathcal{R})}{\Gamma(q_i+1)}(\varphi(t_2') - \varphi(t_1'))^{q_i}, i = 1, 2,$$

where we have used the fact that $(\varphi(t_1')-\varphi(0))^{q_i}-(\varphi(t_2')-\varphi(0))^{q_i} \leq 0, i = 1, 2$. Therefore:

$$\|(\mathcal{H}(x,y))(t_2') - (\mathcal{H}(x, y))(t_1')\| \leq 2M \sum_{i=1}^{2} \frac{\nu_{\phi_i}^* \xi_i(\mathcal{R})}{\Gamma(q_i+1)}(\varphi(t_2') - \varphi(t_1'))^{q_i}.$$

As $t_2' \to t_1'$, the right-hand side of the above inequality tends to zero independently of $(x, y) \in B_{\mathcal{R}}$. Hence, we conclude that $\mathcal{H}(B_{\mathcal{R}}) \subseteq C(I, X)$ is bounded and equicontinuous.

Step 4: Mönch's conditions are still valid.
For this purpose, let $V = V_1 \cap V_2$ and V_i be a subset of $B_{\mathcal{R}}$, such that $V_i \subset \overline{conv}(\mathcal{H}_i(V_i) \cup \{0\}), i = 1, 2$.
V_i is bounded and equicontinuous, and therefore, the function $\gamma_i(t) = \nu(V_i(t))$ is continuous on J. by the properties of the Kuratowski noncompactness measure. Using (2) and (H_3), we get: :

$$\gamma_1(t) = \nu(V_1(t)) \leq \nu(\overline{conv}(\mathcal{H}_1(V_1)(t) \cup \{0\})) \leq \nu(\mathcal{H}_1(V_1)(t))$$

$$\leq M\nu \left\{ \int_0^t \frac{\varphi'(s)(\varphi(t) - \varphi(s))^{q_1-1}}{\Gamma(q_1)} \phi_1(s, x(s), y(s)) ds : (x, y) \in V_1 \right\}$$

$$\leq M \int_0^t \frac{\varphi'(s)(\varphi(t) - \varphi(s))^{q_1-1}}{\Gamma(q_1)} \nu(\phi_1(s, V_1(s))) ds$$

$$\leq M \int_0^t \frac{\varphi'(s)(\varphi(t) - \varphi(s))^{q_1-1}}{\Gamma(q_1)} \nu_{\phi_1}(s)\nu(V_1(s)) ds$$

$$\leq M\mathcal{L}_1 \nu_{\phi_1}^* \|\gamma_1\|.$$

Then:

$$\|\gamma_1\| \leq M\mathcal{L}_1 \nu_{\phi_1}^* \|\gamma_1\|.$$

By (3.4), it follows that $\|\gamma_1\|_\infty = 0$; that is $\gamma_1(t) = 0$ for each $t \in I$. In the similar way, we have $\gamma_2(t) = 0$. Hence, $\nu(V(t)) \leq \nu(V_1(t)) = 0$ and

$\nu(V(t)) \leq \nu(V_2(t)) = 0$, this means that $V(t)$ is relatively compact in $X \times X$. By Ascoli-Arzela theorem, V is relatively compact in $B_{\mathcal{R}}$.

According to theorem (1), the fixed point of $\mathcal{H}$ on $B_{\mathcal{R}}$ is $z = (x, y)$ and it is a solution of (1.1). The theorem (4)'s proof is now complete. $\qquad\square$

4. An application

We provide an example in this section to demonstrate the value of our main result. Let's think about the following time-fractional parabolic partial differential equation with a nonlinear source term: initial-boundary value problem

(4.1)
$$\begin{cases} D^{\frac{1}{2},\exp(t)}v(x,t) = \triangle v(x,t) + \frac{1}{5}\exp(-t)v(t,x), \\ D^{\frac{1}{2},\exp(t)}v(x,t) = \frac{\partial}{\partial x}v(x,t) + v(t,x), \\ v(0,t) = v(1,t) = 0, \quad t \in (0,1], \\ v(x,0) = v_0(x), \quad x \in [0,1]. \end{cases}$$

With $X = L^2([0,1])$, $\Delta : D(\Delta) \subset X \to X$ and $\frac{\partial}{\partial x} : D(\frac{\partial}{\partial x}) \subset X \to X$ are two operators defined by:

$D(\Delta) = H^2(0,1) \cap H_0^1(0,1) = \{v \in H^2(0,1) : v(0) = v(1) = 0\}$,

$D(\frac{\partial}{\partial x}) = H^2(0,1) \cap H_0^1(0,1) = \{v \in H^2(0,1) : v(0) = v(1) = 0\}$,

where $H^2(0,1)$ is the completion of the space $C^2(0,1)$ with respect to the norm

$$\|v\|_{H^2(0,1)} = \left(\int_0^1 \sum_{|\nu| \leq 2} |D^\nu v(x)|^2 dx \right)^{\frac{1}{2}},$$

and C^2 is the set of all continuous defined on $(0,1)$ which have continuous partial derivatives of order less than or equal to 2 and $H_0^1(0,1)$ is the completion of $C^1(0,1)$ with respect to the norm $\|v\|_{H^1(0,1)}$.

Then for any $t \in [0,1]$, we have:

$$\|\phi_1(t,x,y)\| \leq \frac{1}{5}e^{-t}\|(x,y)\| \text{ and } \|\phi_2(t,x,y)\| \leq \|(x,y)\|.$$

$$\|\phi_i(t,x,y) - \phi_i(t,x',y')\| \leq \max\{\|x - x'\|, \|y - y'\|\},$$

for $x, y, x', y' \in \Omega_r = B(0,r)$, for $i = 1,2$

and

$$\nu(\phi_i(t,\mathcal{D})) \leq \nu_{\phi_i}(t)\nu(\mathcal{D}), \ t \in [0,1] \text{ and } \mathcal{D} \in \Omega_r = B(0,r).$$

Therefore, $(H_1), (H_2)$ and (H_3) are satisfied. We take $M = 1$ and $\mathcal{L} = \frac{1}{5}$, this yields

(4.2) $\qquad \dfrac{ML}{\Gamma(1+q)}(\varphi(T) - \varphi(0))^q = \dfrac{1}{5\Gamma(\frac{3}{2})}(\varphi(1) - \varphi(0))^{\frac{1}{2}} \approx 0.1972 < 1,$

and

(4.3) $\qquad\qquad \mathcal{L}_1 = \mathcal{L}_2 = \dfrac{(\varphi(1) - \varphi(0))^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \approx 0.986 < 1.$

Hence, the condition 3.4 is satisfied. Consequently, Theorem 4 implies that problem (4.1) has at least one mild solution on $[0, 1]$.

5.   Conclusion

In this article, we have demonstrated the existence of solutions for a coupled system of $\varphi-$Caputo fractional differential equations in an abstract Banach space. As a preliminary step, we construct a generic structure of solutions associated with our proposed model by using fractional calculus tools and certain fundamental properties of $\varphi-$Caputo fractional derivative and $\varphi-$fractional integral. The existence theorem is established by employing Mönch's fixed point theorem and measures of noncompactness after the fixed point operator equation is provided. Finally, by using an appropriate example, the investigation of our theoretical result has been illustrated.

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