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Haar meager and Haar null subsets of hypergroups

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Abstract. In this paper, we define and study two collections of small subsets of hypergroups: Haar null sets and Haar meager sets.

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1. Introduction and Preliminaries

The concept of Haar null subsets of Abelian Polish groups, as a notion of smallness, first was introduced in [2] (A topological group is called *Polish* if it is separable and completely metrizable). Next, this notion was named *shy* and studied in the paper [8]. In recent decades, many researchers studied the properties of Haar null sets and widely used them regarding other topics in mathematical analysis.

On the other hand, meager sets (also called sets of the first category) are other small subsets of topological spaces which have many applications in topology. A set is called meager if it is a countable union of nowhere dense sets, and a topological space is called a Baire space if the nonempty open sets are non-meager. It is well-known that all completely metrizable spaces and all locally compact Hausdorff spaces are Baire spaces [10]. In particular, meager subsets of a locally compact groups share some properties with the collection of null sets regarding the Haar measure of the locally compact group. Haar meager subsets of abelian Polish groups were introduced by Darji in [3] which are a better analog of Haar null sets in the non-locally-compact case; see [4] for this notion in non-abelian Polish groups. Haar meager sets are the same as meager sets in locally compact Polish groups, while they are different in non-locally-compact abelian groups; see [7] as a survey.

In this paper, we study these two notions of smallness on hypergroups which are a generalization of locally compact groups. In fact, in hypergroups we do not have any action between elements in general, but the space of all Radon measures on a hypergroup is a convolution Banach algebra. In Section 2, we

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introduce the Haar meager subsets of commutative Polish hypergroups and we will show that for each Haar meager subset E of a commutative Polish hypergroup H, there is an element $a \in H$ such that the set $\{a\} * E$ is meager in H. We will show that this result is a generalization of the main result [3, Theorem 2.2]. In Section 3, we give and study some versions of the notion of Haar null sets for hypergroups. Among other facts, we show that a Borel subset $B \subseteq H$ is 1-shy if and only if $\lambda(B) = 0$, where λ is the Haar measure on H.

For the readers' convenience, we recall the definition of hypergroups; see the monographs [1, 9] for basic properties and examples. We denote by $\mathbb{M}(H)$ the space of all regular complex Borel measures on H, and by δ_x the Dirac measure at the point x. The support of a measure $\mu \in \mathbb{M}(H)$ is denoted by $\mathrm{supp}(\mu)$.

Definition 1.1. Suppose that H is a locally compact Hausdorff space, $(\mu, \nu) \mapsto \mu * \nu$ is a bilinear positive-continuous mapping from $\mathbb{M}(H) \times \mathbb{M}(H)$ into $\mathbb{M}(H)$ (called *convolution*), and $x \mapsto x^-$ is an involutive homeomorphism on H (called *involution*) with the following properties:

(i) $\mathbb{M}(H)$ with $*$ is a complex associative algebra;

(ii) if $x, y \in H$, then $\delta_x * \delta_y$ is a probability measure with compact support;

(iii) the mapping $(x, y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ from $H \times H$ into $\mathbf{C}(H)$ is continuous, where $\mathbf{C}(H)$ is the set of all non-empty compact subsets of H equipped with Michael topology;

(iv) there exists a (necessarily unique) element $e \in H$ (called the identity) such that for all $x \in H$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;

(v) for all $x, y \in H$, $e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $x = y^-$;

Then, $H \equiv (H, *, ^-, e)$ is called a *locally compact hypergroup* (or simply *hypergroup*).

A nonzero nonnegative regular Borel measure m on H is called a (left) *Haar measure* if for each $x \in H$, $\delta_x * m = m$.

We recall that the *center* of a hypergroup H is defined by

$$Z(H) := \{x \in H : \delta_x * \delta_{x^-} = \delta_{x^-} * \delta_x = \delta_e\},$$

and introduced in [5] and [9] (see also [11] for more information). In fact, $Z(H)$ is the maximal subgroup of H, and for each $x \in Z(H)$ and $y \in H$, $\mathrm{supp}(\delta_x * \delta_y)$ and $\mathrm{supp}(\delta_y * \delta_x)$ are singleton.

