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A characterization of Ricci solitons on a special golden Riemannian manifolds

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Abstract. The present paper is devoted to characterizing the various kinds of Ricci solitons in a new class of golden Riemannian manifolds.

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1. Introduction

In 1982, Hamilton [11] introduced the concept of Ricci flow to obtain a canonical metric on a smooth manifold. Ricci flow is currently widely used to explore Riemannian manifolds, especially those with positive curvature. Perelman ([16],[17]) used Ricci flow and surgery theory to prove Poincare's conjecture. Ricci flow is a Riemannian metric evolution equation with the following definition:

$$\frac{\partial}{\partial t} g_{\alpha\beta}(t) = -2R_{\alpha\beta},$$

where $g_{\alpha\beta}(t)$ is a one-parameter family of metrics on a specific manifold.

A Ricci soliton appears as the limit of the Ricci flow's solutions. Ricci soliton is a solution to the Ricci flow, If it only moves by a one-parameter diffeomorphism and scaling group. Ricci soliton (g, ν, κ) on a Riemannian manifold (N, g) is a generalisation of an Einstein metric such that

$$L_\nu g + 2S + 2\kappa g = 0 \tag{1.1}$$

where S is the Ricci tensor and L_ν denotes the Lie derivative in ν direction[12].

The Ricci soliton is called:

$$\begin{aligned} \textit{Steady} \quad if \quad & \kappa = 0, \\ \textit{Shrinking} \quad if \quad & \kappa > 0, \\ \textit{and} \quad \textit{Expanding} \quad if \quad & \kappa < 0. \end{aligned}$$

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Pigola et.al [18] introduced the notion of Ricci almost soliton. A Ricci almost soliton (g, ν, κ) on a Riemannian manifold (N, g) satisfies the relation (1.1) for some smooth function κ on N.

A modified version of traditional Ricci flow equation was introduced by Fischer [7], that modifies the unit volume constraint of Ricci flow equation to a scalar curvature constraint. The resulting equations are referred as Conformal Ricci flow equations and are given as:

$$\frac{\partial g}{\partial t} + 2(Ric + \frac{g}{m}) = -pg, \tag{1.2}$$

$$\tau(g) = -1, \tag{1.3}$$

for a non-dynamical scalar field p and dynamically evolving metric g, $\tau(g)$ being the scalar curvature of the manifold. The above equations are comparable to the Navier-Stokes equation of fluid mechanics. Because of this comparibility the time dependent scalar field p is reffered to as conformal pressure.

As a natural generalisation of the standard Ricci soliton, Basu et. al [1] introduced the notion of conformal Ricci soliton.

A Riemannian metric g on a smooth manifold of dimension m is referred to as conformal Ricci soliton if there exists a constant κ and vector field ν such that

$$L_\nu g + 2S = 2\left(\kappa - (\frac{p}{2} + \frac{1}{m})\right) g, \tag{1.4}$$

where p is conformal pressure and $S = Ric$ is the Ricci tensor. Over the years many authors have worked on Conformal Ricci solitons [19, 8, 21].

Furthermore Hamilton introduced the notion of Yamabe flow [12]. It deforms a given manifold by evolving its metric according to

$$\frac{\partial}{\partial t} g(t) = -\tau(t) g(t),$$

with τ being the scalar curvature of metric g. Yamabe soliton corresponds to self similar solutions of Yamabe flow. Let (N, g) be a Riemannian manifold, if Riemannian metric g satisfies

$$\frac{1}{2} L_\nu g = (\tau - \lambda) g \tag{1.5}$$

where L_ν denotes the Lie derivative in ν direction and λ is some real number. The vector field ν is called soliton field for the manifold N. A Yamabe soliton is said to be steady, shrinking and expanding if it admits a soliton vector field for which $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$ respectively. A Riemannian manifold is said to have Quasi Yamabe soliton if it admits a soliton vector field ν such that

$$\frac{1}{2} L_\nu g = (\tau - \lambda) g + \mu \nu^* \otimes \nu^* \tag{1.6}$$

for some constant λ and some function μ, and ν^* being the dual of soliton vector field ν. The notion of quasi-Yamabe soliton was introduced by Chen and

Deshmukh [3]. In particular if $\mu = 0$, then the quasi-Yamabe soliton reduces to Yamabe soliton. We denote Yamabe soliton by (N, g, ν, λ) and quasi-Yamabe soliton by $(N, g, \nu, \lambda, \mu)$.