Let $f : H \to \mathbb{C}$ be a measurable function and $\mu \in M(H)$. For each $a, b \in H$ we put

$$f_b(a) = f(a * b) := \int_H f \, d(\delta_a * \delta_b).$$

Also, we define

$$(\mu * f)(x) := \int_H f(y^- * x)\, d\mu(y) \qquad (f * \mu)(x) := \int_H f(x * y^-)\, d\mu(y)$$

for all $x \in H$, while these integrals exist. Hence,

$$(f * \delta_a)(x) = f(x * a^-) \tag{1.1}$$

for all $a, x \in H$. By [9, Lemma 3.1F], for each $\mu \in M(H)$ we have

$$\int_H f_a\, d\mu = \int_H f\, d(\delta_a * \mu). \tag{1.2}$$

For each subsets $A, B \subseteq H$ we denote

$$A * B := \bigcup_{a\in A,\, b\in B} \operatorname{supp}(\delta_a * \delta_b).$$

Convolution of a compact set and a closed set is closed by [9, Lemma 4.1E].

2. Haar Meager Subsets of Hypergroups

In this section we study meager and Haar meager subsets of hypergroups. Recall that any separable completely metrizable topology is called *Polish*.

Definition 2.1. A hypergroup H is called *Polish* if its topology is Polish.

Example 2.2. Clearly, every polynomial hypergroup is a commutative Polish hypergroup.

Definition 2.3. Let X be a topological space. A subset $E \subseteq X$ is called *nowhere dense* if the interior of the closure of E is empty set (or equivalently for each open set $\varnothing \neq U \subseteq X$, $E \cap U$ is not dense in U). A subset $F \subseteq X$ is called *meager* (or of *first category*) if F equals to a countable union of nowhere dense subsets of X. A set is called *comeager* if it is complement of a meager set.

Clearly, any subset of a meager set is meager.

Theorem 2.4. *Let H be a hypergroup, and $E \subseteq H$. Then the following are equivalent:*

1. *E is a meager subset of H.*
2. *For each $a \in Z(H)$, the set $\{a\} * E$ is meager in H.*
3. *For some $a \in Z(H)$, the set $\{a\} * E$ is meager in H.*

Proof. (1)⇒(2) Let $E \subseteq H$ be a meager set, and let $a \in Z(H)$. Then, there is a sequence $\{E_n\}_n$ of nowhere dense subsets of H such that $E = \bigcup_n E_n$. So, we have

$$\{a\} * E = \bigcup_n (\{a\} * E_n).$$

We claim that for each n, $\{a\} * E_n$ is a nowhere dense set too. In contrast, assume that for some n,

$$\left(\overline{\{a\} * E_n}\right)^\circ \neq \varnothing.$$

Then, there are $x \in H$ and $r > 0$ such that

$$B(x;r) \subseteq \overline{\{a\} * E_n} \subseteq \{a\} * \overline{E_n}$$

because by [9, Lemma 4.1E] the set $\{a\} * \overline{E_n}$ is closed. Hence,

$$\{a^-\} * B(x;r) \subseteq \{a^-\} * \left(\{a\} * \overline{E_n}\right) = \overline{E_n}$$

because a is a center element. But, by Theorem [9, Lemma 4.1D] the set $\{a^-\} * B(x;r)$ is open. This implies that $(\overline{E_n})^\circ \neq \varnothing$, and therefore, E_n is not nowhere dense, a contradiction. So, for each n, $\{a\} * E_n$ is nowhere dense in H, and hence $\{a\} * E$ is a meager set.

(2)⇒(3) Trivial.

(3)⇒(1) Let for an element $a \in Z(H)$, the set $\{a\} * E$ be meager. Then, $a^- \in Z(H)$, and hence by the conclusion (1) ⇒ (2) the set $\{a^-\}*(\{a\} * E) = E$ is meager too, and the proof is complete.

□

This fact implies the next well-known result in the group case, because for each locally compact group G we have $Z(G) = G$. Also, note that in this case, for each $a \in G$ and $E \subseteq G$ we have $\{a\} * E = aE$.

Corollary 2.5. *Let G be a topological group, and $E \subseteq G$. Then the following are equivalent:*

1. *E is a meager subset of G.*
2. *For each $a \in G$, the set aE is meager in G.*
3. *For some $a \in G$, the set aE is meager in G.*

The above results lead us to give the statement 2.8 regarding the relationship between meager and Haar meager subsets of a hypergroup. This would be an extension of [3, Theorem 2.2]. First, we define *Haar meager* subsets of hypergroups as a generalization of this concept in the group case in [3, Definition 2.1].

Definition 2.6. Let H be a commutative Polish hypergroup, and $A \subseteq H$. We say that A is *Haar meager* if there are a Borel set B of H containing A, a compact metric space K, and a continuous function $f : K \to H$ such that for each $x \in H$, $f^{-1}(B * \{x\})$ is a meager subset of K.

For readers' convenience, we recall the following well-know theorem which is a key tool in the proof of Theorem 2.8.

Theorem 2.7 (Kuratowski–Ulam)**.** *Let X and Y be second countable Baire spaces and $A \subseteq X \times Y$ has Baire property. Then, the following are equivalent:*

1. *A is meager in $X \times Y$.*
2. *the set $\{x \in X : A_x$ is meager in $Y\}$ is comeager in X, where*

$$A_x := \{y \in Y : (x, y) \in A\}.$$

Even if A does not have the Baire property, (2) follows from (1).

Theorem 2.8. *Let H be a commutative Polish hypergroup. If E is a Haar meager subset of H, then there is an element $a \in H$ such that the translation $\{a\} * E$ is meager.*

Proof. Let A be a Borel Haar meager subset of H. By Definition 2.6 there are a compact metric space K and a continuous function $f : K \to H$ such that for each $x \in H$, $f^{-1}(A * \{x\})$ is meager in K. Denote

$$\Gamma := \bigcup_{x \in H} \Big[\{x\} \times f^{-1}(A * \{x^-\})\Big].$$

Hence, thanks to Kuratowski–Ulam Theorem 2.7, Γ is a meager subset of $H \times K$. Also, there exists an element $y \in K$ such that the set $E := \{x \in H : (x, y) \in \Gamma\}$ is a meager subset of H. But,

$$E = \{x \in H : f(y) \in A * \{x^-\}\},$$

and by [9, Lemma 4.1B],

$$f(y) \in A * \{x^-\} \quad \text{if and only if} \quad x \in A * \{f(y)^-\}.$$

This implies that $E = A * \{f(y)^-\}$, and the proof is complete. □

U.B. Darji in [3, Theorem 2.4] shows that two concepts meagerness and Haar meagerness are same in *locally compact* groups. Next, we see that in some *locally compact* hypergroups, there are Haar meager sets which are not meager.

Example 2.9. Let $\mathcal{H}_\infty$ be the one-point compactification of $\mathbb{Z}_+ := \{0, 1, 2, ...\}$, and p be a prime number. A convolution, with ∞ as the identity, is defined on $\mathcal{M}(\mathcal{H}_\infty)$ by

$$\delta_m * \delta_n := \begin{cases} \delta_{\min\{n,m\}}, & n \neq m, \\ \frac{p-2}{p-1}\delta_n + \sum_{k=1}^{\infty} \frac{1}{p^k}\delta_{k+n}, & n = m, \end{cases}$$

where $m, n \in \mathbb{Z}_+$. Then, $\mathcal{H}_\infty$ is a Hermitian compact hypergroup. For more details see [6]. Let $E \subseteq \mathbb{Z}_+$. If E is bounded, then $(\overline{E})^\circ = E$. Let E be unbounded. In this case, $(\overline{E})^\circ = \{\infty\} \cup E$. This implies that any $\varnothing \neq E \subseteq \mathbb{Z}_+$ is not nowhere dense in $\mathcal{H}_\infty$. Similarly, every set $E \subseteq \mathcal{H}_\infty$ with $\infty \in E$ and $E \cap \mathbb{Z}_+ \neq \varnothing$, is not nowhere dense. Finally, if $\infty \in \{\infty\}^\circ$, then there is a finite subset K of $\mathbb{Z}_+$ such that $\{\infty\} = \mathcal{H}_\infty \setminus K$, a contradiction. So, $(\overline{\{\infty\}})^\circ = \{\infty\}^\circ = \varnothing$, and hence $\{\infty\}$ is the only nowhere dense (and meager) subset of $\mathcal{H}_\infty$.