The Golden ratio has captivated western intellectuals of varied interests for at least 2,000 years. Some of the greatest mathematical minds of all time have spent endless hours debating this simple ratio and its properties, from ancient Greek mathematicians Pythagoras and Euclid to medieval Italian mathematician Leonardo of Pisa and Renaissance astronomer Johannes Kepler to modern-day scientific figures like Oxford physicist Roger Penrose. In fact, no other number in the history of mathematics has inspired philosophers of all fields like the Golden ratio (see [14]).

Hretcanu [13] introduced the Golden structure to Riemannian manifolds. In a recent work, Crasmareanu and Hretcanu [4] investigated the geometry of the Golden structure on manifolds. They used the matching almost product structure to study the geometry of the Golden structure on a manifold. In [9] Gezer et. al investigated the integrability problem for Golden Riemannian structures and provided significant theorems. Badlaji [10] investigated a novel class of nearly Golden Riemannian structures, as well as a specific form within this class [2]. Many authors have recently written about the topic ([6],[15], [5], [23]).

2. Prelimnaries

Definition 2.1. Let N be a (2m + 1)-dimensional differentiable manifold. The data (Φ, ξ, η, g) is an almost Golden Riemannian nearly contact metric structure on N, where ξ is a global vector field and η is a 1-form on N satisfying $\eta(\xi) = 1$, g is a Riemannian metric that equals $g(U, \xi) = \eta(U)$, and Φ is a tensor field of type (1, 1) that satisfies

$$\Phi U = \phi U - \eta(U)\xi, \tag{2.1}$$

for all vector fields U on N.

A Golden structure is an isomorphism on the tangent space of the manifold, T_xN, for every $x \in N$ and the eigenvalues of Golden structure Φ are the Golden ratio

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \textit{and} \quad 1-\phi = \frac{1-\sqrt{5}}{2}.$$

Furthermore, if Φ is integrable, (Φ, ξ, η, g) is a Golden Riemannian almost contact metric structure, and (N, Φ, ξ, η, g) is a Golden Riemannian almost contact metric manifold [4].

Definition 2.2. [2] An almost Golden Riemannian almost contact manifold (N, Φ, ξ, η, g) is called G^* Golden manifold if it satisfies:

$$(\nabla_U \Phi)V = \sqrt{5}\left(g(U, V)\xi + \eta(V)U - 2\eta(U)\eta(V)\xi\right),$$

for all vector fields U, V on N, where ∇ is the Levi Civita connection on N.

Now on a $(2m+1)$ dimensional G^* Golden manifolds, we have the following results from [2].

$$\nabla_U \xi = -U + \eta(U)\xi, \tag{2.2}$$
$$(\nabla_U \eta)V = -g(U,V) + \eta(U)\eta(V), \tag{2.3}$$
$$R(U,V)\xi = \eta(U)V - \eta(V)U, \tag{2.4}$$
$$R(U,\xi)V = g(U,V)\xi - \eta(V)U, \tag{2.5}$$
$$S(U,\xi) = -2m\eta(U), \tag{2.6}$$

for all vector fields U, V in TN, where R and S are the curvature tensor and Ricci tensor respectively.

3. Main results

In this section we study Ricci almost soliton and conformal Ricci soliton on G^* Golden Riemannian manifolds.

Definition 3.1. ([20], [23]) Let N be a $(2m+1)$-dimensional Riemannian manifold with Riemannian metric g. A smooth vector field ν on N is said to be a conformal Killing vector field if

$$(L_\nu g)(U,W) = 2\rho g(U,W), \tag{3.1}$$

for any vector field U, W on N^{2m+1} where ρ is smooth fuction on N. In particular if ρ is constant then ν is called homothetic vector field and if $\rho = 0$, then ν is called a killing vector field.