As we know, $Z(\mathcal{H}_\infty) = \{\infty\}$. For each $n \in \mathbb{Z}_+$ we have $\{n\} * \{\infty\} = \{n\}$. Hence, if $n \notin Z(\mathcal{H}_\infty)$, $\{n\} * \{\infty\}$ is not meager. In sequel, we show that $\{\infty\}$ is not a Haar meager subset of $\mathcal{H}_\infty$. In contrast, let $\{\infty\}$ be Haar meager. So, there are a non-empty compact metric space K and a continuous function $f : K \to \mathcal{H}_\infty$ such that for each $x \in \mathcal{H}_\infty$, $f^{-1}(\{\infty\} * \{x\}) = f^{-1}(\{x\})$ is meager in K. We have

$$K = f^{-1}(\{\infty\}) \cup f^{-1}(\{1\}) \cup f^{-1}(\{2\}) \cup \ldots,$$

and hence K is a countable union of nowhere dense sets. This contradicts the Baire category theorem.

3. Haar Null Subsets of Hypergroups

In this section, we initiate and study the Haar null subsets of hypergroups, and we introduce a strong version of this notion.

Definition 3.1. Let H be a commutative hypergroup, and let B be a Borel subset of H. For each $n \in \mathbb{N}$,

1. B is called an *n-shy* (or *n-Haar null*) set if there exists a non-negative Borel measure μ on H such that
   - for some compact set $U \subseteq H$ we have $0 < \mu(U) < \infty$, and
   - for each $x_1, \ldots, x_n \in H$,

   $$\int_H \chi_B * \delta_{x_1} * \ldots * \delta_{x_n} \, d\mu = 0. \tag{3.1}$$

   In this case, the measure μ is called an *n-transverse measure* for B.

2. B is called a *strongly n-shy* (or *strongly n-Haar null*) set if there exists a non-zero non-negative Radon measure μ on H such that for each $x_1, \ldots, x_n \in H$,

   $$\int_H \chi_B * \delta_{x_1} * \ldots * \delta_{x_n} \, d\mu = 0. \tag{3.2}$$

   In this case, the measure μ is called a *strong n-transverse measure* for B.

Remark 3.2. Note that if H is a locally compact *group*, then the relation (3.1) holds if and only if $\mu(Bx) = 0$ for all $x \in H$, because in this case, $\chi_{Bx} = \chi_B * \delta_{x^{-1}}$ by (1.1); see [8, Definition 3] and [7, Definition 3.1.1].

Remark 3.3. 1. Let B be an n-shy subset of H with transverse measure μ. Then, clearly any Borel subset of B is n-shy. Also, for each center element a, and each $x_1, \dots, x_n \in H$ we have

$$\int_H \chi_{B*\{a\}} * \delta_{x_1} * \dots \delta_{x_n} \, d\mu = \int_H \chi_B * \delta_a * \delta_{x_1} * \dots \delta_{x_n} \, d\mu = 0.$$

Indeed, since $a \in Z(H)$ we have $\chi_{B*\{a\}} = \chi_B * \delta_a$, and also by [9], the support of $\delta_a * \delta_y$ is singleton. This implies that $B * \{a\}$ is an n-shy set.

2. By considering the restriction of measure, one can easily see that every n-shy set B has a compact supported finite transverse measure, with arbitrarily small diameter of the support. This implies that B has no interior point, and therefore its complement is dense in H.