Theorem 3.2. *Let (g,ν,κ) be an Ricci almost soliton on a $(2m+1)$-dimensional G^* Golden manifold N. If ν is conformal killing vector field then we have*

1. *N is an Einstein manifold,*
2. *$(\kappa+\rho)$ is constant, with both of them being smooth functions,*
3. *N is Symmetric,*
4. *N is Semisymmetric.*

Proof Suppose that $(2m+1)$-dimensional G^* Golden manifold admits Ricci almost soliton (g,ν,κ) and ν be the Conformal killing vector field then from (1.1) and (3.1) we have

$$S(U,W) = -(\kappa+\rho)g(U,W), \tag{3.2}$$

it follows that manifold under consideration is an Einstein manifold and $(\kappa+\rho)$ is always constant by Bianchi's identity, and both of them are smooth functions.

Because the manifold is an Einstein manifold, its Ricci tensor must be parallel, therefore the manifold must be Ricci Symmetric, implying (3). It is commonly known that every Ricci symmetric manifold is also a Ricci semisymmetric manifold, however the converse is not true [22]. Now for any vector fields U, V, W, Z on N, we have

$$(R(U,V).S)(W,Z) = -S(R(U,V)W,Z) - S(W,R(U,V)Z),$$

Using (3.2) in the foregoing equation we get

$$(R(U,V).S)(W,Z) = (\kappa+\rho)[g(R(U,V)W,Z) + g(W,R(U,V)Z)] = 0,$$

which implies the manifold is Ricci semmisymmetric, which implies (4). As N is Ricci semmisymmetric we can write

$$S(R(U,V)W,Z) + S(W,R(U,V)Z) = 0.$$

Putting $Z = \xi$, the previous equation yields

$$S(R(U,V)W,\xi) + S(W,R(U,V)\xi) = 0.$$

Using (2.4) and (2.6) in the foregoing equation we have

$$-2m\eta(R(U,V)W) + S(W,[\eta(U)V - \eta(V)U]) = 0.$$

Replacing V by ξ and then using (2.5) and (2.6) we get

$$S(U,W) = -2mg(U,W).$$

for every vector filed U and W on N.

Equating (3.2) and foregoing equation we have

$$\kappa = 2m - \rho$$

for all vector fields U, V in TN.

The foregoing result leads us to the following theorem:

Theorem 3.3. *In a G^* Golden manifold N^{2m+1}, a Ricci almost soliton (g,κ,ν) with ν being the conformal killing vector filed is (i) steady for $\rho = 2m$, (ii) shrinking for $\rho < 2m$ and (iii) expanding for $\rho > 2m$.*

Theorem 3.4. *If a $(2m+1)$ dimensional G^* Golden manifold N admits a conformal Ricci soliton (g,ν,κ) then the soliton is Steady if $p = 2\left(2m - \frac{1}{2m+1}\right)$, Shrinking if $p > 2\left(2m - \frac{1}{2m+1}\right)$ and Expanding if $p < 2\left(2m - \frac{1}{2m+1}\right)$.*

Proof For all vector filed U, V in $T\hat{N}$, from (1.4) we have

$$L_\nu g(U,V) + 2S(U,V) = 2\left(\kappa - (\frac{p}{2} + \frac{1}{m+1})\right) g(U,V). \tag{3.3}$$

Define a symmetric contravariant rank (2) tensor as:

$$\alpha(U,V) = L_\nu g(U,V) + 2S(U,V). \tag{3.4}$$

As g is a metric connection, we have $\nabla g = 0$ and α is parallel with the Levi-Civita connection, and hence we have $\nabla\alpha$=0, which can also be written as

$$\alpha(R(U,V)W,Z) + \alpha(Z,R(U,V)W) = 0.$$

Replcing V, W and Z by the characteristic vector field ξ then using (2.4) we have;

$$\alpha(U,\xi) = \alpha(\xi,\xi)\eta(U). \tag{3.5}$$

Taking covariant derivative of above equation along arbitrary vector field V we have;

$$\alpha(\nabla_V U,\xi) + \alpha(U,\nabla_V\xi) = \alpha(\xi,\xi)\nabla_V\eta(U).$$

Using (2.2) and (2.3) we get

$$\alpha(\nabla_V U,\xi) + \alpha(U,-V+\eta(V)\xi) = \alpha(\xi,\xi)[-g(U,V)+\eta(U)\eta(V)+\eta(\nabla_V U)].$$

(3.6)