Proposition 3.4. *Let H be a commutative hypergroup with a left Haar measure λ. A Borel set $B \subseteq H$ is 1-shy if and only if $\lambda(B) = 0$.*

Proof. Let B be a Borel subset of H. Let $\lambda(B) = 0$. Since λ is a left Haar measure of H, by [9, Lemma 5.1C], its support equals to H. This guaranties the condition (1) in Definition 3.1. By the relation (1.2) for each $x \in H$ we have

$$\begin{aligned}\int_H \chi_B * \delta_x \, d\lambda &= \int_H (\chi_B)_{x^-} \, d\lambda \\ &= \int_H \chi_B d(\delta_{x^-} * \lambda) \\ &= \int_H \chi_B \, d\lambda \\ &= \lambda(B) = 0.\end{aligned}$$

This implies that B is 1-shy and λ is the related transverse measure. Conversely, assume that B is a 1-shy set. So, there exists a transverse measure μ for B. This implies that for each $x \in H$,

$$\int_H (\chi_B * \delta_x)(y) \, d\mu(y) = 0.$$

Hence,

$$\begin{aligned}
0 &= \int_H \int_H (\chi_B * \delta_x)(y)\, d\mu(y)\, d\lambda(x) \\
&= \int_H \int_H (\chi_B * \delta_x)(y)\, d\lambda(x)\, d\mu(y) \\
&= \int_H \int_H \chi_{B^-}(x * y^-)\, d\lambda(x)\, d\mu(y) \\
&= \int_H \int_H (\chi_{B^-})_{y^-}(x)\, d\lambda(x)\, d\mu(y) \\
&= \int_H \int_H \chi_{B^-}\, d(\delta_{y^-} * \lambda)\, d\mu(y) \\
&= \int_H \int_H \chi_{B^-}\, d\lambda\, d\mu(y) \\
&= \lambda(B^-)\, \mu(H)
\end{aligned}$$

thanks to the relation (1.2). Therefore, $\lambda(B^-) = 0$. But, H is unimodular since it is commutative, and also $\lambda = \Delta\, \lambda^-$ by [9, Theorem 5.3B]. Hence, $\lambda(B) = 0$ and the proof is complete. □

Theorem 3.5. *Let $n \in \mathbb{N}$, and μ be a strongly $(n+1)$-transverse measure for a Borel set $B \subseteq H$. Then, for each non-zero non-negative measure $\nu \in \mathbb{M}(H)$, $\mu * \nu$ is a strongly n-transverse measure for B.*

Proof. Assume that μ is a strongly $(n+1)$-transverse measure for a Borel set $B \subseteq H$. Then, thanks to the relation (1.1),

$$\begin{aligned}
&\int_H \chi_B * \delta_{x_1} * \ldots * \delta_{x_n}\, d(\mu * \nu) \\
&= \int_H \int_H \big(\chi_B * \delta_{x_1} * \ldots * \delta_{x_n}\big)(x * y)\, d\mu(x)\, d\nu(y) \\
&= \int_H \int_H \big(\chi_B * \delta_{x_1} * \ldots * \delta_{x_n} * \delta_{y^-}\big)\, d\mu\, d\nu(y) \\
&= 0
\end{aligned}$$

for all $x_1, \ldots, x_n \in H$. This completes the proof. □

Corollary 3.6. *Let $m, n \in \mathbb{N}$. Then, the union of each m strongly $(n+m-1)$-shy subsets of a commutative hypergroup H is a strongly n-shy set.*

Proof. Let $B_1, \ldots, B_m$ be strongly $(n+m-1)$-shy subsets of H with strongly $(n+m-1)$-transverse measures $\mu_1, \ldots, \mu_m$, respectively. By several times applying Theorem 3.5 one can see that the measure $\mu_1 * \ldots * \mu_m$ is a strongly n-transverse measure for every B_j, where $j = 1, \ldots, m$, and so it is strongly n-transverse measure for the union $B := \bigcup_{j=1}^m B_j$ because

$$\chi_B * \delta_{x_1} * \ldots * \delta_{x_n} \le \sum_{j=1}^m \Big(\chi_{B_j} * \delta_{x_1} * \ldots * \delta_{x_n}\Big)$$

for all $x_1, \ldots, x_n \in H$. □

Definition 3.7. A Borel subset B of a commutative hypergroup H is called *locally shy* if every element of H has a neighborhood whose intersection with B is strongly ∞-shy i.e. it is strongly n-shy for each n.

The following result is a direct conclusion of Corollary 3.6.

Corollary 3.8. *All shy sets are locally shy. All compact locally shy subsets of a commutative hypergroup H are strongly 1-shy.*

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