In view of (3.5) equation (3.6) reduces to

$$\alpha(U,V) = \alpha(\xi,\xi)g(U,V). \tag{3.7}$$

Again from (3.6) we have

$$\alpha(\xi,\xi) = L_\nu g(\xi,\xi) + 2S(\xi,\xi).$$

By the definition of Lie deivative and using (2.6) the equation reduces to

$$\alpha(\xi,\xi) = -4m. \tag{3.8}$$

Using (3.8) in (3.7) and recalling (3.4) we get

$$L_\nu g(U,V) + 2S(U,V) = -4mg(U,V). \tag{3.9}$$

On equating (3.3) and (3.9) we obtain

$$\left(\kappa - (\frac{p}{2} + \frac{1}{2m+1})\right) g(U,V) = -2mg(U,V).$$

It follows that

$$\kappa = [-2m + (\frac{p}{2} + \frac{1}{2m+1})],$$

hence the soliton is Steady if $\kappa = 0$, Shrinking if $\kappa > 0$ and Expanding if $\kappa < 0$. This completes the proof.

4. quasi-Yamabe soliton on new class of Golden manifolds

Theorem 4.1. *If a $(2m+1)$-dimensional G^* Golden manifold N admits a quasi-Yamabe soliton (g, ν, λ, μ) with the potential vector field ν being pointwise collinear with the characteristic vector filed ξ, then*

1. *ν becomes constant multiple of characteristic vector field ξ,*
2. *The soliton vector field ν is strict infinitesimal contact transformation*

Proof. Assume that (g, ν, λ, μ) is a quasi-Yamabe soliton on G^* Golden manifold N, such that the potential vector field is pontwise collinear with the characteristic vector field ξ, then there exists a non-zero smooth function f on N such that $\nu = f\xi$. Then by definition of quasi-Yamabe soliton we have

$$(L_{f\xi}g)(U,V) = 2\left[((\tau-\lambda)g(U,V) + \mu f^2\eta(U)\eta(V)\right]. \tag{4.1}$$

As by the known defintion of Lie derivative and using (2.2) we get

$$(L_{f\xi}g)(U,V) = 2fg(U,V) - 2f\eta(U)\eta(V) + U(f)\eta(V) + V(f)\eta(U). \tag{4.2}$$

On equating foregoing equations we obtain

$$\begin{aligned} 2\left[(\tau-\lambda)g(U,V) + \mu f^2\eta(U)\eta(V)\right] &= 2fg(U,V) - 2f\eta(U)\eta(V) \\ &\quad + U(f)\eta(V) + V(f)\eta(U) \end{aligned} \tag{4.3}$$

Replacing V by ξ in the above equation we have

$$U(f) = [2\left((\tau-\lambda) + \mu f^2\right) - \xi(f)]\eta(U) \tag{4.4}$$

Again replacing U by ξ we get

$$\xi(f) = (\tau-\lambda) + \mu f^2 \tag{4.5}$$

Now consider an orthonormal basis $\{E_i : 1 \le i \le (2m+1)\}$ of the tangent space at each point of the manifold. Then setting $U = V = E_i$ in equation (4.3) and summing over $1 \le i \le (2m+1)$ we get

$$\xi(f) = (2m+1)(\tau-\lambda) + \mu f^2 - 2mf. \tag{4.6}$$

By virtue of (4.5) and (4.6) we have

$$f = \tau - \lambda.$$

As τ and λ both being constant implies f is constant, and hence the soliton vector field ν is constant multiple of ξ which proves the (1). In view of (4.5) we have

$$\mu f^2 = -(\tau - \lambda),$$

which implies μ is constant.

Replacing V by ξ and using $\mu f^2 = -(\tau - \lambda)$ in (4.1) we get

$$(L_\nu g)(U, \xi) = 0,$$

which implies

$$(L_\nu \eta)(U) \quad = \quad g(U, L_\nu \xi). \tag{4.7}$$

As ν is constant multiple of ξ we obtain $Ł_\nu \xi = 0$ and hence (4.7) implies $(L_\nu \eta)(U) = 0$ for any vector field on manifold N, hence ν is strict infinitesimal contact transformation. This proves the second part of the theorem.

5. Concluding Remark

During last decade the researchers did a lot of work on both Golden Riemannian manifolds and Ricci solitons. The present paper will lead researchers to a new direction and obtain some friutful results.

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